IJMMS 28:4 (2001) 189–200
PII. S016117120100686X
http://ijmms.hindawi.com

MATRIX TRANSFORMATIONS OF STARSHAPED SEQUENCES

CHIKKANNA R. SELVARAJ and SUGUNA SELVARAJ

(Received 1 March 2001)

Abstract. We deal with matrix transformations preserving the starshape of sequences. The main result gives the necessary and sufficient conditions for a lower triangular matrix A to preserve the starshape of sequences. Also, we discuss the nature of the mappings of starshaped sequences by some classical matrices.

2000 Mathematics Subject Classification. 40C05, 40D05, 40G05.

1. Introduction. In [1, 2, 4, 5] the authors have studied the convexity preserving matrix transformations. In [6] Toader introduced the notion of starshaped sequences and showed that the set of convex sequences is a subset of starshaped sequences. In this paper, we prove the results on the transformations of starshaped sequences by a lower triangular matrix A. We begin by the following definitions.

Definition 1.1. A real sequence (x_n) is called a starshaped sequence if

$$\frac{x_n - x_0}{n} \geq \frac{x_{n-1} - x_0}{n-1} \tag{1.1}$$

for $n \geq 2$.

Definition 1.2. Let A be a matrix defining a sequence-to-sequence transformation by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k. \tag{1.2}$$

Next, we define the matrices $A^{(1)} = [a_{n,k}^{(1)}]$, $A^{(2)} = [a_{n,k}^{(2)}]$ as

$$a_{n,i}^{(1)} = \sum_{k=i}^{\infty} a_{n,k}, \qquad a_{n,i}^{(2)} = \sum_{k=i}^{\infty} a_{n,k}^{(1)}. \tag{1.3}$$

Throughout we use A to denote a lower triangular matrix.

Also, for any given sequence (x_n) we can find a corresponding sequence (c_k) such that

$$c_0 = x_0, \qquad c_1 = x_1, \tag{1.4}$$

and for $k \geq 2$,

$$c_k = x_k - \frac{k}{k-1} x_{k-1} + \frac{1}{k-1} x_0, \tag{1.5}$$

which implies that (x_n) can be represented by

$$x_n = n\sum_{k=1}^{n}\frac{c_k}{k}-(n-1)c_0. \tag{1.6}$$

As a consequence we get the following lemma.

Lemma 1.3. *For any two given real numbers α and β, we can construct a sequence (x_n) such that the corresponding sequence (c_k) satisfies that $c_k=\alpha$, $c_j=\beta$, $2\le k<j$ and all other c_i's are zero.*

Proof. Consider the following sequence (x_n) defined as

$$x_n = \begin{cases} 0, & \text{if } n<k, \\ \alpha, & \text{if } n=k, \\ \dfrac{n}{k}\alpha, & \text{if } k<n<j, \\ \dfrac{\beta k + j\alpha}{k}, & \text{if } n=j, \\ \dfrac{n(\beta k+j\alpha)}{kj}, & \text{if } n>j. \end{cases} \tag{1.7}$$

It is clear that the sequence (x_n) satisfies the stated condition. Also, Toader proved the following result in [6]. □

Lemma 1.4. *The sequence (x_n) is starshaped if and only if the corresponding sequence (c_k) given in (1.5) satisfies that $c_k\ge 0$ for $k\ge 2$.*

2. Main results.

We give below the sufficient conditions for a matrix A to preserve the starshape of a sequence.

Theorem 2.1. *A matrix $A=[a_{n,k}]$ preserves the starshape of sequences if*

(i) $(a^{(2)}_{n,0}-a_{0,0})/n=(a^{(2)}_{n-1,0}-a_{0,0})/(n-1)$, *for* $n\ge 2$,

(ii) $a^{(2)}_{n,1}/n=a^{(2)}_{n-1,1}/(n-1)$, *for* $n\ge 2$ *and*

(iii) *for each* $k\ge 2$, $\{a^{(1)}_{n,k}\}_{n=0}^{\infty}$ *is starshaped.*

Proof. Assume that conditions (i), (ii), and (iii) are true. Let (x_k) be a starshaped sequence. Since A is a lower triangular matrix, (x_k) is in the domain of A. Denoting the nth term of the transformed sequence by σ_n and using the representation given in (1.6), we have

$$\begin{aligned}\sigma_n=(Ax)_n&=\sum_{k=0}^{n}a_{n,k}\left[k\sum_{i=1}^{k}\frac{c_i}{i}-(k-1)c_0\right]\\ &=c_0\left[a_{n,0}-a_{n,2}-2a_{n,3}-3a_{n,4}\cdots-(n-2)a_{n,n-1}-(n-1)a_{n,n}\right]\\ &\quad+c_1\left[a_{n,1}+2a_{n,2}+3a_{n,3}\cdots+(n-1)a_{n,n-1}+na_{n,n}\right]\end{aligned}$$

$$+\frac{c_2}{2}[2a_{n,2}+3a_{n,3}+\cdots+(n-1)a_{n,n-1}+na_{n,n}]$$
$$+\frac{c_3}{3}[3a_{n,3}+4a_{n,4}+\cdots+(n-1)a_{n,n-1}+na_{n,n}]$$
$$+\cdots+\frac{c_{n-1}}{n-1}[(n-1)a_{n,n-1}+na_{n,n}]+c_n[a_{n,n}]. \tag{2.1}$$

Using the notations given in Definition 1.2, we can write σ_n as

$$\begin{aligned}\sigma_n &= c_0\left[a^{(2)}_{n,0}-2a^{(2)}_{n,1}\right]+c_1\left[a^{(2)}_{n,1}\right]\\ &\quad+\frac{c_2}{2}\left[2a^{(1)}_{n,2}+a^{(1)}_{n,3}+\cdots+a^{(1)}_{n,n}\right]\\ &\quad+\frac{c_3}{3}\left[3a^{(1)}_{n,3}+a^{(1)}_{n,4}+\cdots+a^{(1)}_{n,n}\right]\\ &\quad+\cdots+\frac{c_{n-2}}{n-2}\left[(n-2)a^{(1)}_{n,n-2}+a^{(1)}_{n,n-1}+a^{(1)}_{n,n}\right]\\ &\quad+\frac{c_{n-1}}{n-1}\left[(n-1)a^{(1)}_{n,n-1}+a^{(1)}_{n,n}\right]+c_n\left[a^{(1)}_{n,n}\right].\end{aligned} \tag{2.2}$$

Here, we note that $\sigma_0=c_0a_{0,0}$. In order to show that $\{\sigma_n\}$ is starshaped, we consider for $n\geq 2$,

$$\begin{aligned}\frac{\sigma_n-\sigma_0}{n}-\frac{\sigma_{n-1}-\sigma_0}{n-1} &= c_0\left[\left(\frac{a^{(2)}_{n,0}-a_{0,0}}{n}-\frac{a^{(2)}_{n-1,0}-a_{0,0}}{n-1}\right)-2\left(\frac{a^{(2)}_{n,1}}{n}-\frac{a^{(2)}_{n-1,1}}{n-1}\right)\right]\\ &\quad+c_1\left[\frac{a^{(2)}_{n,1}}{n}-\frac{a^{(2)}_{n-1,1}}{n-1}\right]\\ &\quad+\frac{c_2}{2}\left[2\left(\frac{a^{(1)}_{n,2}}{n}-\frac{a^{(1)}_{n-1,2}}{n-1}\right)+\left(\frac{a^{(1)}_{n,3}}{n}-\frac{a^{(1)}_{n-1,3}}{n-1}\right)\right.\\ &\qquad\left.+\cdots+\left(\frac{a^{(1)}_{n,n-1}}{n}-\frac{a^{(1)}_{n-1,n-1}}{n-1}\right)+\frac{a^{(1)}_{n,n}}{n}\right]\\ &\quad+\frac{c_3}{3}\left[3\left(\frac{a^{(1)}_{n,3}}{n}-\frac{a^{(1)}_{n-1,3}}{n-1}\right)+\left(\frac{a^{(1)}_{n,4}}{n}-\frac{a^{(1)}_{n-1,4}}{n-1}\right)\right.\\ &\qquad\left.+\cdots+\left(\frac{a^{(1)}_{n,n-1}}{n}-\frac{a^{(1)}_{n-1,n-1}}{n-1}\right)+\frac{a^{(1)}_{n,n}}{n}\right]\\ &\quad+\cdots+\frac{c_{n-2}}{n-2}\left[(n-2)\left(\frac{a^{(1)}_{n,n-2}}{n}-\frac{a^{(1)}_{n-1,n-2}}{n-1}\right)\right.\\ &\qquad\left.+\left(\frac{a^{(1)}_{n,n-1}}{n}-\frac{a^{(1)}_{n-1,n-1}}{n-1}\right)+\frac{a^{(1)}_{n,n}}{n}\right]\\ &\quad+\frac{c_{n-1}}{n-1}\left[(n-1)\left(\frac{a^{(1)}_{n,n-1}}{n}-\frac{a^{(1)}_{n-1,n-1}}{n-1}\right)+\frac{a^{(1)}_{n,n}}{n}\right]+\frac{c_n}{n}\left[a^{(1)}_{n,n}\right].\end{aligned} \tag{2.3}$$

Since (x_k) is starshaped, the corresponding $c_k \geq 0$ for $k \geq 2$. Now, using conditions (i), (ii), and (iii) it is easy to see that

$$\frac{\sigma_n - \sigma_0}{n} - \frac{\sigma_{n-1} - \sigma_0}{n-1} \geq 0. \tag{2.4}$$

Hence the theorem is proved. □

We notice that a matrix A which preserves the starshape of sequences need not satisfy all the three conditions of Theorem 2.1. We give below an example of such a matrix.

Example 2.2. Let A be a matrix given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 0 & \cdots \\ 7 & -8 & 4 & 0 & 0 & \cdots \\ 13 & -12 & -3 & 6 & 0 & \cdots \\ 17 & -16 & -4 & 8 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 4n+1 & -4n & -n & 2n & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{2.5}$$

Then the corresponding matrices $A^{(1)}$ and $A^{(2)}$ are

$$\begin{gathered} A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 0 & \cdots \\ 3 & -4 & 4 & 0 & 0 & \cdots \\ 4 & -9 & 3 & 6 & 0 & \cdots \\ 5 & -12 & 4 & 8 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ n+1 & -3n & n & 2n & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \\ A^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 4 & 0 & 0 & \cdots \\ 4 & 0 & 9 & 6 & 0 & \cdots \\ 5 & 0 & 12 & 8 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ n+1 & 0 & 3n & 2n & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \end{gathered} \tag{2.6}$$

For the above matrix A, it is obvious that conditions (i) and (ii) of Theorem 2.1 hold. But condition (iii) fails because

$$\frac{a_{3,2}^{(1)}-a_{0,2}^{(1)}}{3}<\frac{a_{2,2}^{(1)}-a_{0,2}^{(1)}}{2}, \tag{2.7}$$

implying that $\{a_{n,2}^{(1)}\}_{n=0}^{\infty}$ is not starshaped. But A preserves the starshape of sequences. To see this, using conditions (i) and (ii) of Theorem 2.1 we write (2.3) as

$$\begin{aligned}\frac{\sigma_n-\sigma_0}{n}-\frac{\sigma_{n-1}-\sigma_0}{n-1}=\frac{c_2}{2}\Bigg[2\left(\frac{a_{n,2}^{(1)}}{n}-\frac{a_{n-1,2}^{(1)}}{n-1}\right)+\left(\frac{a_{n,3}^{(1)}}{n}-\frac{a_{n-1,3}^{(1)}}{n-1}\right)&\\+\cdots+\left(\frac{a_{n,n-1}^{(1)}}{n}-\frac{a_{n-1,n-1}^{(1)}}{n-1}\right)+\frac{a_{n,n}^{(1)}}{n}\Bigg]&\\+\frac{c_3}{3}\Bigg[3\left(\frac{a_{n,3}^{(1)}}{n}-\frac{a_{n-1,3}^{(1)}}{n-1}\right)+\left(\frac{a_{n,4}^{(1)}}{n}-\frac{a_{n-1,4}^{(1)}}{n-1}\right)&\\+\cdots+\left(\frac{a_{n,n-1}^{(1)}}{n}-\frac{a_{n-1,n-1}^{(1)}}{n-1}\right)+\frac{a_{n,n}^{(1)}}{n}\Bigg]&\\+\cdots+\frac{c_{n-2}}{n-2}\Bigg[(n-2)\left(\frac{a_{n,n-2}^{(1)}}{n}-\frac{a_{n-1,n-2}^{(1)}}{n-1}\right)&\\+\left(\frac{a_{n,n-1}^{(1)}}{n}-\frac{a_{n-1,n-1}^{(1)}}{n-1}\right)+\frac{a_{n,n}^{(1)}}{n}\Bigg]&\\+\frac{c_{n-1}}{n-1}\Bigg[(n-1)\left(\frac{a_{n,n-1}^{(1)}}{n}-\frac{a_{n-1,n-1}^{(1)}}{n-1}\right)+\frac{a_{n,n}^{(1)}}{n}\Bigg]+\frac{c_n}{n}\Big[a_{n,n}^{(1)}\Big].&\end{aligned} \tag{2.8}$$

Therefore, for any starshaped sequence (x_n), using Lemma 1.4 in the above equation we get

$$\begin{aligned}\frac{\sigma_2-\sigma_0}{2}-\frac{\sigma_1-\sigma_0}{1}&=\frac{c_2}{2}\left(a_{2,2}^{(1)}\right)\ge 0,\\ \frac{\sigma_3-\sigma_0}{3}-\frac{\sigma_2-\sigma_0}{2}&=\frac{c_2}{2}\left[2\left(\frac{a_{3,2}^{(1)}}{3}-\frac{a_{2,2}^{(1)}}{2}\right)+\frac{a_{3,3}^{(1)}}{3}\right]+\frac{c_3}{3}\left(a_{3,3}^{(1)}\right)\ge 0,\end{aligned} \tag{2.9}$$

and for $n\ge 4$,

$$\frac{\sigma_n-\sigma_0}{n}-\frac{\sigma_{n-1}-\sigma_0}{n-1}=0. \tag{2.10}$$

We give below the necessary conditions for a matrix A to preserve the starshape of sequences.

Theorem 2.3. *If a matrix $A=[a_{n,k}]$ preserves the starshape of a sequence, then*

(i) $(a_{n,0}^{(2)}-a_{0,0})/n=(a_{n-1,0}^{(2)}-a_{0,0})/(n-1)$, *for $n\ge 2$,*

(ii) $a_{n,1}^{(2)}/n=a_{n-1,1}^{(2)}/(n-1)$, *for $n\ge 2$, and*

(iii) *for each $k\ge 2$, $\{a_{n,k}^{(2)}\}_{n=0}^{\infty}$ is starshaped.*

Proof. Assume that the matrix A preserves the starshape of sequences. Equation (2.3) which is satisfied by any transformed sequence can be written as

$$\begin{aligned}\frac{\sigma_n-\sigma_0}{n}-\frac{\sigma_{n-1}-\sigma_0}{n-1}&=c_0\left[\left(\frac{a^{(2)}_{n,0}-a_{0,0}}{n}-\frac{a^{(2)}_{n-1,0}-a_{0,0}}{n-1}\right)-2\left(\frac{a^{(2)}_{n,1}}{n}-\frac{a^{(2)}_{n-1,1}}{n-1}\right)\right]\\&\quad+c_1\left[\frac{a^{(2)}_{n,1}}{n}-\frac{a^{(2)}_{n-1,1}}{n-1}\right]+\frac{c_2}{2}\left[2\left(\frac{a^{(2)}_{n,2}}{n}-\frac{a^{(2)}_{n-1,2}}{n-1}\right)-\left(\frac{a^{(2)}_{n,3}}{n}-\frac{a^{(2)}_{n-1,3}}{n-1}\right)\right]\\&\quad+\cdots+\frac{c_j}{j}\left[j\left(\frac{a^{(2)}_{n,j}}{n}-\frac{a^{(2)}_{n-1,j}}{n-1}\right)-(j-1)\left(\frac{a^{(2)}_{n,j+1}}{n}-\frac{a^{(2)}_{n-1,j+1}}{n-1}\right)\right]\\&\quad+\cdots+\frac{c_{n-1}}{n-1}\left[(n-1)\left(\frac{a^{(2)}_{n,n-1}}{n}-\frac{a^{(2)}_{n-1,n-1}}{n-1}\right)-(n-2)\frac{a^{(2)}_{n,n}}{n}\right]\\&\quad+\frac{c_n}{n}\left[a^{(2)}_{n,n}\right].\end{aligned}\tag{2.11}$$

Suppose that condition (ii) does not hold. Therefore, there exists an $N\ge 2$ such that

$$\frac{a^{(2)}_{N,1}}{N}-\frac{a^{(2)}_{N-1,1}}{N-1}=\lambda\neq 0.\tag{2.12}$$

Choose a sequence (x_n) such that $x_n=-n\lambda$. Then (x_n) is starshaped since from (1.5) we get that $c_0=x_0=0$, $c_1=x_1=-\lambda$ and for $k\ge 2$, $c_k=x_k-(k/(k-1))x_{k-1}+(1/(k-1))x_0=0$. Thus for the transformed sequence σ_n, we obtain from (2.11) that

$$\frac{\sigma_N-\sigma_0}{N}-\frac{\sigma_{N-1}-\sigma_0}{N-1}=-\lambda^2<0,\tag{2.13}$$

which contradicts that A preserves the starshape of sequences. Therefore, condition (ii) must be true. Consequently, (2.11) reduces to

$$\begin{aligned}\frac{\sigma_n-\sigma_0}{n}-\frac{\sigma_{n-1}-\sigma_0}{n-1}&=c_0\left[\left(\frac{a^{(2)}_{n,0}-a_{0,0}}{n}-\frac{a^{(2)}_{n-1,0}-a_{0,0}}{n-1}\right)\right]\\&\quad+\frac{c_2}{2}\left[2\left(\frac{a^{(2)}_{n,2}}{n}-\frac{a^{(2)}_{n-1,2}}{n-1}\right)-\left(\frac{a^{(2)}_{n,3}}{n}-\frac{a^{(2)}_{n-1,3}}{n-1}\right)\right]\\&\quad+\cdots+\frac{c_{n-1}}{n-1}\left[(n-1)\left(\frac{a^{(2)}_{n,n-1}}{n}-\frac{a^{(2)}_{n-1,n-1}}{n-1}\right)-(n-2)\frac{a^{(2)}_{n,n}}{n}\right]\\&\quad+\frac{c_n}{n}\left[a^{(2)}_{n,n}\right].\end{aligned}\tag{2.14}$$

Suppose that condition (i) is not true. Therefore, there exists an $N\ge 2$ such that

$$\frac{a^{(2)}_{N,0}-a_{0,0}}{N}-\frac{a^{(2)}_{N-1,1}-a_{0,0}}{N-1}=\beta\neq 0.\tag{2.15}$$

Choose a sequence (x_n) such that $x_n=(n-1)\beta$. Then (x_n) is starshaped since from (1.5) we get that $c_0=x_0=-\beta$, $c_1=x_1=0$ and for $k\ge 2$, $c_k=x_k-(k/(k-1))x_{k-1}+$

$(1/(k-1))x_0 = 0$. Thus for the transformed sequence σ_n, we obtain from (2.14), that

$$\frac{\sigma_N - \sigma_0}{N} - \frac{\sigma_{N-1} - \sigma_0}{N-1} = -\beta^2 < 0, \tag{2.16}$$

which contradicts that A preserves the starshape of sequences. Therefore, the condition (i) must be true. We will now show that condition (iii) is necessary for the matrix A to preserve the starshape of sequences. Since we have established conditions (i) and (ii), (2.14) reduces for $n \ge 2$ to

$$\begin{aligned} \frac{\sigma_n - \sigma_0}{n} - \frac{\sigma_{n-1} - \sigma_0}{n-1} = {} & \frac{c_2}{2}\left[2\left(\frac{a_{n,2}^{(2)}}{n} - \frac{a_{n-1,2}^{(2)}}{n-1}\right) - \left(\frac{a_{n,3}^{(2)}}{n} - \frac{a_{n-1,3}^{(2)}}{n-1}\right)\right] \\ & + \frac{c_3}{3}\left[3\left(\frac{a_{n,3}^{(2)}}{n} - \frac{a_{n-1,3}^{(2)}}{n-1}\right) - 2\left(\frac{a_{n,4}^{(2)}}{n} - \frac{a_{n-1,4}^{(2)}}{n-1}\right)\right] \\ & + \cdots + \frac{c_j}{j}\left[j\left(\frac{a_{n,j}^{(2)}}{n} - \frac{a_{n-1,j}^{(2)}}{n-1}\right) - (j-1)\left(\frac{a_{n,j+1}^{(2)}}{n} - \frac{a_{n-1,j+1}^{(2)}}{n-1}\right)\right] \\ & + \cdots + \frac{c_{n-1}}{n-1}\left[(n-1)\left(\frac{a_{n,n-1}^{(2)}}{n} - \frac{a_{n-1,n-1}^{(2)}}{n-1}\right) - (n-2)\frac{a_{n,n}^{(2)}}{n}\right] \\ & + \frac{c_n}{n}\left[a_{n,n}^{(2)}\right]. \end{aligned} \tag{2.17}$$

To show that $\{a_{n,k}^{(2)}\}_{n=0}^{\infty}$ is starshaped for $k \ge 2$, we need to prove that for each fixed $j \ge 2$, $a_{n,j}^{(2)}/n - a_{n-1,j}^{(2)}/(n-1) \ge 0$, for $n \ge 2$. So, it is sufficient to show that for each fixed $n \ge 2$, $a_{n,j}^{(2)}/n - a_{n-1,j}^{(2)}/(n-1) \ge 0$, for $2 \le j \le n$. Fix $n \ge 2$.

Case 1. When $j = n$ we will show that $a_{n,n}^{(2)} \ge 0$. If $a_{n,n}^{(2)} = \alpha < 0$, then using Lemma 1.3 we choose a starshaped sequence (x_k) such that $c_n = 1$ and other c_k's are zero. Thus from (2.17) we obtain

$$\frac{\sigma_n - \sigma_0}{n} - \frac{\sigma_{n-1} - \sigma_0}{n-1} = \frac{1}{n}a_{n,n}^{(2)} = \frac{\alpha}{n} < 0, \tag{2.18}$$

which is a contradiction. Thus, $a_{n,n}^{(2)} \ge 0$.

Case 2. When $j = n-1$ we will prove that $a_{n,n-1}^{(2)}/n - a_{n-1,n-1}^{(2)}/(n-1) \ge 0$.

Suppose not. If $a_{n,n-1}^{(2)}/n - a_{n-1,n-1}^{(2)}/(n-1) = \beta < 0$, then as before we choose a starshaped sequence (x_k) such that $c_k = 0$ for $k \ne n-1$ and $c_{n-1} = 1$. Thus using Case 1 in (2.17) we obtain

$$\begin{aligned} \frac{\sigma_n - \sigma_0}{n} - \frac{\sigma_{n-1} - \sigma_0}{n-1} &= \frac{1}{n-1}\left[(n-1)\left(\frac{a_{n,n-1}^{(2)}}{n} - \frac{a_{n-1,n-1}^{(2)}}{n-1}\right) - (n-2)\left(\frac{a_{n,n}^{(2)}}{n}\right)\right] \\ &= \frac{1}{n-1}\left[(n-1)\beta - (n-2)\left(\frac{a_{n,n}^{(2)}}{n}\right)\right] \\ &< 0, \end{aligned} \tag{2.19}$$

which is a contradiction. Continuing in this manner with $j = n-2, n-3, \ldots, 3, 2$, we can establish that $a_{n,j}^{(2)}/n - a_{n-1,j}^{(2)}/(n-1) \ge 0$. This completes the proof. □

We give below an example to show that the three conditions of Theorem 2.3 are not sufficient for a matrix A to preserve the starshape of sequences.

Example 2.4. Let A be a matrix given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 0 & 0 & 0 & \dots \\ 3 & 3 & -6 & 3 & 0 & \dots \\ 4 & 8 & -16 & 8 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ n & 2n & -4n & 2n & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{2.20}$$

Then the corresponding matrices $A^{(1)}$ and $A^{(2)}$ are given by

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 0 & 0 & 0 & \dots \\ 3 & 0 & -3 & 3 & 0 & \dots \\ 4 & 0 & -8 & 8 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ n & 0 & -2n & 2n & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \qquad A^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 0 & 0 & 0 & \dots \\ 3 & 0 & 0 & 3 & 0 & \dots \\ 4 & 0 & 0 & 8 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ n & 0 & 0 & 2n & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{2.21}$$

The above matrix A satisfies all three conditions of Theorem 2.3. But A does not preserve the starshape of sequences. To see this, we choose a starshaped sequence (x_n) such that $c_k = 0$ for $k \neq 2$ and $c_2 = 1$. Then the sequence transformed by A is not starshaped, because when $n = 4$, (2.17) yields

$$\frac{\sigma_4 - \sigma_0}{4} - \frac{\sigma_3 - \sigma_0}{3} = \frac{1}{2}\left[2\left(\frac{a^{(2)}_{4,2}}{4} - \frac{a^{(2)}_{3,2}}{3}\right) - \left(\frac{a^{(2)}_{4,3}}{4} - \frac{a^{(2)}_{3,3}}{3}\right)\right] < 0. \tag{2.22}$$

In Theorem 2.1 we stated the sufficient conditions for a matrix to preserve the starshape of sequences, and in Theorem 2.3 we stated the necessary conditions. Now, we give the necessary and sufficient conditions for a matrix A to preserve the starshape of sequences.

Theorem 2.5. *A matrix $A = [a_{n,k}]$ preserves the starshape of sequences if and only if*

(i) $(a^{(2)}_{n,0} - a_{0,0})/n = (a^{(2)}_{n-1,0} - a_{0,0})/(n-1)$, *for* $n \geq 2$,

(ii) $a^{(2)}_{n,1}/n = a^{(2)}_{n-1,1}/(n-1)$, *for* $n \geq 2$, *and*

(iii) *for each* $k \geq 2$, *the sequence* $\{k a^{(2)}_{n,k} - (k-1) a^{(2)}_{n,k+1}\}_{n=0}^{\infty}$ *is starshaped.*

Proof. Assume that conditions (i), (ii), and (iii) are true.

From condition (iii) we obtain, using Definition 1.1 that for each $k \geq 2$,

$$k\left(\frac{a_{n,k}^{(2)}}{n}-\frac{a_{n-1,k}^{(2)}}{n-1}\right)-(k-1)\left(\frac{a_{n,k+1}^{(2)}}{n}-\frac{a_{n-1,k+1}^{(2)}}{n-1}\right)\geq 0. \tag{2.23}$$

If a sequence (x_n) is starshaped, then using Lemma 1.4 and the above inequality in (2.17) we see that the transformed sequence $\{\sigma_n\}$ is also starshaped.

Conversely, let A preserve the starshape of sequences. Then conditions (i) and (ii) follow from Theorem 2.3. Suppose that condition (iii) is not true for some $k = j \geq 2$. Therefore, the sequence

$$\left\{ja_{n,j}^{(2)}-(j-1)a_{n,j+1}^{(2)}\right\}_{n=0}^{\infty} \tag{2.24}$$

is not starshaped. So, there exists an $N > j$ such that

$$j\left(\frac{a_{N,j}^{(2)}}{N}-\frac{a_{N-1,j}^{(2)}}{N-1}\right)-(j-1)\left(\frac{a_{N,j+1}^{(2)}}{N}-\frac{a_{N-1,j+1}^{(2)}}{N-1}\right)=\beta<0. \tag{2.25}$$

Choose a starshaped sequence (x_k) such that $c_k = 0$ for $k \neq j$ and $c_j = -\beta > 0$. Then from (2.17) we see that the transformed sequence satisfies that

$$\frac{\sigma_N-\sigma_0}{N}-\frac{\sigma_{N-1}-\sigma_0}{N-1}=-\frac{\beta^2}{j}<0, \tag{2.26}$$

which is a contradiction. Hence the theorem is proved. □

In the next theorem, we give a sufficient condition for a starshape preserving matrix A to map a non-starshaped sequence into a starshaped one. For simplicity we introduce the following notation

$$\omega_{n,k}^{(2)}:=k\left(\frac{a_{n,k}^{(2)}}{n}-\frac{a_{n-1,k}^{(2)}}{n-1}\right)-(k-1)\left(\frac{a_{n,k+1}^{(2)}}{n}-\frac{a_{n-1,k+1}^{(2)}}{n-1}\right). \tag{2.27}$$

Theorem 2.6. *Let a matrix A preserve the starshape of sequences. If there exist k and j with $j > k \geq 2$ and a constant $\lambda > 0$ such that*

$$\lambda\omega_{n,k}^{(2)}\geq\omega_{n,j}^{(2)} \tag{2.28}$$

for all $n \geq 2$, then A maps a non-starshaped sequence into a starshaped sequence.

Proof. From (2.17) any transformed sequence satisfies for $n \geq 2$,

$$\begin{aligned}\frac{\sigma_n-\sigma_0}{n}-\frac{\sigma_{n-1}-\sigma_0}{n-1}&=\frac{c_2}{2}\left[\omega_{n,2}^{(2)}\right]+\frac{c_3}{3}\left[\omega_{n,3}^{(2)}\right]+\cdots+\frac{c_k}{k}\left[\omega_{n,k}^{(2)}\right]\\&\quad+\cdots+\frac{c_j}{j}\left[\omega_{n,j}^{(2)}\right]+\cdots+\frac{c_{n-1}}{n-1}\left[\omega_{n,n-1}^{(2)}\right]+\frac{c_n}{n}\left[\omega_{n,n}^{(2)}\right].\end{aligned} \tag{2.29}$$

Now, using Lemma 1.3 we choose a non-starshaped sequence (x_k) such that $c_k = \lambda$, $c_j = -1$ and all other c_i's are zero. Thus when $n < k$,

$$\frac{\sigma_n - \sigma_0}{n} - \frac{\sigma_{n-1} - \sigma_0}{n-1} = 0. \tag{2.30}$$

When $k \le n < j$,

$$\frac{\sigma_n - \sigma_0}{n} - \frac{\sigma_{n-1} - \sigma_0}{n-1} = \frac{\lambda}{k}\left[\omega_{n,k}^{(2)}\right] \ge 0, \tag{2.31}$$

by Theorem 2.5. When $n \ge j$,

$$\frac{\sigma_n - \sigma_0}{n} - \frac{\sigma_{n-1} - \sigma_0}{n-1} = \frac{\lambda}{k}\left[\omega_{n,k}^{(2)}\right] - \frac{1}{j}\left[\omega_{n,j}^{(2)}\right] \ge 0, \tag{2.32}$$

by our assumption. Thus $\{\sigma_n\}$ is starshaped. □

3. Examples. In this section, we study the starshape preserving nature of some well-known matrices. Some of the results will not be proved here, as either their proof follows in an obvious manner or is straightforward but more tedious.

One of the most familiar summability matrices is the Cesáro matrix [3, page 44]. This matrix is a lower triangular matrix given by

$$C_{n,k} = \begin{cases} \dfrac{1}{n+1}, & \text{if } k \le n, \\ 0, & \text{if } k > n. \end{cases} \tag{3.1}$$

The corresponding matrices $C^{(1)}$ and $C^{(2)}$ are given by

$$\begin{gathered} C_{n,k}^{(1)} = \begin{cases} \dfrac{n+1-k}{n+1}, & \text{if } k \le n, \\ 0, & \text{if } k > n, \end{cases} \\ C_{n,k}^{(2)} = \begin{cases} \dfrac{(n+1-k)(n+2-k)}{2(n+1)}, & \text{if } k \le n, \\ 0, & \text{if } k > n. \end{cases} \end{gathered} \tag{3.2}$$

Theorem 3.1. *The Cesáro matrix preserves the starshape of the sequences. Also, it maps a non-starshaped sequence into a starshaped one.*

Proof. It is easy to verify conditions (i) and (ii) of Theorem 2.5. To see condition (iii), consider

$$k\left(\frac{C_{n,k}^{(2)}}{n} - \frac{C_{n-1,k}^{(2)}}{n-1}\right) - (k-1)\left(\frac{C_{n,k+1}^{(2)}}{n} - \frac{C_{n-1,k+1}^{(2)}}{n-1}\right), \tag{3.3}$$

which simplifies to $k(k-1)/n(n^2-1) \ge 0$, for $n, k \ge 2$. It is not difficult to verify that $3\omega_{n,2}^{(2)} = \omega_{n,3}^{(2)}$ for all $n \ge 2$. Therefore, by Theorem 2.6 we conclude that the Cesáro matrix is stronger than the identity matrix. □

For $j \in \mathbb{N}$, the jth order of Cesáro matrix is given by [3, page 45]

$$C_j[n,k] = \begin{cases} \dfrac{\binom{n-k+j+1}{j-1}}{\binom{n+j}{j}}, & \text{if } k \le n, \\ 0, & \text{if } k > n. \end{cases} \tag{3.4}$$

Lemma 3.2. *For each $j \in \mathbb{N}$, the matrix $C_{j+1}C_j^{-1}$ preserves the starshape of sequences.*

Proof. If $C_{j+1}C_j^{-1}$ is represented by A_j, it is easy to see that

$$A_j[n,k] = \begin{cases} \dfrac{\binom{j+k}{j}}{\binom{n+j+1}{j+1}}, & \text{if } k \le n, \\ 0, & \text{if } k > n. \end{cases} \tag{3.5}$$

Let (x_n) be a starshaped sequence. Then a simple calculation shows that

$$\binom{j+n}{j} x_n = \binom{n+j+1}{j+1} (A_j x)_n - \binom{n+j}{j+1} (A_j x)_{n-1}. \tag{3.6}$$

Writing

$$(A_j x)_n = n \sum_{i=1}^{n} \frac{c_i}{i} - (n-1)c_0, \tag{3.7}$$

equation (3.6) can be simplified to

$$x_n = \frac{1}{j+1}\left[(n+j+1)c_n + n(j+2)\sum_{i=1}^{n-1}\frac{c_i}{i} + (j+1-n(j+2))c_0\right]. \tag{3.8}$$

Therefore for each $n \ge 2$, we get

$$\frac{x_n - x_0}{n} - \frac{x_{n-1} - x_0}{n-1} = \frac{(n+j+1)}{n(j+1)} c_n + \frac{(2-n)}{(n-1)(j+1)} c_{n-1}, \tag{3.9}$$

which is nonnegative by assumption. Considering the values for $n = 2, 3, \ldots$ successively, we see that $c_n \ge 0$ for $n \ge 2$. This completes the proof. □

Thus the matrix C_j is included by the matrix C_{j+1} in the starshape sense for each $j \in \mathbb{N}$. Combining this result with Theorem 3.1, we obtain the following theorem.

Theorem 3.3. *For each $j \in \mathbb{N}$, the jth order of Cesáro matrix C_j preserves the starshape of sequences.*

Another well-known lower triangular matrix is the Nörlund matrix [3, page 43]

$$N_p[n,k] = \begin{cases} \dfrac{p_{n-k}}{P_n}, & \text{if } k \le n, \\ 0, & \text{if } k > n, \end{cases} \tag{3.10}$$

where $\{p_n\}$ is a nonnegative sequence with $p_0 > 0$ and $P_n = \sum_{k=0}^{n} p_k$. It is obvious that

$$N_p^{(2)}[n,0] = \frac{1}{P_n}\sum_{i=0}^{n} P_i, \tag{3.11}$$

which in turn yields that

$$\frac{N_p^{(2)}[n,0]-N_p[0,0]}{n} - \frac{N_p^{(2)}[n-1,0]-N_p[0,0]}{n-1} \tag{3.12}$$

is negative. Therefore condition (i) of Theorem 2.3 fails. This results in the following theorem.

Theorem 3.4. *The Nörlund matrix N_p does not preserve the starshape of sequences.*

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Chikkanna R. Selvaraj: Penn State University, Shenango Campus, 147, Shenango Avenue, Sharon, PA 16146, USA
E-mail address: ulf@psu.edu

Suguna Selvaraj: Penn State University, Shenango Campus, 147, Shenango Avenue, Sharon, PA 16146, USA
E-mail address: sxs32@psu.edu