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PROPERTIES OF THE FUNCTION $f(x) = x/\pi(x)$

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ABSTRACT. We obtain the asymptotic estimations for $\sum_{k=2}^{n} f(k)$ and $\sum_{k=2}^{n} 1/f(k)$, where $f(k) = k/\pi(k)$, $k \geq 2$. We study the expression $2f(x+y) - f(x) - f(y)$ for integers $x, y \geq 2$ and as an application we make several remarks in connection with the conjecture of Hardy and Littlewood.

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1. Introduction. We denote by $\pi(x)$ the number of all prime numbers $\leq x$. We denote also $f(x) = x/\pi(x)$ for $x \geq 2$. Since $\pi(x) \sim x/\log x$, it follows that $f(x) \sim \log x$. We could expect that the function $f(x)$ behaves like $\log x$. However, we will see that $\log x$ possesses several properties that $f(x)$ does not possess.

Indeed, the function $\log x$ is increasing and concave, while $f(x)$ does not have these properties. Denoting by p_n the nth prime number, we remark that $f(p_n) - f(p_n - 1) = p_n/n - (p_n - 1)/(n-1) = (n - p_n)/n(n-1) < 0$, so the function f is not increasing.

As shown also in [3], the function f is not concave because for $x_1 = p_n - 1$ and $x_2 = p_n + 1$ it follows that $f(x_1) + f(x_2) \geq 2f((x_1 + x_2)/2)$. The following fact was proved in [1]:

$$f(ax) + f(bx) < 2f\left(\frac{a+b}{2} \cdot x\right) \tag{1.1}$$

for $a, b > 0$ and x sufficiently large.

A property of the function log is given by Stirling's formula asserting that $n! \sim n^n e^{-n} \sqrt{2n\pi}$, that is,

$$\sum_{k=1}^{n} \log k \sim n \log n - n + \frac{1}{2} \log n + \log \sqrt{2\pi}. \tag{1.2}$$

2. A property that is neighbor to Stirling's formula. Related to (1.2) we prove the following theorem.

THEOREM 2.1. *For fixed $m \geq 1$ and $n \geq 2$,*

$$S(n) = \sum_{k=2}^{n} f(k) = n\left(\log n - 2 - \sum_{i=2}^{m} \frac{h_i}{\log^i n} + O\left(\frac{1}{\log^{m+1} n}\right)\right), \tag{2.1}$$

where $h_1, h_2, \ldots, h_m$ are computable constants.

Proof. As proved in [2], for fixed $m \geq 2$ there exist $k_1, k_2, \ldots, k_m$, such that $k_i + 1!k_{i-1} + 2!k_{i-2} + \cdots + (i-1)!k_1 = i! \cdot i$ for $i \in \overline{1,m}$ and

$$\pi(x) = \frac{x}{\log x - 1 - \sum_{i=1}^{m} (k_i / \log^i x)} + O\left(\frac{x}{\log^{m+1} x}\right). \tag{2.2}$$

Denoting $S_i(n) = \sum_{i=2}^{n} 1/\log^i n$, we have

$$S(n) = \sum_{k=2}^{n} \log k - (n-1) - \sum_{i=1}^{m} k_i S_i(n) + O\left(\frac{n}{\log^{m+1} n}\right). \tag{2.3}$$

Since

$$\frac{1}{\log^i 3} + \frac{1}{\log^i 4} + \cdots + \frac{1}{\log^i n} < \int_2^n \frac{dt}{\log^i t} < \frac{1}{\log^i 2} + \frac{1}{\log^i 3} + \cdots + \frac{1}{\log^i (n-1)}, \tag{2.4}$$

it follows that $S_i(n) = \int_2^n dt/\log^i t + O(1)$. Denote $I_i(n) = \int_2^n dt/\log^i t$. Then

$$S_i(n) = I_i(n) + O(1). \tag{2.5}$$

From (1.2), (2.3), and (2.5) it follows that

$$S(n) = n\log n - 2n - \sum_{i=1}^{m} k_i I_i(n) + O\left(\frac{n}{\log^{m+1} n}\right). \tag{2.6}$$

The integration by parts then implies that

$$I_i(n) = \frac{n}{\log^i n} + iI_{i+1}(n) + O(1). \tag{2.7}$$

By (2.6) and (2.7) we deduce that

$$S(n) = n\log n - 2n - \sum_{i=1}^{m} h_i \cdot \frac{n}{\log^i n} + O\left(\frac{n}{\log^{m+1} n}\right). \tag{2.8}$$

In view of (2.7), the relation (2.8) becomes

$$S(n) = n\log n - 2n - \sum_{i=1}^{m} h_i \left(I_i(n) - iI_{i+1}(n)\right) + O\left(\frac{n}{\log^{m+1} n}\right). \tag{2.9}$$

Comparing this relation with (2.6), we get

$$\begin{gathered} h_1 = k_1, \\ h_2 - 1 \cdot h_1 = k_2, \\ \cdots \\ h_m - (m-1)h_{m-1} = k_m, \end{gathered} \tag{2.10}$$

hence we have

$$h_j = k_j + (j-1)k_{j-1} + (j-1)(j-2)k_{j-2} + (j-1)(j-2)\cdots 1 \cdot k_1 \tag{2.11}$$

for $j \in \overline{1,m}$. We get $h_1 = 1$, $h_2 = 4$, $h_3 = 21$, and so forth. □

By means of a similar method we now prove the following theorem.

Theorem 2.2. *For fixed $m \geq 1$ the relation*

$$S(n) = \sum_{k=2}^{n} \frac{1}{f(k)} = n\left(\sum_{i=1}^{m} \frac{i!}{\log^i n} + O\left(\frac{1}{\log^{m+1} n}\right)\right) \tag{2.12}$$

holds for $n \geq 2$.

Proof. In [2], the following relation was used:

$$\pi(n) = n\sum_{i=1}^{m} \frac{(i-1)!}{\log^i n} + O\left(\frac{n}{\log^{m+1} n}\right). \tag{2.13}$$

With the notation from the proof of Theorem 2.1, we have

$$S(n) = \sum_{i=1}^{m}\left(\sum_{k=2}^{n} \frac{(i-1)!}{\log^i k}\right) + O\left(\sum_{k=2}^{m} \frac{1}{\log^{m+1} k}\right), \tag{2.14}$$

that is,

$$S(n) = \sum_{i=1}^{m} (i-1)! S_i(n) + O(S_{m+1}(n)). \tag{2.15}$$

In view of (2.5) and of the fact that $I_{m+1}(n) = O(n/\log^{m+1} n)$, we get

$$S(n) = \sum_{i=1}^{m} (i-1)! I_i(n) + O\left(\frac{n}{\log^{m+1} n}\right). \tag{2.16}$$

It easily follows from (2.7) that $S(n) = \sum_{i=1}^{m} q_i/\log^i n + O(n/\log^{m+1} n)$ and

$$S(n) = \sum_{i=1}^{m} q_i (I_i - i I_{i+1}) + O\left(\frac{n}{\log^{m+1} n}\right). \tag{2.17}$$

Comparing the above relation with (2.16), we get

$$\begin{aligned} q_1 &= 0!, \\ q_2 &= 1! + 1 \cdot q_1, \\ &\dots \\ q_{i+1} &= i! + i q_i. \end{aligned} \tag{2.18}$$

Consequently $q_i = i!$ and the proof is finished. □

3. An inequality for the function $f(x)$. We have shown in the introduction that the function f is not concave. In particular, it follows neither that $f(x+y) \geq f(x)$ nor that $f(x+y) \geq f(y)$. However, we can prove the following theorem.

Theorem 3.1. *The inequality*

$$2f(x+y) \geq f(x) + f(y) \tag{3.1}$$

holds for all integers $x \geq y \geq 2$, except for the pairs $(3,2)$ and $(5,2)$.

Proof. In [3], it was proved that

$$\begin{aligned} \pi(x) &< \frac{x}{\log x - 1 - (\log x)^{-0.5}} \quad \text{whenever } x \geq 6, \\ \pi(x) &> \frac{x}{\log x - 1 + (\log x)^{-0.5}} \quad \text{whenever } x \geq 59. \end{aligned} \tag{3.2}$$

In view of these inequalities, it follows that for $x, y \geq 59$ it suffices to prove that

$$2\left(\log(x+y) - 1 - \frac{1}{\sqrt{\log(x+y)}}\right) > \log x + \log y - 2 + \frac{1}{\sqrt{\log x}} + \frac{1}{\sqrt{\log y}}, \tag{3.3}$$

that is,

$$\log \frac{(x+y)^2}{xy} > \frac{2}{\sqrt{\log(x+y)}} + \frac{1}{\sqrt{\log x}} + \frac{1}{\sqrt{\log y}}. \tag{3.4}$$

Since $(x+y)^2 \geq 4xy$ and $x \geq y$, it suffices to have the inequality

$$\log 4 \geq \frac{2}{\sqrt{\log 2y}} + \frac{2}{\sqrt{\log y}}. \tag{3.5}$$

This inequality holds whenever $y \geq 2960$.

For $y < 2960$, consider $x \geq 6000$. Then $x/y > 1.5085$, hence we have $(x+y)^2/xy > 25/4$ and $\log(x+y)^2/xy > 1.5085$. To verify the relation (3.4) it suffices to have $1.5085 > 1/\sqrt{\log y} + 3/\sqrt{\log 5000}$. This holds whenever $y \geq 63$.

It remains to treat the cases (a) $y < 63$, $x \geq y$, and (b) $y < 2960$, $x < 6000$, $x \geq y$.

(a) If $y \leq 62$, then $\min(x, y) \leq 146$. In this case, Schinzel [4] proved that

$$\pi(x+y) \leq \pi(x) + \pi(y), \tag{3.6}$$

so $2f(x+y) \geq 2 \cdot (x+y)/\pi(x) + \pi(y)$. It remains to prove that $2 \cdot (x+y)/(\pi(x)+\pi(y)) > x/\pi(x) + y/\pi(y)$, that is, $x/\pi(x) \geq y/\pi(y)$. Remark that $\max_{2 \leq y \leq 62} y/\pi(y) = 58/16$. Since $\min_{x \geq 80} x/\pi(x) \geq 58/16$, it remains to study the situation $y \leq x \leq 80$, that is contained in the case (b).

By means of a personal computer, one can verify the cases when $y < 2960$, $x < 6000$ and $x \geq y$. Then one finds out the exceptions indicated in the statement of the theorem, namely $y = 2$ and either $x = 3$ or $x = 5$. □

4. A consequence for the Hardy-Littlewood conjecture. Related to the celebrated conjecture

$$\pi(x+y) \leq \pi(x) + \pi(y) \quad \text{for integers } x, y \geq 2, \tag{4.1}$$

several facts are known (see [4, pages 231–237]). However these results are far from solving the problem.

We can draw from Theorem 3.1 the following.

Consequence. *If the inequality from (4.1) is false for some integers* $x \geq y \geq 2$, *then* $f(x) > f(y)$ *and* $f(x) > f(x+y)$.

Proof. From Theorem 3.1 it follows that $\pi(x+y) \le 2(x+y)/(x/\log x + y/\log y)$. To prove (4.1), it would suffice that $2(x+y)/(x/\log x + y/\log y) \le \pi(x)+\pi(y)$, that is, $x/\pi(x) \le y/\pi(y)$. Thus, if the inequality from (4.1) is false, then $f(x) > f(y)$.

Now assume that $f(x) > f(y)$ and $f(x+y) \le f(x)$. It then follows that

$$\pi(x+y) < \frac{(x+y)\pi(x)}{x} = \pi(x) + \frac{y\pi(x)}{x} < \pi(x)+\pi(y). \tag{4.2}$$

Consequently, if inequality (4.1) does not hold, then $f(x) > f(y)$ and $f(x) > f(x+y)$. □

Remark that for $x = y$ the statement of Theorem 3.1 reduces to $\pi(2x) \le 2\pi(x)$. This is just Landau's theorem, that is a special case of the Hardy-Littlewood conjecture.

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