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ON THE NUMBER OF ZEROS OF ITERATED OPERATORS ON ANALYTIC LEGENDRE EXPANSIONS

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ABSTRACT. Let $L = (1-z^2)D^2 - 2zD$, $D = d/dz$ and $f(z) = \sum_{n=0}^{\infty} c_n P_n(z)$, with P_n being the nth Legendre polynomial and f analytic in an ellipse with foci ± 1. Set $L^k = L(L^{k-1})$, $k \geq 2$. Then the number of zeros of $L^k f(z)$ in this ellipse is $O(k \ln k)$.

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1. Introduction. In [3], Erdös and Rényi showed that for a function analytic in $|z| \leq R$, the number of zeros of the kth derivative $f^{(k)}(z)$ in $|z| \leq r < R$ is $O(k)$. This result includes an earlier result of Pólya [8] that for a function that is real on the real axis and is the restriction to a closed interval I of an analytic function, the number of zeros of $f^{(k)}$ in I is $O(k)$.

Let

$$L = (1-z^2)D^2 - 2zD, \tag{1.1}$$

with $D = d/dz$. Let $f(z)$ be analytic in an ellipse E_R with foci at ± 1, where the sum of the semiaxes is $R > 1$. Now, $f(z)$ can be represented as

$$f(z) = \sum_{n=0}^{\infty} c_n P_n(z), \tag{1.2}$$

where P_n is the nth Legendre polynomial [17, Theorem 9.1.1]. Moreover, by [17, formula (9.1.4)]

$$\liminf_{n\to\infty} \left(|c_n|^{-1/n}\right) = R. \tag{1.3}$$

Calculation shows that

$$(L^k f)(z) = \sum_{n=0}^{\infty} (-\lambda_n)^k c_n P_n(z), \tag{1.4}$$

where $\lambda_n = n(n+1)$. Formula (1.4) holds for $x \in (-1,1)$ and hence in E_R by analytic continuation. Moreover,

$$\liminf_{n\to\infty} \left(|\lambda_n^k c_n|^{-1/n}\right) = R \tag{1.5}$$

for every positive integer k so that $(L^k f)(z)$ is also analytic in E_R.

2. The main theorem and lemmas

Theorem 2.1. *Let f be analytic in E_R of the form (1.2). Let $1 < T < R$. Then the number of zeros of $(L^k f)(z)$ in E_T is $O(k\ln k)$.*

The above theorem implies our next result. For the next corollary we consider the operator L to be restricted to the real axis. That is,

$$L = (1-x^2)D^2 - 2xD \tag{2.1}$$

with

$$D = \frac{d}{dx}. \tag{2.2}$$

Corollary 2.2. *Let L be given by (2.1) and (2.2), and let f be analytic in E_R of the form (1.2), with real c_n. Then the number of sign changes of $(L^k f)(x)$ in $(-1,1)$ is $O(k\ln k)$.*

We next give the lemmas needed. The first is a version of Jensen's formula for functions analytic in an ellipse [7, page 58].

Lemma 2.3. *Let $f(z)$ be analytic inside the ellipse $z = (Se^{i\theta} + (Se^{i\theta})^{-1})/2$, for $R > S > 1$. For $1 < r \le S$, denote by $N(r)$ the number of zeros of f (counting multiplicities) inside and on the ellipse*

$$z = \frac{1}{2}\left(re^{i\theta} + (re^{i\theta})^{-1}\right). \tag{2.3}$$

Then

$$\begin{aligned}\int_1^S \frac{1}{r}N(r)\,dr = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f\left(\frac{1}{2}\left(Se^{i\theta} + (Se^{i\theta})^{-1}\right)\right)\right| d\theta \\ - \frac{1}{2\pi}\int_0^{2\pi} \ln|f(\cos\theta)|\,d\theta.\end{aligned} \tag{2.4}$$

We also need Laplace's method [9, Part 2, Chapter 5, no. 201] and [1, Section 5.1].

Lemma 2.4. *Suppose that the functions $\phi(x)$ and $\exp(h(x))$ are defined and satisfy the following conditions on $(0,\infty)$:*

(1) $\phi(x)\exp(kh(x))$ is absolutely integrable over $(0,\infty)$ for every $k = 0,1,2,\ldots$.

(2) The function $h(x)$ attains its maximum only at the point $x_0 \in (0,\infty)$. Moreover, $h(x) < h(x_0)$ on any closed integral that does not contain the point x_0. Furthermore, there is a neighborhood of x_0 where $h''(x)$ exists and is continuous with $h''(x_0) < 0$.

(3) $\phi(x)$ is continuous at x_0, $\phi(x_0) \ne 0$.

Then

$$\int_0^\infty \phi(x)\exp(kh(x))\,dx \sim \sqrt{2\pi}\,\phi(x_0)\exp(kh(x_0))\left(-kh''(x_0)\right)^{-1/2} \tag{2.5}$$

as $k \to \infty$.

We also need an expansion for Legendre polynomials [2, Lemma 12.4.1].

Lemma 2.5. *Given* P_n,

$$P_n\left(\frac{(Re^{i\theta}+(Re^{i\theta})^{-1})}{2}\right)=\sum_{j=0}^{n} a_j a_{n-j}(Re^{i\theta})^{n-2j}, \tag{2.6}$$

where

$$a_j = 2^{-2j}\binom{2j}{j}. \tag{2.7}$$

3. Proof of the main theorem

Proof. We will use Jensen's formula in the form (3.15). Let $1<S<R$ and $z=(Se^{i\theta}+(Se^{i\theta})^{-1})/2$. By (1.3), for a fixed ϵ, $0<\epsilon<R-S$, there exists $N=N(\epsilon)$ such that $n\ge N$ implies that

$$|c_n| \le (R-\epsilon)^{-n}. \tag{3.1}$$

Now,

$$(L^k f)(z)=\sum_{n=0}^{\infty}\lambda_n^k c_n P_n(z) \tag{3.2}$$

which, by Lemma 2.5, equals

$$\sum_{n=0}^{\infty}\frac{\lambda_n^k c_n}{4^n}\sum_{j=0}^{n}\binom{2j}{j}\binom{2n-2j}{n-j}(Se^{i\theta})^{n-2j}. \tag{3.3}$$

Taking the modulus,

$$|(L^k f)(z)| \le \sum_{n=0}^{\infty}\lambda_n^k|c_n|\left(\frac{S}{4}\right)^n\sum_{j=0}^{n}\binom{2j}{j}\binom{2n-2j}{n-j}S^{-2j}. \tag{3.4}$$

We now employ an identity that is a special case of the Chu-Vandermonde sum, which is

$$\sum_{j=0}^{n}\binom{2j}{j}\binom{2n-2j}{n-j}=4^n. \tag{3.5}$$

Since $S>1$,

$$|(L^k f)(z)| \le \sum_{n=0}^{\infty}\lambda_n^k|c_n|S^n, \tag{3.6}$$

which, by (3.1) and $R>S$, is less than or equal to

$$\sum_{n=0}^{N-1}\lambda_n^k|c_n|S^n+\sum_{n=N}^{\infty}\lambda_n^k\left(\frac{S}{R-\epsilon}\right)^n. \tag{3.7}$$

The second term in (3.7) is less than

$$\begin{aligned}\int_0^\infty (n(n+1))^k \exp\Big(-n\ln\frac{R-\epsilon}{S}\Big)dn\\ = \int_0^\infty \exp\Big(k\Big(\ln(n(n+1))-\frac{n}{k}\ln\frac{R-\epsilon}{S}\Big)\Big)dn.\end{aligned} \tag{3.8}$$

In (3.8), n is considered a continuous variable.

Next we employ Laplace's method as in Lemma 2.4. We set

$$h(n) = \ln\Big(n(n+1)-\frac{n}{k}\ln\frac{R-\epsilon}{S}\Big), \tag{3.9}$$

with $n \in (0,\infty)$.

Calculation shows that $h'(n_0) = 0$ where

$$n_0 = \Big(\frac{2k}{\ln((R-\epsilon)/S)}\Big) - 1 + \Big(\frac{(1+4k^2)}{\ln((R-\epsilon)/S)}\Big)^{1/2}. \tag{3.10}$$

Now, for all sufficiently large k, the term n_0 is positive. Note also that

$$n_0 \sim \alpha k \quad \text{as } k \longrightarrow \infty, \tag{3.11}$$

where the constant α is independent of k.

Further calculation shows that

$$h''(n_0) = -2\frac{\lambda_{n_0}+1}{\lambda_{n_0}^2} < 0, \tag{3.12}$$

with λ_n given by $h(n+1)$.

By Lemma 2.4, the integral in (3.8) is asymptotic to

$$(\lambda_{n_0})^k\Big(\frac{S}{R-\epsilon}\Big)^{n_0}\Big(\frac{\pi(\lambda_{n_0})^2}{k(\lambda_{n_0}+1)}\Big)^{1/2} \quad \text{as } k \longrightarrow \infty. \tag{3.13}$$

The first term in (3.7) is

$$\sum_{n=0}^{N-1}\lambda_n^k|c_n|S^n \le c(N-1)(\lambda_{N-1})^k\Big(\sum_{n=0}^{N-1}S^n\Big) = c(N-1)(\lambda_{N-1})^k\frac{S^N-1}{S-1}, \tag{3.14}$$

where $c = \max\{|c_j|\}$, for $j = 0,\dots,N-1$.

We next take $1 < T < S$. We use Jensen's formula in (2.4) with f replaced by $L^k f$. This yields

$$\begin{aligned}N(T)\ln\frac{S}{T} &\le \int_T^S \frac{1}{r}N(r)dr\\ &\le \frac{1}{2\pi}\int_0^{2\pi}\ln\Big|(L^k f)\Big(\frac{1}{2}(Se^{i\theta}+(Se^{i\theta})^{-1})\Big)\Big|d\theta\\ &\quad -\frac{1}{2\pi}\int_0^{2\pi}\ln|(L^k f)(\cos\theta)|d\theta.\end{aligned} \tag{3.15}$$

We first use the estimates in (3.6), (3.7), (3.13), and (3.14) to estimate the first integral on the right-hand side of inequality (3.15). In light of these estimates, we choose a constant $M > 1$ independent of k such that for all sufficiently large k,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\ln\left|(L^k f)\left(\frac{1}{2}(Se^{i\theta}+(Se^{i\theta})^{-1})\right)\right|d\theta \\ \le \ln\left\{Mc(N-1)(\lambda_{N-1})^k\frac{S^N-1}{S-1}+M(\lambda_{n_0})^k\left(\frac{S}{R-\epsilon}\right)^{n_0}\left(\frac{\pi(\lambda_{n_0})^2}{k(\lambda_{n_0}+1)}\right)^{1/2}\right\}.\end{aligned} \tag{3.16}$$

By (3.11), for all sufficiently large k,

$$n_0 > N-1. \tag{3.17}$$

Accordingly, rewrite the term on the right-hand side of (3.16) as

$$\ln\left\{k^{1/2}(\lambda_{n_0})^k\left[M\left(\frac{S}{R-\epsilon}\right)^{n_0}\left(\frac{\pi(\lambda_{n_0})^2}{k^2(\lambda_{n_0}+1)}\right)^{1/2}+k^{-1/2}Mc(N-1)\left(\frac{\lambda_{N-1}}{\lambda_{n_0}}\right)^k\frac{S^N-1}{S-1}\right]\right\}. \tag{3.18}$$

Now, for the first term inside the bracket in (3.18), by (3.11) this term is $O(1)$ as $k\to\infty$. Next, by (3.17),

$$\left(\frac{\lambda_{N-1}}{\lambda_{n_0}}\right)^k = O(1) \quad \text{as } k \longrightarrow \infty. \tag{3.19}$$

In summary, by (1.4), (3.11), (3.16), (3.18), and (3.19), the integral in (3.16) that appears in (3.15) as well is $O(k\ln k)$.

Finally, we estimate the second integral on the right-hand side of (3.15). This is, of course, the case $S = 1$ in the integral just estimated. So, we fix ϵ, with

$$0 < \epsilon < R-1 \tag{3.20}$$

which is possible as $R > 1$. Inequality (3.20) is equivalent to $1/(R-\epsilon) < 1$. First, we replace the estimate in (3.14) by

$$\sum_{n=0}^{N-1}(\lambda_n)^k|c_n| \le c(N-1)(\lambda_{N-1})^k, \tag{3.21}$$

where, again, $c = \max\{|c_j|\}$ for $j = 0,\dots,N-1$.

We first note that Jensen's formula as used in (3.15) is independent of the estimates done in (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.16), (3.17), and (3.18). Next, set $z = x = \cos\theta = (e^{i\theta}+e^{-i\theta})/2$. The estimates in (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.16), (3.17), and (3.18) are replaced by

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\ln|(L^k f)(\cos\theta)|\,d\theta \\ \le \ln\left[k^{1/2}(\lambda_{n_0})^k\left[M\left(\frac{1}{R-\epsilon}\right)^{n_0}\left(\frac{\pi\lambda_{n_0}^2}{k^2(\lambda_{n_0}+1)}\right)^{1/2}+k^{-1/2}Mc(N-1)\left(\frac{\lambda_{N-1}}{\lambda_{n_0}}\right)^k\right]\right],\end{aligned} \tag{3.22}$$

this inequality holds for all sufficiently large k. Again, by (3.11), the first term inside the bracket in (3.22) is $O(1)$ as $k \to \infty$. Then, because of (3.11) and (3.19), the integral in (3.22) is $O(k\ln k)$.

In summary, by Jensen's formula as in (3.15), $N(T)$, which equals the number of zeros of $(L^k f)(z)$ in E_T, satisfies

$$N(T) = O(k\ln k). \tag{3.23}$$

□

4. Commentary. The order of growth $O(k\ln k)$ that appears in the conclusions of Theorem 2.1 and Corollary 2.2 is due to the method of the proof used. The inspiration for this method was corresponding methods used by Erdös and Rényi [3]. The correct order of growth, namely $O(k)$, should be possible to obtain in the conclusion of this theorem and corollary.

In this paper, we have assumed a function to be analytic in an ellipse with foci at ± 1 and obtained asymptotic bounds on the number of zeros of $L^k f(z)$ in this ellipse, which in particular bounds the number of sign changes of $(L^k f)(x)$ in $(-1,1)$. The definition of L appears in the introduction.

Much work has been done in various contexts addressing the converse of the problem posed here. That is, one assumes for a real function that is C^∞ on a real interval I an asymptotic bound on the number of sign changes of $(L^k f)(x)$ in I, where L is the appropriate differential operator. One then deduces extendability by analytic continuation of the function to be analytic, or even to be an entire function, or even entire of a certain growth, depending on the frequency of sign changes of $(L^k f)(x)$ in I. Work on problems of this type can be found in [4, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16].

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