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# VARIATIONAL-LIKE INEQUALITIES FOR PSEUDOMONOTONE OPERATORS

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ABSTRACT. The aim of this note is to use a fixed point theorem to prove results for variational-like inequalities for pseudomonotone operators.

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**1. Introduction.** Recently, Singh et al. [10] studied pseudomonotone operators and derived interesting results in variational inequality and complementarity problems using a recent fixed point theorem of Tarafdar [13], which is equivalent to F-KKM theorem [13]. They derived a few interesting results as corollaries and gave an application in minimization problems. Earlier, Parida et al. [7] studied a variational-like inequality problem and developed a theory for the existence of its solution using Kakutani's fixed point theorem, and also established the relationship between the variational-like inequality problem and some mathematical programming problems. Further results on existence theorem for variational-like inequality problems were obtained by Wadhwa and Ganguly [14] using Tarafdar's fixed point theorem [11], which is equivalent to the KKM fixed point theorem [13].

In this note, we use Tarafdar's result [13] and prove an existence theorem for variational-like inequality problem for $g$-pseudomonotone operators and then derive some interesting results and corollaries.

We need the following definitions:

Let $E$ stand for a real locally convex Hausdorff topological vector space and $X$ a nonempty convex subset of $E$ with $E^* \neq \{0\}$, being the continuous dual of $E$. Let $T : X \to E^*$ be a nonlinear map. The mapping $T : X \to E^*$ is hemicontinuous if $T$ is continuous from the line segment of $X$ to the weak topology of $E^*$. A point $y \in X$ is said to be a solution of the variational inequality if

$$\langle Ty, x-y\rangle \geq 0 \quad \forall x \in X. \tag{1.1}$$

Let $g$ be a continuous map, $g : X\times X \to E$. A point $y \in X$ is said to be a solution of the variational-like inequality problems if

$$\langle Ty, g(x,y)\rangle \geq 0 \quad \forall x \in X. \tag{1.2}$$

If $g(x,y) = x-y$, (1.2) reduces to (1.1) [7].

A map $T : X \to E^*$ is said to be monotone if

$$\langle Ty-Tx, y-x\rangle \geq 0 \quad \forall x,y \in X. \tag{1.3}$$

Here, $(\cdot,\cdot)$ denotes the pairing between $E^*$ and $E$.

The map $T$ is called pseudomonotone if

$$\langle Ty, y-x\rangle \geq 0 \quad \text{whenever } \langle Tx, y-x\rangle \geq 0\ \forall x,y \in X. \tag{1.4}$$

**Definition 1.1.** A map $T : X \to E^*$ is said to be $g$-monotone on $X$ if

$$\langle Tx, g(y,x)\rangle + \langle Ty, g(x,y)\rangle \leq 0 \quad \forall x,y \in X. \tag{1.5}$$

For $g(y,x) = y-x$, we get the definition of monotone operators.

**Definition 1.2.** A map $T : X \to E^*$ is said to be $g$-pseudomonotone if

$$\langle Tx, g(y,x)\rangle \geq 0 \quad \text{whenever } \langle Ty, g(x,y)\rangle \geq 0\ \forall x,y \in X. \tag{1.6}$$

For $g(y,x) = y-x$, we get the definition of pseudomonotone operators.

We are interested in the following:

Find $x \in X$ such that

$$\langle Tx, g(y,x)\rangle + hy - hx \geq 0 \quad \forall y \in X, \tag{1.7}$$

where $T : X \to E^*$ is a nonlinear mapping and $h : X \to \mathbb{R}$ is a low semi-continuous and convex functional.

We need the following fixed point theorem [13].

**Theorem 1.3.** *Let $X$ be a nonempty, convex subset of a Hausdorff topological vector space $E$. Let $F : X \to 2^X$ be a set-valued mapping such that*

(i) *for each $x \in X$, $f(x)$ is a nonempty, convex subset of $X$;*

(ii) *for each $y \in X$, $F^{-1}(y) = \{x \in X : y \in F(x)\}$ contains a relatively open subset $O_y$ of $X$ ($O_y$ may be empty for some $y$);*

(iii) *$U_{x\in X} O_x = X$; and*

(iv) *$X$ contains a nonempty subset $X_0$ contained in a compact convex subset $X_1$ of $X$ such that the set $D = \bigcap_{x\in X_0} O_x^c$ is compact ($D$ may be empty and $O_x^c$ denotes the complement of $O_x$ in $X$).*

*Then there exists a point $x_0 \in X$ such that $x_0 \in F(x_0)$.*

We make the following hypothesis.

**Condition 1.4.** For $X \subset E$, let $T : X \to E^*$ and $g : X \times X \to E$ satisfy the following:

(i) for each $x \in X$, $g(y,x)$ is convex $y \in X$;

(ii) $g(x,y) + g(y,z) = g(x,z)$ for all $x,y,z \in X$;

(iii) $g(x,x) = 0$;

(iv) for every $x \in E^*$, $\langle Tx, y\rangle$ is monotone increasing in $y \in E^*$.

## 2. Main results

First, we give the following result.

**Lemma 2.1.** *If $X$ is a nonempty convex subset of a topological vector space $E$ and $T : X \to E^*$ is a $g$-pseudomonotone and hemicontinuous, then $x \in X$ is a solution of*

$$\langle Tx, g(y,x)\rangle + hy - hx \geq 0 \quad \forall y \in X \tag{2.1}$$

*if and only if $x \in X$ is a solution of*

$$\langle Ty, g(y,x)\rangle + hy - hx \geq 0 \quad \forall y \in X, \tag{2.2}$$

*where $h : X \to \mathbb{R}$ is a convex function and $g : X \times X \to E$ is such that it satisfies Condition 1.4.*

**Proof.** Let $x \in X$ be a solution of (2.1). Then, by Condition 1.4(i), (ii) and the $g$-pseudomonotonicity of $T$, we have

$$\langle Ty, g(y,x)\rangle + hy - hx \geq 0 \quad \forall y \in X. \tag{2.3}$$

Now, assume that $x$ satisfies (2.2) and let $y \in X$ be arbitrary. Then, using Minty's technique [5],

$$yt = (1-t)x + ty \in X \quad \forall t \in (0,1) \tag{2.4}$$

since $X$ is convex. Hence, we have

$$\langle Ty_t, g(y_t,x)\rangle + hy_t - hx \geq 0. \tag{2.5}$$

So, by Condition 1.4(ii), (iii),

$$t\langle Ty_t, g(y,x)\rangle + t(hy - hx) \geq 0 \tag{2.6}$$

since $T$ is hemicontinuous. Letting $t \to 0$, we get

$$\langle Tx, g(y,x)\rangle + hy - hx \geq 0. \tag{2.7}$$

□

Now, we state the following result.

**Theorem 2.2.** *Let $X$ be a nonempty closed convex subset of a real Hausdorff topological vector space $E$ with $E^* \neq \{0\}$. Let $T : X \to E^*$ be $g$-pseudomonotone and hemicontinuous map such that Condition 1.4 is satisfied, and $h : X \to \mathbb{R}$ is a lower semicontinuous and convex function. Further, assume that there exists a nonempty set $X_0$ contained in a compact convex subset $X_1$ of $X$ such that the set*

$$D = \bigcap_{x \in X_0} \{y \in X : \langle Tx, g(x,y)\rangle + hx - hy \geq 0\} \tag{2.8}$$

*is either empty or compact.*

*Then, there exists an $x_0 \in X$ such that*

$$\langle Tx_0, g(y,x_0)\rangle + hy - hx_0 \geq 0 \quad \forall y \in X. \tag{2.9}$$

**Proof.** Suppose that, for each $y \in X$, there exists an $x \in X$ such that

$$\langle Tx, g(y,x)\rangle + hx - hy < 0. \tag{2.10}$$

First, suppose that (2.10) does not hold. This means that there exists at least one $y_0 \in X$ such that

$$\langle Tx, g(y_0, x)\rangle + hx - hy_0 \geq 0 \quad \forall x \geq X, \tag{2.11}$$

that is, $y_0 \geq X$ is a solution of (2.2). Then, by Lemma 2.1, $y_0 \in X$ is a solution of (2.1).

Next, assume that there is no solution of (2.1) under condition (2.10) given that (2.10) holds. Then, for each $x \in X$, the set

$$F(x) = \{y \in X : \langle Tx, g(y, x)\rangle + hy - hx < 0\} \tag{2.12}$$

must be nonempty. It also follows from the convexity of $h$ and by Condition 1.4 that the set $F(x)$ is convex for each $x \in X$. Thus, $F : X \to 2^X$ is a set-valued map with $F(x)$ nonempty and convex for each $x \in X$.

Now, for each $x \in X$,

$$F^{-1}(x) = \{y \in X : x \in (y)\} = \{y \in X : \langle Ty, g(x, y)\rangle + hx - hy < 0\}. \tag{2.13}$$

For each $x \in X$,

$$\begin{aligned} \{F^{-1}(x)\}^c = {} & \text{complement of } F^{-1}(x) \text{ in } X \\ & = \{y \in X : \langle Ty, g(x, y)\rangle + hx - hy \geq 0\} \\ & \subset \{y \in X : \langle Tx, g(x, y)\rangle + hx - hy \geq 0\} \end{aligned} \tag{2.14}$$

by the $g$-pseudomonotonicity of $T = G(x)$.

Again, using Condition 1.4 and the convexity of $h$, we can show that $G(x)$ is convex for each $x \in X$. Since $g$ is continuous and $h$ is lower semi-continuous, $G(x)$ is a relatively closed subset of $X$.

Hence, for each $x \in X$,

$$F^{-1}(x) \supset [G(x)]^c = 0_x \quad \text{is a relatively open subset of } X. \tag{2.15}$$

Now, by condition (2.10), we can easily see that $\bigcup_{x\in X} O_x = X$. (Indeed, if $y \in X$, by (2.10), there exists an $x \in X$ such that $y \in [G(x)]^c = O_x$. Thus, $y \in \bigcup_{x\in X} O_x$. Hence, $\bigcup_{x\in X} O_x = X$.)

Finally, $D = \bigcap_{x\in X_0} G(x) = \bigcap_{x\in X_0} O_x^c$ is compact or empty by the given condition. Hence, by Theorem 1.3, there exists an $x \in X$ such that $\langle Tx, g(x, x)\rangle + hx - hx < 0$, which is impossible. Hence, there is a solution in this case as well. □

Here, we give a few results that are special cases of Theorem 2.2.

**Corollary 2.3.** *Let $T : X \to E^*$ be $g$-monotone and hemicontinuous, where $g$-satisfies Condition 1.4, $h : X \to \mathbb{R}$ is convex and lower semi-continuous. Further, assume that there exists a nonempty set $X_0$ contained in a compact convex subset $X_1$ of $X$ such that $D = \bigcap_{x\in X_0} \{y \in X : \langle Tx, g(x, y)\rangle + hx - hy \geq 0\}$ is either empty or compact. Then there is an $x \in X$ satisfying (2.1).*

**Remark 2.4.** For $g(x, y) = x - y$, Corollary 2.3 implies Corollary 1.2 of Singh et al. [10] which, in turn, implies a well-known result of Tarafdar [12].

**Corollary 2.5.** *Let $X$ be a compact convex subset of $E$ and $T : X \to E^*$ be $g$-pseudomonotone and hemicontinuous where $g$ satisfies Condition 1.4. Suppose that $h : X \to \mathbb{R}$ is lower semicontinuous and convex. Then there is an $x \in X$ satisfying (2.1).*

**Remark 2.6.** For $g(x,y) = x - y$,

(i) Corollary 2.5 implies [10, Corollary 1.3].

(ii) If we take $T = A - B$, where $A$ is a monotone map and $B$ is antimonotone and both are hemicontinuous, then we derive a result due to Siddiqui et al. [8]. Here, we need only two conditions, the lower semicontinuity, and the convexity of the function $h$.

**Remark 2.7.** For $h = 0$, Corollary 2.5 implies Theorem 2 and Corollary 1 of Wadhwa and Ganguly [14] which implies, respectively, Theorem 2 and Corollary of Tarafdar [11]. Tarafdar's result covered the result of Browder [1] and Theorem 1.1 of Hartman and Stampacchia [3].

Now, we prove a result similar to Theorem 2.1 of Singh et al. [9]. For $A \subset E$, $\text{int}(A)$ and $\partial(A)$ denote, respectively, the interior and the boundary of $A$, while for $A$, $X \subset E$, $\text{int}_X(A)$ and $\partial(A)$ denote, respectively, the relative interior and the relative boundary of $A$ in $X$. A subset of a Banach space is said to be solid if it has a nonempty interior.

**Theorem 2.8.** *Let $X$ be a closed convex subset of a reflexive Banach space $E$ and $T : X \to E^*$ a $g$-pseudomonotone and hemicontinuous mapping, $g : X \times X \to E$ satisfy Condition 1.4, and h is convex and lower semicontinuous. Then the following conditions are equivalent:*

(i) *There exists $\bar{x} \in X$ such that $\langle T\bar{x}, g(x,\bar{x})\rangle + hx - h\bar{x} \ge 0$ for all $x \in X$, that is, $x$ is a solution of (2.1).*

(ii) *There exists a $u \in X$ and a constant $r > \|u\|$ such that $X\langle T(x), g(x,u)\rangle + hx - hu \ge 0$ for all $x \in X$ with $\|x\| = r$.*

(iii) *There exists $r > 0$ such that the set $\{x \in X : \|x\| \le r\}$ is nonempty with the property that, for each $x \in X$ with $\|x\| = r$, there exists a $u \in X$ with $\|u\| < r$ and $\langle T(x), g(x,u)\rangle hx\, hu \ge 0$.*

**Proof.** This can be proved following Cottle and Yao [2, Theorem 2.2] as well as Parida et al. [7, Theorem 3.4]. □

**Remark 2.9.** For a monotone $T$ operator and $h = 0$:

(1) Theorem 2.8(i), (ii), and (iii) were obtained by Parida et al. [7].

(2) For $g(x,\bar{x}) = x - \bar{x}$, Theorem 2.8(ii) and (iii) reduce to the results of Theorems 2.3 and 2.4 of Moré [6], respectively.

**Remark 2.10.** For $g(x,x) = x - \bar{x}$ and $h = 0$, Theorem 2.8(i), (ii), and (iii) were obtained as Theorem 2.1(i), (ii), and (iii) by Singh et al. [9] and, in Hilbert spaces, similar results were obtained by Cottle and Yao (see [1, Theorem 2.2]).

Let $H,K$ be nonempty, closed subsets of $\mathbb{R}^n$, then we denote, by $B_H(K)$, the set of $z \in K$ such that $U(z) \bigcap (H - K) \ne \Phi$ and, by $I_H(K)$, the set of $z \in K$ such that $U(z) \bigcap (H - K) = \Phi$, for some neighbourhood $U(z)$ of $z$.

Finally, we present a result similar to Hirano and Takahashi [4] for unbounded subsets in $\mathbb{R}^n$. Before that, we present the following result of Singh et al. [9, Corollary 1.12].

**Corollary 2.11.** *Let $X$ be a closed bounded convex subset of a reflexive Banach space $E$ and $T : X \to E^*$ a pseudomonotone and hemicontinuous mapping. Then the set of solutions of variational inequality for a point $x_0 \in X$, $\langle Tx_0, y - x_0\rangle \geq 0$ for all $y \in X$; $y \in x$; is a nonempty weakly compact convex subset of $X$.*

**Theorem 2.12.** *Let $X$ be a nonempty closed convex subset of $\mathbb{R}^n$ and $T : X \to \mathbb{R}^n$ be $g$-pseudomonotone such that Condition 1.4 is satisfied; $h : X \to \mathbb{R}$ a lower semicontinuous and convex function. Then there exists a solution of (2.1) in $X$ if and only if there exists a bounded closed convex subset $K$ of $X$ such that, for each $z \in B_x(K)$, there exists $y \in I_x(K)$ such that*

$$\langle Tz, g(y^*, z)\rangle + hz - hy \longrightarrow 0. \tag{2.16}$$

**Proof.** Using Corollary 2.11, with little modification, it can be shown that if there exists a solution of (2.1), then there exists a weakly compact convex subset $K$ of $X$ such that (2.16) is satisfied. Conversely, let $K$ be a weakly compact convex subset and there exists $x^* \in K$ such that

$$\langle Tx^*, g(x, x^*)\rangle \geq 0 \quad \forall x \geq K, \tag{2.17}$$

where $T$ is a $g$-pseudomonotone operator. The rest of the proof is similar to that of Theorem 3 of Wadhwa and Ganguly [14]. □

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## References


[1] F. E. Browder, *Nonlinear monotone operators and convex sets in Banach spaces*, Bull. Amer. Math. Soc. **71** (1965), 780–785. MR 31#5112. Zbl 138.39902.

[2] R. W. Cottle and J. C. Yao, *Pseudo-monotone complementarity problems in Hilbert space*, J. Optim. Theory Appl. **75** (1992), no. 2, 281–295. MR 93i:47098. Zbl 795.90071.

[3] P. Hartman and G. Stampacchia, *On some non-linear elliptic differential-functional equations*, Acta Math. **115** (1966), 271–310. MR 34#6355. Zbl 142.38102.

[4] N. Hirano and W. Takahashi, *Existence theorems on unbounded sets in Banach spaces*, Proc. Amer. Math. Soc. **80** (1980), no. 4, 647–650. MR 82b:47068. Zbl 471.49011.

[5] G. J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. **29** (1962), 341–346. MR 29#6319. Zbl 111.31202.

[6] J. J. Moré, *Coercivity conditions in nonlinear complementarity problems*, SIAM Rev. **16** (1974), 1–16. MR 49#1270. Zbl 272.65041.

[7] J. Parida, M. Sahoo, and A. Kumar, *A variational-like inequality problem*, Bull. Austral. Math. Soc. **39** (1989), no. 2, 225–231. MR 90c:49013. Zbl 649.49007.

[8] A. H. Siddiqui, Q. H. Ansari, and K. R. Kazmi, *On nonlinear variational inequalities*, Indian J. Pure Appl. Math. **25** (1994), no. 9, 969–973. Zbl 817.49012.

[9] S. P. Singh, E. Tarafdar, and B. Watson, *Variational inequalities for a pair of pseudomonotone functions*, Far East J. Math. Sci. Special Volume, Part I (1996), 31–52. MR 98d:49011. Zbl 931.47054.

[10] ———, *Variational inequalities and applications*, Indian J. Pure Appl. Math. **28** (1997), no. 8, 1083–1089. CMP 1 470 123. Zbl 880.49009.

[11] E. Tarafdar, *On nonlinear variational inequalities*, Proc. Amer. Math. Soc. **67** (1977), no. 1, 95–98. MR 57#7267. Zbl 369.47029.

[12] ———, *Variational problems via a fixed point theorem*, Indian J. Math. **28** (1986), no. 3, 229–240 (1987). MR 88j:47081. Zbl 641.49005.

[13] ———, *A fixed point theorem equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz theorem*, J. Math. Anal. Appl. **128** (1987), no. 2, 475–479. MR 89a:47084. Zbl 644.47050.

[14] K. K. Wadhwa and A. Ganguly, *Variational like inequality problem*, Bull. Calcutta Math. Soc. **88** (1996), no. 1, 71–74. CMP 1 448 983. Zbl 887.49009.

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