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CENTER CONDITIONS FOR A SIMPLE CLASS OF QUINTIC SYSTEMS

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We obtain center conditions for an O-symmetric system of degree 5 for which the origin is a uniformly isochronous singular point.

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1. Consider a planar differential system

$$\dot{x} = y + xR_{n-1}(x,y), \qquad \dot{y} = -x + yR_{n-1}(x,y), \tag{1.1}$$

where $R_{n-1}(x,y)$ is a polynomial in x,y of degree $n-1$.

System (1.1) has a unique singular point $O(0,0)$ whose linear part is of center type.

Orbits of (1.1) move around the origin with a constant angular velocity and the origin is a uniformly isochronous singular point.

In [3], the following problem was proposed.

Problem 1.1. Identify (1.1) of odd degree that are O-symmetric (not necessarily quasi-homogeneous) having O as a (uniformly isochronous) center.

We solve this problem for $n = 5$ and derive necessary and sufficient center conditions for the system

$$\begin{gathered}\dot{x} = y + x\left(ax^2 + bxy + cy^2 + dx^4 + ex^3y + fx^2y^2 + gxy^3 + hy^4\right),\\ \dot{y} = -x + y\left(ax^2 + bxy + cy^2 + dx^4 + ex^3y + fx^2y^2 + gxy^3 + hy^4\right),\\ a,b,c,d,e,f,g,h \in \mathbb{R}.\end{gathered} \tag{1.2}$$

Theorem 1.2. *The origin is a center of (1.2) if and only if one of the following sets of conditions is satisfied:*

$$a = b = c = 0, \qquad f = -3(d+h); \tag{1.3a}$$

$$a = c = d = f = h = 0; \tag{1.3b}$$

$$\begin{gathered} a \neq 0, \qquad c = -a, \qquad f = \frac{3b(ae-bd)}{(2a^2)},\\ g = \frac{(2a^2bd + (2a^2-b^2)(bd-ae))}{(2a^3)},\\ h = \frac{(-2a^2d + b(bd-ae))}{(2a^2)}.\end{gathered} \tag{1.3c}$$

Proof

Necessity. To describe the behaviour of trajectories of (1.2) near the origin, we construct the comparison function (see [6])

$$F(x,y) = \frac{(x^2+y^2)}{2} + f_3(x,y) + f_4(x,y) + \cdots, \tag{1.4}$$

where f_k is a homogeneous polynomial of degree k whose derivative is

$$\frac{dF}{dt} = D_1(x^4+y^4) + D_2(x^6+y^6) + D_3(x^8+y^8) + \cdots. \tag{1.5}$$

The number of the first coefficient D_i other than zero defines the multiplicity of a complex focus and the sign of this coefficient defines stability of a focus; if $D_i = 0$ for all i, the origin is a center of (1.2). We refer to coefficients D_i as the Poincaré-Lyapunov constants.

To find the Poincaré-Lyapunov constants of a system $\dot{x} = p(x,y)$, $\dot{y} = q(x,y)$ with a linear center, we used computer algebra and wrote a Mathematica code that rests on the Poincaré algorithm in [6]; (see [9] for more details)

```
PLconst[n_] :=
Module[{dF, ff, fF, x, y, pP, qQ, dD},
       fF[2] := (x^2+y^2)/2;
       fF[i_] :=  Sum[ff[i-j, j]*x^(i-j)*y^j, {j, 0, i}];
       pP[1] := y;
       pP[i_] := Sum[p[i-j, j]*x^(i-j)*y^j, {j, 0, i}];
       qQ[1] := -x;
       qQ[i_] := Sum[q[i-j, j]*x^(i-j)*y^j, {j, 0, i}];
       dF[k_] := (Sum[D[fF[i], x]*pP[k+1-i], {i, 2, k}]+
          Sum[D[fF[i], y]*qQ[k+1-i], {i, 2, k}])//Expand;
       Do[
          Solve[Table[Coefficient[dF[k], x^(k-j) y^j],
                {j, 0, k}]
                ==Table[0, {k+1}],
                Table[ff[k-j, j], {j, 0, k}]
               ]/.Rule->Set;
          Solve[Table[Coefficient[dF[k+1], x^(k+1-j)*y^j],
                {j, 0, k+1}]
                ==Flatten[{dD[k], Table[0, {k}], dD[k]}]
                &&ff[0, k+1]==0,
                Flatten[{Table[ff[k+1-j, j],{j, 0, k+1}], dD[k]}]
               ]/.Rule->Set,
          {k, 3, 2n+1, 2}];
          Table[Numerator[Together[dD[k]]],{k, 3, 2n+1, 2}]
      ]
```

The procedure `PLconst[n]` returns a list $\{D_1,\ldots,D_n\}$ of the Poincaré-Lyapunov constants if we define the coefficients p_{ij}, q_{ij} $(2 \le i+j \le 2n+1)$ in the Taylor series expansion of the functions $p(x,y)$ and $q(x,y)$ beforehand.

Using this procedure, we found the first four Poincaré-Lyapunov constants of (1.2).

$$\begin{aligned}
D_1 &= 2(a+c),\\
D_2 &= -4ab-4bc+3d+f+3h,\\
D_3 &= 2(-85a^3+15ab^2-67a^2c+15b^2c+61ac^2+43c^3-24bd-34ae\\
&\qquad -22ce-12bf-50ag-38cg-48bh),\\
D_4 &= 44600a^3b+2736ab^3+84696a^2bc+2736b^3c+47688abc^2+7592bc^3\\
&\quad -37120a^2d-1782b^2d-32552acd-2704c^2d+2364abe+1284bce\\
&\quad -2673de-6120a^2f-234b^2f-3384acf+792c^2f-891ef\\
&\quad +6876abg+5076bcg-3807dg-1269fg+4720a^2h+1098b^2h\\
&\quad +31448ach+19456c^2h-2673eh-3807gh.
\end{aligned} \tag{1.6}$$

It is easy to verify that the equalities $D_i = 0$; $i = 1,2,3,4$, are equivalent to the following relations:

$$\begin{gathered}
a+c=0, \qquad 3d+f+3h=0,\\
3ce-bf+3cg-6bh=0, \qquad 2c^2f-3bcg+3b^2h=0.
\end{gathered} \tag{1.7}$$

If $a = 0$ then our simultaneous polynomial equations have two sets of solutions indicated in (1.3a) and (1.3b). If $a \neq 0$ then, in view of the condition $c = -a$, we see that the other three equations constitute a nondegenerate linear system for determining the variables f, g, h. The solution is given by (1.3c).

The necessity part of the theorem is proved.

Sufficiency

Case 1. System (1.2) now takes the form

$$\begin{aligned}
\dot{x} &= y+x(dx^4+ex^3y+fx^2y^2+gxy^3+hy^4) \equiv y+xp_4(x,y),\\
\dot{y} &= -x+y(dx^4+ex^3y+fx^2y^2+gxy^3+hy^4) \equiv x+yp_4(x,y).
\end{aligned} \tag{1.8}$$

This is a quasi-homogeneous system of degree 5 whose coefficients satisfy the equality $f = -3(d+h)$, that is, the necessary and sufficient center condition in the case we study (see [2]).

Case 2. System (1.2) now takes the form

$$\begin{aligned}
\dot{x} &= y+x^2y(b+ex^2+gy^2),\\
\dot{y} &= -x+xy^2(b+ex^2+gy^2).
\end{aligned} \tag{1.9}$$

The planar differential system

$$\dot{x} = p(x,y), \qquad \dot{y} = q(x,y) \tag{1.10}$$

is said to be reversible (in the sense of Żołądek), if its orbits are symmetric with respect to a line passing through the origin.

System (1.10) is reversible if there is a linear transformation $S:\mathbb{R}^2 \to \mathbb{R}^2$, sending a point (x,y) to the point (x',y') symmetric to (x,y) with respect to the line $\alpha x + \beta y = 0$ and satisfying the condition $S(p(x,y),q(x,y)) = -(p(S(x,y)),q(S(x,y)))$.

A more general condition of reversibility is as follows:

$$\begin{aligned}2\alpha\beta\big(p(x,y)p(x',y')-q(x,y)q(x',y')\big)\\ +(\beta^2-\alpha^2)\big(p(x,y)q(x',y')+p(x',y')q(x,y)\big)=0.\end{aligned}\tag{1.11}$$

It is well known that if (1.10) is reversible and has a linear center at the origin then the origin is a center of this system (cf. [6]).

Obviously, system (1.9) is reversible because its trajectories are symmetric with respect to both coordinate axes. So, the origin is a center for (1.9).

Case 3. System (1.2) now takes the form

$$\begin{aligned}(2a^3)\dot x &= (2a^3)y+x(ax^2+bxy-ay^2)\\ &\quad\times(2a^3+2a^2dx^2-2abdxy+2a^2exy+2a^2dy^2-b^2dy^2+abey^2),\\ (2a^3)\dot y &= -(2a^3)x+y(ax^2+bxy-ay^2)\\ &\quad\times(2a^3+2a^2dx^2-2abdxy+2a^2exy+2a^2dy^2-b^2dy^2+abey^2).\end{aligned}\tag{1.12}$$

It turns out that system (1.12) is reversible. Its trajectories are symmetric with respect to each of the two perpendicular lines defined by the equation $ax^2+bxy-ay^2=0$. The appropriate linear transformation S is given by each of the two matrices

$$S_{1,2}=\pm(4a^2+b^2)^{-1/2}\begin{pmatrix}-b & 2a\\ 2a & b\end{pmatrix}.\tag{1.13}$$

This fact is confirmed by straight calculations. We used Mathematica here.

With the coordinate change $x\mapsto x\cos\varphi+y\sin\varphi$, $y\mapsto -x\sin\varphi+y\cos\varphi$, where the angle φ is defined from the condition $a\tan^2\varphi+b\tan\varphi-a=0$, system (1.12) becomes as follows:

$$\begin{aligned}\dot x &= y+x^2y(b_1+e_1x^2+g_1y^2),\\ \dot y &= -x+xy^2(b_1+e_1x^2+g_1y^2).\end{aligned}\tag{1.14}$$

Hence the origin is a center for (1.2) in this case once again.

The theorem is proved. □

2. It is known that isochronism of a center of a planar polynomial system is equivalent to the existence of an analytic transversal system commuting with a given system in a neighborhood of a center [7]; observe that an arbitrary polynomial system with isochronous center does not necessarily commute with a polynomial system [4, 8].

It is proved in [1] that if the systems

$$\begin{aligned}\dot x &= p(x,y), & \dot y &= q(x,y),\\ \dot x &= r(x,y), & \dot y &= s(x,y)\end{aligned}\tag{2.1}$$

commute, then $\mu(x,y)=1/(p(x,y)s(x,y)-q(x,y)r(x,y))$ is an integrating factor of both systems.

Thereby, if both commuting systems are polynomial then we can find the integrating Darboux factor for the given system and integrate the latter, (about the method of Darboux and the relevant definitions see, for example, [5]).

We now state the following fact that will be useful later.

Considering (2.1), assume that

$$\begin{aligned} p(x,y) &= y + xR(x,y), \qquad & q(x,y) &= -x + yR(x,y), \\ r(x,y) &= xQ(x,y), \qquad & s(x,y) &= yQ(x,y), \end{aligned} \tag{2.2}$$

where $R(x,y)$, $Q(x,y)$ are polynomials in x,y. Then the algebraic curves $x^2+y^2=0$, $Q(x,y)=0$ are invariants for each of these systems.

Indeed, it is immediately obvious that $x^2+y^2=0$ is an invariant of both systems with the cofactor $2R(x,y)$ and $2Q(x,y)$, respectively. The curve $Q(x,y)=0$ is an invariant of the second system with the cofactor $xQ_x(x,y)+yQ_y(x,y)$.

Because our systems commute, the Lie bracket of vector fields (p,q) and (r,s) vanishes and we have

$$\begin{aligned} &p_x(x,y)r(x,y)+p_y(x,y)s(x,y)-r_x(x,y)p(x,y)-r_y(x,y)q(x,y)=0, \\ &xQ(x,y)(R(x,y)+xR_x(x,y))+yQ(x,y)(1+xR_y(x,y)) \\ &\qquad\qquad -p(x,y)(Q(x,y)+xQ_x(x,y))-xq(x,y)Q_y(x,y)=0, \end{aligned} \tag{2.3}$$

or

$$\begin{aligned} &x(Q_x(x,y)p(x,y)+Q_y(x,y)q(x,y)) \\ &\quad = (R(x,y)+xR_x(x,y))xQ(x,y)+(1+xR_y(x,y))yQ(x,y)-Q(x,y)p(x,y) \\ &\quad = (R(x,y)+xR_x(x,y))xQ(x,y)+(1+xR_y(x,y))yQ(x,y) \\ &\qquad -Q(x,y)(y+xR(x,y)) \\ &\quad = x(xR_x(x,y)+yR_y(x,y))Q(x,y). \end{aligned} \tag{2.4}$$

We see that the curve $Q(x,y)=0$ is an invariant with the cofactor $xR_x(x,y)+yR_y(x,y)$.

In this case, $\mu(x,y)=1/(Q(x,y)(x^2+y^2))$ is an integrating Darboux factor.

3. In each of the three cases, we have found a nontrivial polynomial system commuting with the respective system.

In Case 1 such a system is

$$\begin{aligned} \dot{x} &= x(1+ex^4-4dx^3y+4hxy^3-gy^4) \equiv x(1+q_4(x,y)), \\ \dot{y} &= y(1+ex^4-4dx^3y+4hxy^3-gy^4) \equiv y(1+q_4(x,y)). \end{aligned} \tag{3.1}$$

The function

$$\mu(x,y) = \frac{1}{(x^2+y^2)(1+q_4(x,y))} \tag{3.2}$$

is the integrating Darboux factor of (1.8) and the function

$$H(x,y) = \frac{(x^2+y^2)^2}{1+q_4(x,y)} \tag{3.3}$$

is the first rational integral of (1.8).

The algebraic curves $x^2+y^2 = 0$ and $1+q_4(x,y) = 0$ are invariant curves for (1.8).

According to [2], system (1.1) has a center of type B^k, $1 \le k \le n-1$, whose boundary is a finite union of k unbounded open trajectories. Using (3.3), in Case 1 we can describe this boundary explicitly:

$$\varrho = \frac{1}{(c_0 - q_4(\cos\varphi, \sin\varphi))^{1/4}}, \tag{3.4}$$

where $c_0 = \max_{[0,2\pi]} q_4(\cos\varphi, \sin\varphi)$, $x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$.

A straight analysis of this expression allows us to conclude that in our case a center may be of type B^2 or B^4 only.

In Case 2, (1.9) commutes with the system

$$\begin{aligned}\dot{x} &= (e-g)x + x(ex^2+gy^2)(b+ex^2+gy^2),\\ \dot{y} &= (e-g)y + y(ex^2+gy^2)(b+ex^2+gy^2).\end{aligned} \tag{3.5}$$

This permits us to find an integrating Darboux factor

$$\mu(x,y) = \frac{1}{(x^2+y^2)(e-g+(ex^2+gy^2)(b+ex^2+gy^2))}. \tag{3.6}$$

The algebraic curves $x^2+y^2 = 0$, $e-g+(ex^2+gy^2)(b+ex^2+gy^2) = 0$ are the invariant ones for (1.9).

If $b = 0$, then (1.9) is a system of the form (1.8) for which the condition $f = -3(d+h)$ is obviously fulfilled. Then its first integral is

$$H(x,y) = \frac{(x^2+y^2)^2}{1+ex^4-gy^4}. \tag{3.7}$$

If $b \ne 0$ then we may suppose that $b = 1$. The general case reduces to this by the change of variables $x \to x/\sqrt{b}$, $y \to y/\sqrt{b}$ for $b > 0$ or $x \to y/\sqrt{-b}$, $y \to x/\sqrt{-b}$, $t \to -t$ for $b < 0$.

Then our system takes the form

$$\begin{aligned}\dot{x} &= y + x^2y(1+ex^2+gy^2) \equiv X_1(x,y),\\ \dot{y} &= -x + xy^2(1+ex^2+gy^2) \equiv Y_1(x,y).\end{aligned} \tag{3.8}$$

The function $\mu_1(x,y)$, which is equal to $\mu(x,y)$ from (3.6) for $b = 1$, is an integrating factor of (3.8). The first integral $H_1(x,y)$ of (3.8) associated to the integrating factor $\mu_1(x,y)$ can be computed via the integral

$$H_1(x,y) = \int \mu_1(x,y)Y_1(x,y)dx + m(y) \tag{3.9}$$

imposing the condition $\partial H_1(x,y)/\partial y = -\mu_1(x,y)X_1(x,y)$.

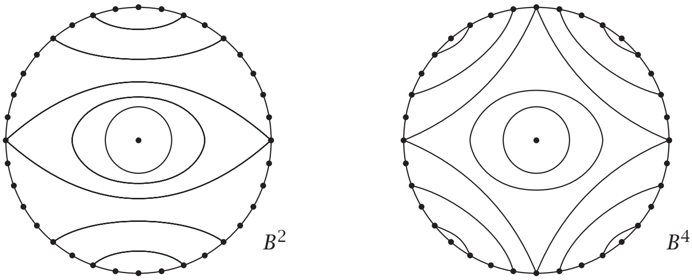

Figure 3.1

We have

$$H_1(x,y)=\frac{1}{2(e-g)}\left(\frac{1}{2}\ln\left(e-g+(ex^2+gy^2)(1+ex^2+gy^2)\right)\right.$$
$$\left.-\ln(x^2+y^2)+\frac{1}{\sqrt{4(e-g)-1}}\arctan\frac{1+2ex^2+2gy^2}{\sqrt{4(e-g)-1}}\right). \tag{3.10}$$

We used Mathematica here.

Then the function $H(x,y)=\exp(-4(e-g)H_1(x,y))$ is the first integral of (3.8) also. It has the form

$$H(x,y)=\frac{(x^2+y^2)^2}{e-g+(ex^2+gy^2)(1+ex^2+gy^2)}$$
$$\times\exp\left(-\frac{2}{\sqrt{4(e-g)-1}}\arctan\frac{1+2ex^2+2gy^2}{\sqrt{4(e-g)-1}}\right). \tag{3.11}$$

Since (3.8) has a unique finite singular point at the origin, the phase portraits are obtained by studying the points at infinity. A standard inspection of the location and types of such points on the equator of the Poincaré sphere allows us to conclude that (3.8) has phase portraits of two types only: a center is of type B^2 when $eg \geq 0$ or of type B^4 when $eg < 0$.

The relevant phase portraits are presented in Figure 3.1. These portraits are fairly typical (cf. [5]) and we do not supply explanations for them.

In Case 3, a commuting system and integrating Darboux factor and first integral may be found on considering that (1.12) is equivalent to (3.8).

Observe that for $d=e=0$, (1.12) is a quasi-homogeneous O-symmetric cubic system of the form

$$\begin{aligned}\dot{x}&=y+x(ax^2+bxy-ay^2),\\ \dot{y}&=-x+y(ax^2+bxy-ay^2).\end{aligned} \tag{3.12}$$

It commutes with the system

$$\begin{aligned}\dot{x}&=x+x(bx^2-2axy),\\ \dot{y}&=y+y(bx^2-2axy),\end{aligned} \tag{3.13}$$

and has the first integral

$$H(x,y) = \frac{x^2+y^2}{1+bx^2-2axy}. \tag{3.14}$$

Summarizing, we conclude that Figure 3.1 presents all possible phase portraits of (1.2) having the origin as a center.

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