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A NOTE ON RUSCHEWEYH TYPE OF INTEGRAL OPERATORS FOR UNIFORMLY α-CONVEX FUNCTIONS

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We prove that the class of uniformly α-*convex* functions introduced by Kanas is closed under the generalized Ruscheweyh integral operator for $0 < \alpha \le 1$.

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We denote by $\mathcal{A}$ the class of functions $f(z) = z + a_2 z^2 + \cdots$ which are analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the class of functions in $\mathcal{A}$ that are univalent in Δ. The subclasses of S containing functions which are uniformly convex and uniformly starlike, introduced by Goodman [1, 2], are denoted by *UCV* and *UST*, respectively.

The class of uniformly α-convex functions was introduced by Kanas [3] and she gave an analytic condition for such functions as follows: $f(z)$ is a uniformly α-convex function if and only if

$$\operatorname{Re}\left\{(1-\alpha)\frac{(z-\zeta)f'(z)}{f(Z)-f(\zeta)} + \alpha\left(1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right)\right\} > 0 \tag{1}$$

for all $z, \zeta \in \Delta$ and $0 \le \alpha \le 1$. For $\zeta = 0$, this class of functions reduces to Mocanu's class $M(\alpha)$ of α-convex functions [4].

In this note, for $\alpha > 0$, we consider the integral operator

$$F(z) = \frac{F_\alpha(z,\zeta) - F_\alpha(0,\zeta)}{F'_\alpha(0,\zeta)}, \tag{2}$$

where

$$F_\alpha(z,\zeta) = \left\{\frac{c+1/\alpha}{(z-\zeta)^c}\int_\zeta^z (t-\zeta)^{c-1}\left(f(t)-f(\zeta)\right)^{1/\alpha} dt\right\}^\alpha \tag{3}$$

for all $z \in \Delta$ and for fixed $\zeta \in \Delta$ with $z \ne \zeta$. We prove that this normalized function $F(z)$ is a uniformly α-convex function when $f(z)$ is a uniformly α-convex function in the sense of Kanas [3].

For $\zeta = 0$ the operator $F(z)$ reduces to Ruscheweyh's integral operator [5]. It is well known that Mocanu's class $M(\alpha)$ of α-convex functions is closed under Ruscheweyh's integral operator for $\alpha > 0$.

Theorem 1. *Let $f(z) = z + a_2 z^2 + \cdots$ be a uniformly α-convex function in Δ and let $c > 0$. Then, for $0 < \alpha \le 1$, the function*

$$F(z) = \frac{F_\alpha(z,\zeta) - F_\alpha(0,\zeta)}{F'_\alpha(0,\zeta)}, \quad z,\zeta \in \Delta, \tag{4}$$

is uniformly α-convex where $F_\alpha(z,\zeta)$ is defined as in (3).

Proof. We have from (3) that

$$F_\alpha^{1/\alpha}(z,\zeta) = \frac{c + 1/\alpha}{(z-\zeta)^c} \int_\zeta^z (t-\zeta)^{c-1} \big(f(t) - f(\zeta)\big)^{1/\alpha} dt. \tag{5}$$

Differentiating with respect to z, we have

$$\begin{aligned} &(z-\zeta)^c \frac{1}{\alpha} F_\alpha^{1/\alpha-1}(z,\zeta) F'_\alpha(z,\zeta) + c(z-\zeta)^{c-1} F_\alpha^{1/\alpha}(z,\zeta) \\ &\qquad = \left(c + \frac{1}{\alpha}\right)(z-\zeta)^{c-1}\big(f(z) - f(\zeta)\big)^{1/\alpha} \end{aligned} \tag{6}$$

and again differentiating with respect to z we get

$$\begin{aligned} &\frac{1}{\alpha}\Big\{(z-\zeta) F_\alpha^{1/\alpha-1}(z,\zeta) F''_\alpha(z,\zeta) + (z-\zeta)\left(\frac{1}{\alpha} - 1\right) F_\alpha^{1/\alpha-2}(z,\zeta)\big(F'_\alpha(z,\zeta)\big)^2 \\ &\qquad + F_\alpha^{1/\alpha-1}(z,\zeta) F'_\alpha(z,\zeta)\Big\} + \frac{c}{\alpha} F_\alpha^{1/\alpha-1}(z,\zeta) F'_\alpha(z,\zeta) \\ &\quad = \left(c + \frac{1}{\alpha}\right)\frac{1}{\alpha} f'(z)\big(f(z) - f(\zeta)\big)^{1/\alpha-1}; \\ &F_\alpha^{1/\alpha-1}(z,\zeta) F'_\alpha(z,\zeta)\left\{\alpha \frac{(z-\zeta)F''_\alpha(z,\zeta)}{F'_\alpha(z,\zeta)} + (1-\alpha)(z-\zeta)\frac{F'_\alpha(z,\zeta)}{F_\alpha(z,\zeta)} + \alpha(1+c)\right\} \\ &\quad = (\alpha c + 1) f'(z)\big(f(z) - f(\zeta)\big)^{1/\alpha-1}. \end{aligned} \tag{7}$$

Thus we get

$$\begin{aligned} &F_\alpha^{1/\alpha-1}(z,\zeta) F'_\alpha(z,\zeta)\left\{\alpha\left[1 + \frac{(z-\zeta)F''_\alpha(z,\zeta)}{F'_\alpha(z,\zeta)} - \frac{(z-\zeta)F'_\alpha(z,\zeta)}{F_\alpha(z,\zeta)}\right]\right. \\ &\qquad\qquad \left. + \frac{(z-\zeta)F'_\alpha(z,\zeta)}{F_\alpha(z,\zeta)} + c\alpha\right\} \\ &\quad = (c\alpha + 1) f'(z)\big(f(z) - f(\zeta)\big)^{1/\alpha-1}. \end{aligned} \tag{8}$$

From (2) we have

$$F'(z) = \frac{F'_\alpha(z,\zeta)}{F'_\alpha(0,\zeta)}, \tag{9}$$

showing that $F(0) = 0$ and $F'(0) = 1$.

Considering

$$\frac{(z-\zeta)F'(z)}{F(z) - F(\zeta)} = \frac{(z-\zeta)F'_\alpha(z,\zeta)}{F_\alpha(z,\zeta)} \tag{10}$$

and differentiating with respect to z, we have

$$\frac{F''(z)}{F'(z)}+\frac{1}{z-\zeta}-\frac{F'(z)}{F(z)-F(\zeta)}=\frac{1}{z-\zeta}+\frac{F''_\alpha(z,\zeta)}{F'_\alpha(z,\zeta)}-\frac{F'_\alpha(z,\zeta)}{F_\alpha(z,\zeta)}; \tag{11}$$

$$\frac{(z-\zeta)F''(z)}{F'(z)}+1-\frac{(z-\zeta)F'(Z)}{F(z)-F(\zeta)}=1+\frac{(z-\zeta)F''_\alpha(z,\zeta)}{F'_\alpha(z,\zeta)}-\frac{F'_\alpha(z,\zeta)(z-\zeta)}{F_\alpha(z,\zeta)}. \tag{12}$$

Substituting (10) and (12) in (8), we obtain

$$\begin{aligned}
&F_\alpha^{1/\alpha-1}(z,\zeta)F'_\alpha(z,\zeta)\left\{\alpha\left[\frac{(z-\zeta)F''(z)}{F'(z)}+1-\frac{(z-\zeta)F'(z)}{F(z)-F(\zeta)}\right]+\frac{(z-\zeta)F'(z)}{F(z)-F(\zeta)}+c\alpha\right\}\\
&\quad=(c\alpha+1)f'(z)\left(f(z)-f(\zeta)\right)^{1/\alpha-1};\\
&F_\alpha^{1/\alpha-1}(z,\zeta)F'_\alpha(z,\zeta)\left\{(1-\alpha)\frac{(z-\zeta)F'(z)}{F(z)-F(\zeta)}+\alpha\left(1+\frac{(z-\zeta)F''(z)}{F'(z)}\right)+c\alpha\right\}\\
&\quad=(c\alpha+1)f'(z)\left(f(z)-f(\zeta)\right)^{1/\alpha-1}.
\end{aligned} \tag{13}$$

Setting

$$P(z,\zeta)=(1-\alpha)\frac{(z-\zeta)F'(z)}{F(z)-F(\zeta)}+\alpha\left(\frac{(z-\zeta)F''(z)}{F'(z)}+1\right), \tag{14}$$

equation (13) becomes

$$F_\alpha^{1/\alpha-1}(z,\zeta)F'_\alpha(z,\zeta)\{P(z,\zeta)+c\alpha\}=(c\alpha+1)f'(z)\left(f(z)-f(\zeta)\right)^{1/\alpha-1}. \tag{15}$$

Taking logarithmic differentiation with respect to z, we get

$$\begin{aligned}
&(1-\alpha)(z-\zeta)\frac{F'_\alpha(z,\zeta)}{F_\alpha(z,\zeta)}+\alpha(z-\zeta)\frac{F''_\alpha(z,\zeta)}{F'_\alpha(z,\zeta)}+\frac{\alpha(z-\zeta)P'(z,\zeta)}{P(z,\zeta)+c\alpha}+\alpha\\
&\quad=\alpha+\alpha\frac{(z-\zeta)f''(z)}{f'(z)}+(1-\alpha)\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)};\\
&\alpha\left[(z-\zeta)\frac{F''_\alpha(z,\zeta)}{F'_\alpha(z,\zeta)}+1-\frac{(z-\zeta)F'_\alpha(z,\zeta)}{F_\alpha(z,\zeta)}\right]+\frac{(z-\zeta)F'_\alpha(z,\zeta)}{F_\alpha(z,\zeta)}+\frac{\alpha(z-\zeta)P'(z,\zeta)}{P(z,\zeta)+c\alpha}\\
&\quad=\alpha\left(1+\frac{(z-\zeta)f''(z)}{f'(z)}\right)+(1-\alpha)\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}.
\end{aligned} \tag{16}$$

Equations (10) and (12) give

$$\begin{aligned}
&\alpha\left[\frac{(z-\zeta)F''(z)}{F'(z)}+1-\frac{(z-\zeta)F'(z)}{F(z)-F(\zeta)}\right]+\frac{(z-\zeta)F'(z)}{F(z)-F(\zeta)}+\frac{\alpha(z-\zeta)P'(z,\zeta)}{P(z,\zeta)+c\alpha}\\
&\quad=\alpha\left(1+\frac{(z-\zeta)f''(z)}{f'(z)}\right)+(1-\alpha)\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}.
\end{aligned} \tag{17}$$

That is

$$\left[\alpha\left(1+\frac{(z-\zeta)F''(z)}{F'(z)}\right)+(1-\alpha)\frac{(z-\zeta)F'(z)}{F(z)-F(\zeta)}\right]+\frac{\alpha(z-\zeta)P'(z,\zeta)}{P(z,\zeta)+c\alpha} = \alpha\left(1+\frac{(z-\zeta)f''(z)}{f'(z)}\right)+(1-\alpha)\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}. \tag{18}$$

Hence, we have

$$P(z,\zeta)+\frac{\alpha(z-\zeta)P'(z,\zeta)}{P(z,\zeta)+c\alpha}=\alpha\left(1+\frac{(z-\zeta)f''(z)}{f'(z)}\right)+(1-\alpha)\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}. \tag{19}$$

Since $f(z)$ is uniformly α-convex, we have

$$\mathrm{Re}\left\{P(z,\zeta)+\frac{\alpha(z-\zeta)P'(z,\zeta)}{P(z,\zeta)+c\alpha}\right\}>0 \tag{20}$$

for all $z,\zeta\in\Delta$, $0\le\alpha\le 1$.

We show that $\mathrm{Re}\,P(z,\zeta)>0$. Suppose that there exists a point $\zeta_0\in\Delta$ such that the image of the arc $\Gamma : z(t)=\zeta_0+re^{it}$ is tangent to the imaginary axis. Let w_0 be the point of contact and let $z_0\in\Delta$ such that $w_0=P(z_0,\zeta_0)$. Then $\mathrm{Re}\,P(z_0,\zeta_0)=0$ and therefore $P(z_0,\zeta_0)=ix$, where $x\in R$. Hence the outer normal to $F(\Gamma)$ is

$$(z_0-\zeta_0)P'(z_0,\zeta_0)=y<0. \tag{21}$$

For such ζ_0, we have

$$\begin{aligned}\mathrm{Re}\left\{P(z_0,\zeta_0)+\frac{\alpha(z_0-\zeta_0)P'(z_0,\zeta_0)}{P(z_0,\zeta_0)+c\alpha}\right\}&=\mathrm{Re}\left\{ix+\frac{\alpha y}{c\alpha+ix}\right\}\\&=\mathrm{Re}\left\{ix+\frac{\alpha y(c\alpha-ix)}{c^2\alpha^2+x^2}\right\}\\&=\frac{c\alpha^2 y}{(c\alpha)^2+x^2}<0\quad\text{for } c>0\end{aligned} \tag{22}$$

which contradicts (20) and hence $\mathrm{Re}\,P(z,\zeta)>0$ in Δ showing that $F(z)$ is a uniformly α-convex function. □

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