IJMMS 29:5 (2002) 271–277
PII. S0161171202007378
http://ijmms.hindawi.com

RADIUS OF STRONGLY STARLIKENESS FOR CERTAIN ANALYTIC FUNCTIONS

OH SANG KWON and SHIGEYOSHI OWA

Received 13 April 2001

For analytic functions $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$ in the open unit disk $\mathbb{U}$ and a polynomial $Q(z)$ of degree $n > 0$, the function $F(z) = f(z)[Q(z)]^{\beta/n}$ is introduced. The object of the present paper is to determine the radius of p-valently strongly starlikeness of order γ for $F(z)$.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let $\mathcal{A}_p$ (p is a fixed integer $\geqq 1$) denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let Ω denote the class of bounded functions $w(z)$ analytic in $\mathbb{U}$ and satisfying the conditions $w(0) = 0$ and $|w(z)| \leqq |z|$, $z \in \mathbb{U}$. We use $\mathcal{P}$ to denote the class of functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ which are analytic in $\mathbb{U}$ and satisfy $\operatorname{Re} p(z) > 0$ ($z \in \mathbb{U}$).

For $0 \leqq \alpha < p$ and $|\lambda| < \pi/2$, we denote by $\mathcal{S}_p^{\lambda}(\alpha)$, the family of functions $g(z) \in \mathcal{A}_p$ which satisfy

$$\frac{zg'(z)}{g(z)} \prec \frac{p + \{2(p-\alpha)e^{-i\lambda}\cos\lambda - p\}z}{1-z}, \quad z \in \mathbb{U}, \tag{1.2}$$

where $\prec$ means the subordination. From the definition of subordinations, it follows that $g(z) \in \mathcal{A}_p$ has the representation

$$\frac{zg'(z)}{g(z)} = \frac{p + \{2(p-\alpha)e^{-i\lambda}\cos\lambda - p\}w(z)}{1-w(z)}, \tag{1.3}$$

where $w(z) \in \Omega$. Clearly, $\mathcal{S}_p^{\lambda}(\alpha)$ is a subclass of p-valent λ-spiral functions of order α. For $\lambda = 0$, we have the class $\mathcal{S}_p^{*}(\alpha)$, $0 \leqq \alpha < p$, of p-valent starlike functions of order α, investigated by Goluzina [5].

A function $f(z) \in \mathcal{A}_p$ is said to be p-valently strongly starlike of order γ, $0 < \gamma \leqq 1$, if it satisfies

$$\left| \arg\left\{ \frac{zf'(z)}{f(z)} \right\} \right| \leqq \frac{\pi}{2}\gamma. \tag{1.4}$$

Başgöze [1, 2] has obtained sharp inequalities of univalence (starlikeness) for certain polynomials of the form $F(z) = f(z)[Q(z)]^{\beta/n}$, where β is real and $Q(z)$ is a polynomial of degree $n > 0$ all of whose zeros are outside or on the unit circle $\{z : |z| = 1\}$. Rajasekaran [7] extended Başgöze's results for certain classes of analytic functions of the form $F(z) = f(z)[Q(z)]^{\beta/n}$. Recently, Patel [6] generalized some of the work of Rajasekaran and Başgöze for functions belonging to the class $\mathcal{S}_p^{\lambda}(\alpha)$. That is, determine the radius of starlikeness for some classes of p-valent analytic functions of the polynomial form $F(z)$.

In the present paper, we extend the results of Patel [6]. Thus, we determine the radius of p-valently strongly starlike of order γ for polynomials of the form $F(z)$ in such problems.

2. Some lemmas.

Before proving our next results, we need the following lemmas.

Lemma 2.1 (see Gangadharan [4]). *For $|z| \leqq r < 1$, $|z_k| = R > r$,*

$$\left| \frac{z}{z - z_k} + \frac{r^2}{R^2 - r^2} \right| \leqq \frac{Rr}{R^2 - r^2}. \tag{2.1}$$

Lemma 2.2 (see Ratti [8]). *If $\phi(z)$ is analytic in $\mathbb{U}$ and $|\phi(z)| \leqq 1$ for $z \in \mathbb{U}$, then for $|z| = r < 1$,*

$$\left| \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right| \leqq \frac{1}{1 - r}. \tag{2.2}$$

Lemma 2.3 (see Causey and Merkes [3]). *If $p(z) = 1 + c_1 z + c_2 z + \cdots \in \mathcal{P}$, then for $|z| = r < 1$,*

$$\left| \frac{zp'(z)}{p(z)} \right| \leqq \frac{2r}{1 - r^2}. \tag{2.3}$$

This estimate is sharp.

Lemma 2.4 (see Patel [6]). *Suppose $g(z) \in \mathcal{S}_p^{\lambda}(\alpha)$. Then for $|z| = r < 1$,*

$$\left| \frac{zg'(z)}{g(z)} - \left(p + \frac{2(p - \alpha)e^{i\lambda} r^2 \cos\lambda}{1 - r^2} \right) \right| \leqq \frac{2(p - \alpha) r \cos\lambda}{1 - r^2}. \tag{2.4}$$

This result is sharp.

Lemma 2.5 (see Gangadharan [4]). *If $R_a \leqq \operatorname{Re}(a)\sin((\pi/2)\gamma) - \operatorname{Im}(a)\cos((\pi/2)\gamma)$, $\operatorname{Im}(a) \geqq 0$, then the disk $|w - a| \leqq R_a$ is contained in the sector $|\arg w| \leqq (\pi/2)\gamma$, $0 < \gamma \leqq 1$.*

3. Main results.

Our first theorem is the following one.

Theorem 3.1. *Suppose that*

$$F(z) = f(z)[Q(z)]^{\beta/n}, \tag{3.1}$$

where β is real and $Q(z)$ is a polynomial of degree $n > 0$ with no zeros in $|z| < R$,

$R \geqq 1$. *If* $f(z) \in \mathcal{A}_p$ *satisfies*

$$\operatorname{Re}\left[\left(\frac{f(z)}{g(z)}\right)^{1/\delta}\right] > 0, \quad 0 < \delta \leqq 1,\ z \in \mathbb{U}, \tag{3.2}$$

$$\operatorname{Re}\left[\frac{g(z)}{h(z)}\right] > 0, \quad z \in \mathbb{U}, \tag{3.3}$$

for some $g(z) \in \mathcal{A}_p$ *and* $h(z) \in \mathcal{S}_p^{\lambda}(\alpha)$, *then* $F(z)$ *is* p*-valently strongly starlike of order* γ *in* $|z| < R(\gamma)$, *where* $R(\gamma)$ *is the smallest positive root of the equation*

$$\begin{aligned} &r^4\left[(p+\beta)\sin\left(\frac{\pi}{2}\gamma\right)+2(p-\alpha)\cos\lambda\sin\left(\lambda-\frac{\pi}{2}\gamma\right)\right] \\ &\quad +r^3\left[|\beta|R+2(p-\alpha)\cos\lambda+2(\delta+1)\right] \\ &\quad -r^2\left[(p(1+R^2)+\beta)\sin\left(\frac{\pi}{2}\gamma\right)+2(p-\alpha)R^2\cos\lambda\sin\left(\lambda-\frac{\pi}{2}\gamma\right)\right] \\ &\quad -r\left[|\beta|R+2(p-\alpha)R^2\cos\lambda+2(\delta+1)R^2\right]+pR^2\sin\left(\frac{\pi}{2}\gamma\right)=0. \end{aligned} \tag{3.4}$$

Proof. We choose a suitable branch of $(f(z)/g(z))^{1/\delta}$ so that $(f(z)/g(z))^{1/\delta}$ is analytic in $\mathbb{U}$ and takes the value 1 at $z = 0$. Thus from (3.2) and (3.3), we have

$$F(z) = p_1^{\delta}(z)p_2 h(z)[Q(z)]^{\beta/n}, \tag{3.5}$$

where $p_j(z) \in \mathcal{P}$ $(j = 1,2)$.

Then

$$\frac{zF'(z)}{F(z)} = \delta\frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)} + \frac{zh'(z)}{h(z)} + \frac{\beta}{n}\sum_{k=1}^{n}\frac{z}{z-z_k}. \tag{3.6}$$

Since $h(z) \in \mathcal{S}_p^{\lambda}(\alpha)$, by Lemma 2.4, we have

$$\left|\frac{zh'(z)}{h(z)} - \left(p + \frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2}\right)\right| \leqq \frac{2(p-\alpha)r\cos\lambda}{1-r^2}. \tag{3.7}$$

Using (3.6) and (3.7) with Lemmas 2.1 and 2.3, we get

$$\begin{aligned} &\left|\frac{zF'(z)}{F(z)} - \left(p + \frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2} - \frac{\beta r^2}{R^2-r^2}\right)\right| \\ &\qquad \leqq \frac{2\{(p-\alpha)r\cos\lambda + r(\delta+1)\}}{1-r^2} + \frac{|\beta|Rr}{R^2-r^2}. \end{aligned} \tag{3.8}$$

Using Lemma 2.5, we get that the above disk is contained in the sector $|\arg w| < (\pi/2)\gamma$ provided the inequality

$$\begin{aligned} &\frac{2\{(p-\alpha)r\cos\lambda + r(\delta+1)\}}{1-r^2} + \frac{|\beta|Rr}{R^2-r^2} \\ &\qquad \leqq \left(p + \frac{2(p-\alpha)r^2\cos^2\lambda}{1-r^2} - \frac{\beta r^2}{R^2-r^2}\right)\sin\left(\frac{\pi}{2}\gamma\right) \\ &\qquad\quad - \frac{2(p-\alpha)r^2\sin\lambda\cos\lambda}{1-r^2}\cos\left(\frac{\pi}{2}\gamma\right) \end{aligned} \tag{3.9}$$

is satisfied. The above inequality is simplified to $T(r) \geqq 0$, where

$$\begin{aligned} T(r) = r^4\Big[&(p-2(p-\alpha)\cos^2\lambda+\beta)\sin\Big(\frac{\pi}{2}\gamma\Big)+(p-\alpha)\sin 2\lambda\cos\Big(\frac{\pi}{2}\gamma\Big)\Big] \\ &+r^3[|\beta|R+2(p-\alpha)\cos\lambda+2(\delta+1)] \\ &+r^2\Big[(-pR^2-p+2(p-\alpha)R^2\cos^2\lambda-\beta)\sin\Big(\frac{\pi}{2}\gamma\Big)-(p-\alpha)R^2\sin 2\lambda\cos\Big(\frac{\pi}{2}\gamma\Big)\Big] \\ &-r[|\beta|R+2(p-\alpha)R^2\cos\lambda+2(\delta+1)R^2]+pR^2\sin\Big(\frac{\pi}{2}\gamma\Big). \end{aligned} \tag{3.10}$$

Since $T(0) > 0$ and $T(1) < 1$, there exists a real root of $T(r) = 0$ in $(0,1)$. Let $R(\gamma)$ be the smallest positive root of $T(r) = 0$ in $(0,1)$. Then $F(z)$ is p-valent strongly starlike of order γ in $|z| < R(\gamma)$. □

Remark 3.2. For $R = 1$ and $\gamma = 1$, Theorem 3.1 reduces to a result by Patel [6].

Theorem 3.3. *Suppose that $F(z)$ is given by (3.1). If $f(z) \in \mathcal{A}_p$ satisfies (3.2) for some $g(z) \in \mathcal{S}_p^\lambda(\alpha)$, then $F(z)$ is p-valently strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation*

$$\begin{aligned} r^4\Big[&(p+\beta)\sin\Big(\frac{\pi}{2}\gamma\Big)+2(p-\alpha)\cos\lambda\sin\Big(\lambda-\frac{\pi}{2}\gamma\Big)\Big] \\ &+r^3[|\beta|R+2(p-\alpha)\cos\lambda+2\delta] \\ &-r^2\Big[(p(1+R^2)+\beta)\sin\Big(\frac{\pi}{2}\gamma\Big)+2(p-\alpha)R^2\cos\lambda\sin\Big(\lambda-\frac{\pi}{2}\gamma\Big)\Big] \\ &-r[|\beta|R+2(p-\alpha)R^2\cos\lambda+2\delta R^2]+pR^2\sin\Big(\frac{\pi}{2}\gamma\Big)=0. \end{aligned} \tag{3.11}$$

Proof. If $f(z) \in \mathcal{A}_p$ satisfies (3.2) for some $g(z) \in \mathcal{S}_p^\lambda(\alpha)$, then

$$\frac{zF'(z)}{F(z)} = \delta\frac{zp'(z)}{p(z)}+\frac{zg'(z)}{g(z)}+\frac{\beta}{n}\sum_{k=1}^{n}\frac{z}{z-z_k}. \tag{3.12}$$

Using Lemma 2.4, we get

$$\left|\frac{zg'(z)}{g(z)}-\left(p+\frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2}\right)\right| \leqq \frac{2(p-\alpha)r\cos\lambda}{1-r^2}. \tag{3.13}$$

By (3.12) and (3.13) with Lemmas 2.1 and 2.3, we have

$$\begin{aligned} \left|\frac{zF'(z)}{F(z)}-\left(p+\frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2}-\frac{\beta r^2}{R^2-r^2}\right)\right| \\ \leqq \frac{2\{(p-\alpha)r\cos\lambda+r\delta\}}{1-r^2}+\frac{|\beta|Rr}{R^2-r^2}. \end{aligned} \tag{3.14}$$

The remaining parts of the proof can be proved by a method similar to the one given in the proof of Theorem 3.1. □

With $\lambda = 0$, $\beta = 0$, $\delta = 1$, $R = 1$, and $\gamma = 1$, Theorem 3.3 gives the following corollary.

Corollary 3.4. *Suppose that $f(z)$ is in $\mathcal{A}_p$. If $\operatorname{Re}(f(z)/g(z)) > 0$ for $z \in \mathbb{U}$ and $g(z) \in \mathcal{S}_p^*(\alpha)$, then $f(z)$ is p-valently starlike for*

$$|z| < \frac{p}{(p+1-\alpha)+\sqrt{\alpha^2-2\alpha+2p+1}}. \tag{3.15}$$

Theorem 3.5. *Suppose that $F(z)$ is given by (3.1). If $f(z) \in \mathcal{A}_p$ satisfies*

$$\left|\left(\frac{f(z)}{g(z)}\right)^{1/\delta}-1\right| < 1, \quad 0 < \delta \leqq 1, \qquad p\sin\left(\frac{\pi}{2}\gamma\right) > \delta, \tag{3.16}$$

$$\operatorname{Re}\left(\frac{g(z)}{h(z)}\right) > 0, \quad z \in \mathbb{U} \tag{3.17}$$

for some $g(z) \in \mathcal{A}_p$ and $h(z) \in \mathcal{S}_p^\lambda(\alpha)$, then $F(z)$ is p-valently strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$\begin{aligned} &r^4\left[(p+\beta)\sin\left(\frac{\pi}{2}\gamma\right)+2(p-\alpha)\cos\lambda\sin\left(\lambda-\frac{\pi}{2}\gamma\right)\right] \\ &\quad +r^3\left[|\beta|R+2(p-\alpha)\cos\lambda+2+\delta\right] \\ &\quad -r^2\left[(p(1+R^2)+\beta)\sin\left(\frac{\pi}{2}\gamma\right)+2(p-\alpha)R^2\cos\lambda\sin\left(\lambda-\frac{\pi}{2}\gamma\right)+\delta\right] \\ &\quad -r\left[|\beta|R+2(p-\alpha)R^2\cos\lambda+2(\delta+1)R^2\right]+pR^2\sin\left(\frac{\pi}{2}\gamma\right)-\delta R^2=0. \end{aligned} \tag{3.18}$$

Proof. We choose a suitable branch of $(f(z)/g(z))^{1/\delta}$ so that $(f(z)/g(z))^{1/\delta}$ is analytic in $\mathbb{U}$ and takes the value 1 at $z=0$. From (3.16), we deduce that

$$f(z)=g(z)(1+w(z))^{\delta}, \quad w(z) \in \Omega. \tag{3.19}$$

So that

$$F(z)=p(z)h(z)(1+z\phi(z))^{\delta}[Q(z)]^{\beta/n}, \tag{3.20}$$

where $\phi(z)$ is analytic in $\mathbb{U}$ and satisfies $|\phi(z)| \leqq 1$ and $p \in \mathcal{P}$ for $z \in \mathbb{U}$.

We have

$$\frac{zF'(z)}{F(z)}=\frac{zh'(z)}{h(z)}+\frac{zp'(z)}{p(z)}+\delta\left(\frac{z\phi'(z)+\phi(z)}{1+z\phi(z)}\right)+\frac{\beta}{n}\sum_{k=1}^{n}\frac{z}{z-z_k}. \tag{3.21}$$

Using Lemma 2.4 and (3.21), we have

$$\begin{aligned} &\left|\frac{zF'(z)}{F(z)}-\left(p+\frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2}\right)\right| \\ &\qquad \leqq \frac{2\{(p-\alpha)r\cos\lambda+r\}+\delta(1+r)}{1-r^2}+\frac{|\beta|Rr}{R^2-r^2}. \end{aligned} \tag{3.22}$$

So, using Lemma 2.5 and (3.22), the result can be proved by using a method similar to the one given in the proof of Theorem 3.1. □

Theorem 3.6. *Suppose that $F(z)$ is given by (3.1). If $f(z) \in \mathcal{A}_p$ satisfies (3.16) for some $g(z) \in \mathcal{S}_p^\lambda(\alpha)$, then $F(z)$ is p-valently strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation*

$$\begin{aligned} r^4&\left[(p+\beta)\sin\left(\frac{\pi}{2}\gamma\right)+2(p-\alpha)\cos\lambda\sin\left(\lambda-\frac{\pi}{2}\gamma\right)\right] \\ &+r^3\left[|\beta|R+2(p-\alpha)\cos\lambda+\delta\right] \\ &-r^2\left[(p(1+R^2)+\beta)\sin\left(\frac{\pi}{2}\gamma\right)+2(p-\alpha)R^2\cos\lambda\sin\left(\lambda-\frac{\pi}{2}\gamma\right)+\delta\right] \\ &-r\left[|\beta|R+2(p-\alpha)R^2\cos\lambda+\delta R^2\right]+pR^2\sin\left(\frac{\pi}{2}\gamma\right)-\delta R^2=0. \end{aligned} \tag{3.23}$$

Proof. We choose a suitable branch of $(f(z)/g(z))^{1/\delta}$ so that $(f(z)/g(z))^{1/\delta}$ is analytic in $\mathbb{U}$ and takes the value 1 at $z=0$. Since $f(z) \in \mathcal{A}_p$ satisfies (3.16) for some $g(z) \in \mathcal{S}_p^\lambda(\alpha)$, we have

$$F(z)=g(z)(1+z\phi(z))[Q(z)]^{\beta/n}, \tag{3.24}$$

where $\phi(z)$ is analytic in $\mathbb{U}$ and satisfies the condition $|\phi(z)| \leqq 1$ for $z \in \mathbb{U}$. Thus, we have

$$\frac{zF'(z)}{F(z)}=\frac{zg'(z)}{g(z)}+\delta\left(\frac{z\phi'(z)+\phi(z)}{1+z\phi(z)}\right)+\frac{\beta}{n}\sum_{k=1}^{n}\frac{z}{z-z_k}. \tag{3.25}$$

Using Lemma 2.4 and (3.25), we get

$$\begin{aligned} \left|\frac{zF'(z)}{F(z)}-\left(p+\frac{2(p-\alpha)e^{i\lambda}r^2\cos\lambda}{1-r^2}\right)\right| \\ \leqq \frac{2(p-\alpha)r\cos\lambda+\delta(1+r)}{1-r^2}+\frac{|\beta|Rr}{R^2-r^2}. \end{aligned} \tag{3.26}$$

Using Lemma 2.5 and (3.26) and a method similar to the one given in the proof of Theorem 3.1, we complete the proof of the theorem. □

Remark 3.7. Some of the results of Patel [6] can be obtained from Theorem 3.6 by taking $R=1$ and $\gamma=1$.

References

[1] T. Başgöze, *On the univalence of certain classes of analytic functions*, J. London Math. Soc. (2) **1** (1969), 140–144.

[2] ______, *On the univalence of polynomials*, Compositio Math. **22** (1970), 245–252.

[3] W. M. Causey and E. P. Merkes, *Radii of starlikeness of certain classes of analytic functions*, J. Math. Anal. Appl. **31** (1970), 579–586.

[4] A. Gangadharan, V. Ravichandran, and T. N. Shanmugam, *Radii of convexity and strong starlikeness for some classes of analytic functions*, J. Math. Anal. Appl. **211** (1997), no. 1, 301–313.

[5] E. C. Goluzina, *On the coefficients of a class of functions, regular in a disk and having an integral representation in it*, J. Soviet Math. **2** (1974), 606–617.

[6] J. Patel, *Radii of p-valent starlikeness for certain classes of analytic functions*, Bull. Calcutta Math. Soc. **85** (1993), no. 5, 427–436.

[7] S. Rajasekaran, *A study on extremal problems for certain classes of univalent analytic functions*, Ph.D. thesis, Indian Institute of Technology Kanpur, India, 1985.
[8] J. S. Ratti, *The radius of univalence of certain analytic functions*, Math. Z. **107** (1968), 241–248.

Oh Sang Kwon: Department of Mathematics, Kyungsung University, Pusan 608-736, Korea
E-mail address: oskwon@star.kyungsung.ac.kr

Shigeyoshi Owa: Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan
E-mail address: owa@math.kindai.ac.jp