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GRACEFUL NUMBERS

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We construct a labeled graph $D(n)$ that reflects the structure of divisors of a given natural number n. We define the concept of graceful numbers in terms of this associated graph and find the general form of such a number. As a consequence, we determine which graceful numbers are perfect.

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1. Introduction. In [2], Gallian presented a detailed survey of various types of graph labeling, the two best known being graceful and harmonious. Recall that a graph G with q edges is called graceful if one can label its vertices with distinct numbers from the set $\{0,1,\dots,q\}$ and mark the edges with differences of the labels of the end vertices in such a way that the resulting edge labels are distinct. A number of interesting results on graceful and graceful-like labelings are obtained in [1, 3, 4] and some other works. In this note, we give a description of natural numbers whose associated graph of divisors satisfies certain graceful-like conditions. For any natural number n, we construct a labeled graph $D(n)$ that reflects the structure of divisors of n. We define the concept of graceful number in terms of this associated graph and find the general form of such a number. As a consequence, we determine which graceful numbers are perfect.

2. Main results. Given a natural number n one can generate a graph $D(n)$ that reflects the structure of divisors of n as follows. The vertices of the graph represent all the divisors of the number n, each vertex is labeled by a certain divisor. (In what follows, we refer to the vertex of the graph $D(n)$ with label k as the "vertex k.") If r and s are two divisors of n and $r > s$, then there is an edge between the vertices s and r if and only if s divides r and the ratio r/s is a prime number. As in the theory of graceful graphs, we label such an edge by the difference $r-s$ of the labels of its vertices. In what follows, the sum of the labels of all edges of the graph $D(n)$ is denoted by $\overline{SD}(n)$ while $SD(n)$ denotes the sum of labels of all edges of $D(n)$ except the edges terminating at n. (Clearly, if $n = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ is the prime factorization of a natural number n, then $SD(n) = \overline{SD}(n) - \sum_{i=1}^{k}(n - n/p_i)$.)

Example 2.1. It is easy to see that if $n = p^r$, where p is a prime number and r is any positive integer, then $\overline{SD}(n) = \sum_{i=1}^{r}(p^i - p^{i-1}) = p^r - 1$ and $SD(n) = \sum_{i=1}^{r-1}(p^i - p^{i-1}) = p^{r-1} - 1$, so that $SD(n) < n$. The graph $D(n)$ is shown in Figure 2.1.

FIGURE 2.1. The graph $D(p^r)$.

The following example shows that there are numbers n such that $SD(n) > n$, as well as numbers that satisfy the condition $SD(n) = n$.

EXAMPLE 2.2. Let $n = 24$ and $m = 12$. Then $SD(n) = (12-6)+(12-4)+(8-4)+(6-3)+(6-2)+(4-2)+(3-1)+(2-1) = 30 > n$ and $SD(m) = (6-3)+(6-2)+(4-2)+(3-1)+(2-1) = 12 = m$.

DEFINITION 2.3. A natural number n is called *graceful* if $SD(n) = n$.

In order to obtain the description of graceful numbers, we first find the value of $SD(n)$ when n is a product of powers of two different prime numbers.

EXAMPLE 2.4. Let $n = p^r q^s$ where p and q are different prime numbers, $r \geq 1$, and $s \geq 1$. In this case the graph $D(n)$ is of the form

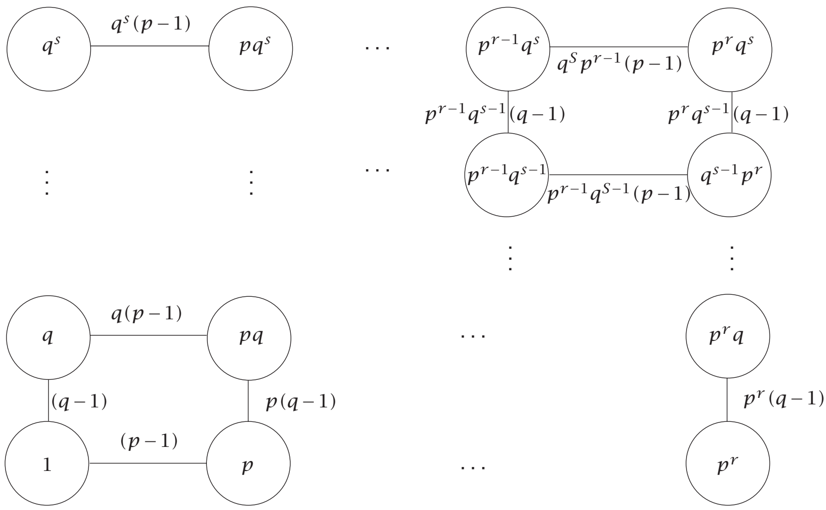

FIGURE 2.2. The graph $D(p^r q^s)$.

and $\overline{SD}(n) = \sum_{i=0}^{r}\sum_{j=1}^{s}(p^i q^j - p^i q^{j-1}) + \sum_{i=1}^{r}\sum_{j=0}^{s}(p^i q^j - p^{i-1} q^j) = \sum_{i=0}^{r} p^i (q^s - 1) + \sum_{j=0}^{s} q^j (p^r - 1)$ (the first sum corresponds to the differences of the consecutive divisors of n when the exponent of q decreases, and the second sum takes care about the differences of consecutive divisors of n when the exponent of p decreases). Thus,

$$\overline{SD}(n) = (q^s - 1)\sum_{i=0}^{r} p^i + (p^r - 1)\sum_{j=0}^{s} q^j = (q^s - 1)\frac{p^{r+1}-1}{p-1} + (p^r - 1)\frac{q^{s+1}-1}{q-1}, \quad (2.1)$$

so that

$$SD(n)=\overline{SD}(n)-\left[\left(n-\frac{n}{p}\right)+\left(n-\frac{n}{q}\right)\right]. \tag{2.2}$$

It follows from formulas (2.1) and (2.2) that a number $n=p^r q^s$ (p and q are prime, $r\ge 1$, and $s\ge 1$) is graceful if and only if $p=2$ and $s=1$, that is, $n=4q$ for some odd prime number q.

Indeed, equality $SD(n)=n$ can hold only for even numbers n (if n is odd, then (2.1) shows that $SD(n)$ is even, whence $SD(n)\ne n$). If $n=2^r q^s$, where $r\ge 2$, $s\ge 2$, then

$$\begin{aligned}SD(n)-n&=(2^r-1)\sum_{i=0}^{s}q^i+(q^s-1)(2^{r+1}-1)-2^{r+1}q^s+2^{r-1}q^s+2^rq^{s-1}-2^rq^s\\&>(2^{r-1}-2)q^s+(2^{r+1}q^{s-1}-q^{s-1}-2^{r+1})+(2^r-1)\sum_{i=0}^{s-2}q^i\\&>0,\end{aligned} \tag{2.3}$$

so that $SD(n)>n$. Finally, if $n=2^rq$ $(r\ge 1)$, then $SD(n)-n=(q-1)(2^{r+1}-1)+(2^r-1)(q+1)-2^{r+1}q+2^{r-1}q+2^r-2^rq=q(2^{r-1}-2)$, so that $SD(2^rq)=2^rq$ if and only if $r=2$. Thus, for any two different prime numbers p and q, $p<q$, and for any two nonnegative integers r and s, the number p^rq^s is graceful if and only if $p=2$, $r=2$, and $s=1$.

Now, we generalize formula (2.1) to the case of arbitrary number n. More precisely, we show that if $n=p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ is a prime decomposition of a positive integer n ($p_1,\dots,p_k$ are different primes and $r_1,\dots,r_k$ are positive integers), then

$$\overline{SD}(n)=\sum_{i=1}^{k}(p_i^{r_i}-1)\prod_{1\le j\le k,\,j\ne i}\left(\frac{p_j^{r_j+1}-1}{p_j-1}\right). \tag{2.4}$$

We proceed by induction on n. We have seen that the formula is true if n is a power of a prime number or a product of two powers of primes. In order to perform the step of induction, notice that

$$\overline{SD}(n)=\overline{SD}\left(\frac{n}{p_1}\right)+(p^{r_1}-p^{r_1-1})\sum_{i_2=0}^{r_2}\cdots\sum_{i_k=0}^{r_k}p_2^{i_2}\cdots p_k^{i_k}+p_1^{r_1}\overline{SD}\left(\frac{n}{p_1^{r_1}}\right). \tag{2.5}$$

Applying the inductive hypothesis and taking into account that

$$\overline{SD}(n)=\sum_{i_2=0}^{r_2}\cdots\sum_{i_k=0}^{r_k}p_2^{r_2}\cdots p_k^{r_k}=\prod_{j=2}^{k}\sum_{i=0}^{r_j}p_j^i=\prod_{j=2}^{k}\left(\frac{\left(p_j^{r_j+1}-1\right)}{(p_j-1)}\right), \tag{2.6}$$

we obtain that

$$\begin{aligned}\overline{SD}(n)=&\left(p_1^{r_1-1}-1\right)\prod_{j=2}^{k}\left(\frac{\left(p_j^{r_j+1}-1\right)}{(p_j-1)}\right)+\sum_{i=2}^{k}(p_i^{r_i}-1)\left(\frac{(p_1^{r_1}-1)}{(p_1-1)}\right)\prod_{2\le j\le k,j\ne i}\left(\frac{\left(p_j^{r_j+1}-1\right)}{(p_j-1)}\right)\\&+\left(p_1^{r_1}-p_1^{r_1-1}\right)\prod_{j=2}^{k}\left(\frac{\left(p_j^{r_j+1}-1\right)}{(p_j-1)}\right)+p_1^{r_1}\sum_{i=2}^{k}(p_i^{r_i}-1)\prod_{2\le j\le k,j\ne i}\left(\frac{\left(p_j^{r_j+1}-1\right)}{(p_j-1)}\right)\end{aligned}$$

$$= \left(p_1^{r_1-1}\right)\prod_{j=2}^{k}\left(\frac{\left(p_j^{r_j+1}-1\right)}{(p_j-1)}\right)+\sum_{i=2}^{k}(p_i^{r_i}-1)\prod_{1\le j\le k, j\ne i}\left(\frac{\left(p_j^{r_j+1}-1\right)}{(p_j-1)}\right)$$
$$= \sum_{i=1}^{k}(p_i^{r_i}-1)\prod_{1\le j\le k, j\ne i}\left(\frac{\left(p_j^{r_j+1}-1\right)}{(p_j-1)}\right), \tag{2.7}$$

so formula (2.4) is proved.

Now, formulas (2.2) and (2.4) imply that

$$SD(n) = \sum_{i=1}^{k}(p_i^{r_i}-1)\prod_{1\le j\le k, j\ne i}\left(\frac{p_j^{r_j+1}-1}{p_j-1}\right)-\sum_{i=1}^{k}\left(n-\frac{n}{p_i}\right). \tag{2.8}$$

Formula (2.8) shows, in particular, that if a number n is odd, then $SD(n)$ is even (it is easily seen that both sums in the right side of the formula are even if n is odd). Therefore, every graceful number must be even, that is,

$$n = 2^r q_1^{s_1}\cdots q_m^{s_m} \tag{2.9}$$

for some odd primes $q_1,\dots,q_m$ $(m \ge 1, s_i \ge 1$ for $i = 1,\dots,m)$. As we have seen, if $m = 1$, then the number n is graceful if and only if $s_1 = 1$ and $r = 2$, that is, $n = 4q_1$. We show that if $m \ge 2$, then $SD(n) > n$, so the only graceful numbers are the numbers of the form $4q$ where q is an odd prime.

First of all, notice that $SD(2^r q_1^{s_1}) \ge 2^r q_1^{s_1}$ for $r \ge 1$, $s \ge 2$ (see Example 2.4) and $SD(2q_1q_2) \ge 2q_1q_2$ for any two different primes q_1 and q_2 (applying formula (2.1) we obtain that $SD(2q_1q_2) = (q_1+1)(q_2+1)+3(q_1-1)(q_2+1)+3(q_2-1)(q_1+1)-6q_1q_2+q_1q_2+2q_1+2q_2 = 2q_1q_2+3(q_1+q_2)-5 > 2q_1q_2$). Therefore, in order to prove that $SD(n) > n$ for any number n of the form (2.9) with $m \ge 2$, it is sufficient to prove that $SD(n) > q_m^{s_m} SD(n/q_m^{s_m})$. But the last inequality is a consequence of equality (2.5). Indeed,

$$\begin{aligned} SD(n) &= \overline{SD}(n)-n = \overline{SD}\left(\frac{n}{q_m}\right)+q^{s_m}-q^{s_m-1}\sum_{i=0}^{r}\sum_{i_1=0}^{s_1}\cdots\sum_{i_{m-1}=0}^{s_{m-1}}2^i q_1^{i_1}\cdots q_{m-1}^{i_{m-1}} \\ &\quad + q_m^{s_m}\overline{SD}\left(\frac{n}{q_m^{s_m}}\right)-n > q_m^{s_m}\left(\overline{SD}\left(\frac{n}{q_m}\right)-\frac{n}{q_m}\right) = q_m^{s_m} SD\left(\frac{n}{q_m^{s_m}}\right). \end{aligned} \tag{2.10}$$

We arrive at the following result.

Theorem 2.5. *A natural number n is graceful if and only if $n = 4q$ where q is an odd prime.*

Recall that a positive integer m is called a *perfect number* if it is equal to the sum of all its proper divisors (i.e., of all divisors of m except of the number m itself). It is known (cf. [4, Theorem 5.10]) that every even perfect number is of the form $2^{k-1}(2^k-1)$, where the number 2^k-1 is prime. Thus, our theorem implies the following result.

Corollary 2.6. *The only perfect graceful number is* 28.

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