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ASYMPTOTIC EXPANSION OF SMALL ANALYTIC SOLUTIONS TO THE QUADRATIC NONLINEAR SCHRÖDINGER EQUATIONS IN TWO-DIMENSIONAL SPACES

NAKAO HAYASHI and PAVEL I. NAUMKIN

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We study asymptotic behavior in time of global small solutions to the quadratic nonlinear Schrödinger equation in two-dimensional spaces $i\partial_t u + (1/2)\Delta u = \mathcal{N}(u)$, $(t,x) \in \mathbb{R}\times\mathbb{R}^2$; $u(0,x) = \varphi(x)$, $x \in \mathbb{R}^2$, where $\mathcal{N}(u) = \sum_{j,k=1}^{2}(\lambda_{jk}(\partial_{x_j}u)(\partial_{x_k}u)+\mu_{jk}(\partial_{x_j}\overline{u})(\partial_{x_k}\overline{u}))$, where $\lambda_{jk},\mu_{jk} \in \mathbb{C}$. We prove that if the initial data φ satisfy some analyticity and smallness conditions in a suitable norm, then the solution of the above Cauchy problem has the asymptotic representation in the neighborhood of the scattering states.

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1. Introduction. We consider the large time asymptotic behavior of small analytic solutions to the Cauchy problem for the derivative nonlinear Schrödinger equation in two-dimensional spaces

$$\begin{aligned} i\partial_t u + \frac{1}{2}\Delta u &= \mathcal{N}(u), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^2, \\ u(0,x) &= \varphi(x), \quad x \in \mathbb{R}^2, \end{aligned} \tag{1.1}$$

with quadratic nonlinearity

$$\mathcal{N}(u) = \sum_{j,k=1}^{2}\Big(\lambda_{jk}\big(\partial_{x_j}u\big)\big(\partial_{x_k}u\big) + \mu_{jk}\big(\partial_{x_j}\overline{u}\big)\big(\partial_{x_k}\overline{u}\big)\Big), \tag{1.2}$$

where $\lambda_{jk},\mu_{jk} \in \mathbb{C}$. In [8], we proved the global in time existence of small analytic solutions to the Cauchy problem (1.1) and showed that the usual scattering states exist. In [3], a global existence theorem of small solutions to (1.1) with $\lambda_{jk} = 0$ was shown in the usual weighted Sobolev space by using the method of normal forms by Shatah [12]. In the present paper, we continue to study the asymptotic behavior in time of solutions to the Cauchy problem (1.1) and obtain the asymptotic expansion of solutions in the neighborhood of the scattering states.

We use the following classification of the scattering problem. If the usual scattering states exist in $\mathbf{L}^2$ sense, then we call the scattering problem a super-critical problem. If the usual scattering states do not exist and the $\mathbf{L}^2$ norm of the nonlinearity decays like $Ct^{-\delta}$, then we call the problem a critical one, when $\delta = 1$ and a sub-critical one, when $0 < \delta < 1$. The problem under consideration is classified as super-critical since the usual scattering states were shown, in [8], to exist in $\mathbf{L}^2$. In [10], the asymptotic expansion was obtained in the neighborhood of scattering states for small solutions to the nonlinear nonlocal Schrödinger equations with nonlinearities of Hartree type

$$\mathcal{N}(u) = u(t,x)\int dy|x-y|^{-\delta}|u(t,y)|^2 \tag{1.3}$$

in the super-critical case $1 < \delta < n$. The critical case $\delta = 1$ was treated in [13], where the asymptotic expansion of small solutions in the neighborhood of the modified scattering states was obtained. In the case of critical power nonlinearity $\mathcal{N}(u) = |u|^2u$ in one-dimensional spaces, the asymptotic expansion of solutions was constructed in [11]. In [5, 6], the sub-critical scattering problem in one-dimensional spaces was studied for the nonlinear Schrödinger equation with power nonlinearity $\mathcal{N}(u) = t^{1-\delta}|u|^2u$ and Hartree type nonlinearity (1.3) with $0 < \delta < 1$. Roughly speaking, they used the asymptotic expansion in the neighborhood of the final states to the transformed equations for the new dependent variable

$$w = \mathcal{F}\mathcal{U}(-t)u(t)\exp\left(i\int_1^t t^{-\delta}|\mathcal{F}\mathcal{U}(-t)u(t)|^2dt\right) \tag{1.4}$$

(in the case of the power type nonlinearity).

Thus the asymptotic expansions of solutions to the nonlinear Schrödinger equations were studied extensively in the case of the nonlinear terms without derivatives of unknown function and satisfying the gauge condition (i.e., having the self-conjugate property $\mathcal{N}(u) = e^{-i\theta}\mathcal{N}(e^{i\theta}u)$ for any $\theta \in \mathbb{R}$). The present paper is concerned with the derivative nonlinear Schrödinger equations which do not satisfy the gauge condition. The presence of derivatives in the nonlinear term implies the so-called derivative loss and the absence of the gauge condition makes it difficult to estimate the norm involving the operator $\mathcal{J} = x + it\nabla$, which plays a crucial role in the large time asymptotic behavior of solutions to the nonlinear Schrödinger equations. To overcome these obstacles, we use the analytic function spaces $\mathbf{A}_b^{m,p}$ defined in (1.9) and the operators $\mathcal{P} = x\cdot\nabla + 2t\partial_t$ and $\mathcal{Q} = x\cdot\nabla + it\Delta$.

To state our result precisely, we now give *notation and function spaces*. We denote $\partial_{x_j} = \partial/\partial_{x_j}$ and $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$, where $\alpha \in (\mathbb{N}\cup\{0\})^2$. We define the following differential operators $\mathcal{P} = x\cdot\nabla + 2t\partial_t$, $\mathcal{Q} = x\cdot\nabla + it\Delta$, $\mathcal{J} = x + it\nabla$ and the vector $\Omega = (\Omega^{(j,k)})_{(j,k=1,2)}$, where the operators $\Omega^{(j,k)} = x_j\partial_k - x_k\partial_j$ act as the angular derivatives. These operators help us to obtain the time decay properties of the linear Schrödinger evolution group

$$\mathcal{U}(t)\phi = \frac{1}{2\pi it}\int e^{(i/2t)(x-y)^2}\phi(y)dy = \mathcal{F}^{-1}e^{-(it/2)\xi^2}\mathcal{F}\phi, \tag{1.5}$$

where $\mathcal{F}\phi \equiv \hat{\phi}(\xi) = (1/2\pi)\int e^{-i(x\cdot\xi)}\phi(x)dx$ denotes the Fourier transform of the function $\phi(x)$, and $\mathcal{F}^{-1}$ is the inverse Fourier transformation defined by $\mathcal{F}^{-1}\phi \equiv \check{\phi}(x) = (1/2\pi)\int e^{i(x\cdot\xi)}\phi(\xi)d\xi$. Note that the free Schrödinger evolution group $\mathcal{U}(t)$

also can be represented as $\mathcal{U}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{F}\mathcal{M}(t)$, where $\mathcal{M}(t) = \exp(ix^2/2t)$, the dilation operator is $(\mathcal{D}(t)\phi)(x) = (i/t)\phi(x/t)$, then the inverse free Schrödinger evolution group is written as $\mathcal{U}(-t) = -\mathcal{M}(-t)i\mathcal{F}^{-1}\mathcal{D}(1/t)\mathcal{M}(-t)$, where $\mathcal{D}^{-1}(t) = -i\mathcal{D}(1/t)$ is the inverse dilation operator. We define the extended vectors $\Gamma = (\mathcal{P},\Omega,\nabla)$, $\widetilde{\Gamma} = (\mathcal{P}+2,\Omega,\nabla)$, and $\Theta = (\mathcal{Q},\Omega,\nabla)$. We have the following relations:

$$\mathcal{Q} = \mathcal{P} - 2it\mathcal{L} = \mathcal{J}\cdot\nabla = \mathcal{U}(t)x\mathcal{U}(-t)\cdot\nabla = it\mathcal{M}(t)\nabla\overline{\mathcal{M}(t)}\cdot\nabla, \tag{1.6}$$

where $\mathcal{M}(t) = e^{ix^2/2t}$, $\mathcal{L} = i\partial_t + (1/2)\Delta$. The commutation relations

$$\begin{gathered}
[\mathcal{Q},\nabla] = [\mathcal{P},\nabla] = -\nabla, \qquad [\mathcal{Q},\mathcal{J}] = [\mathcal{P},\mathcal{J}] = \mathcal{J}, \\
[\mathcal{P},\mathcal{Q}] = [\Omega,\mathcal{P}] = [\Omega,\mathcal{Q}] = 0, \qquad [\partial_k,\mathcal{J}_l] = \delta_l^{(k)}, \\
\left[\Omega_x^{(j,k)},\partial_l\right] = \delta_l^{(k)}\partial_j - \delta_l^{(j)}\partial_k,
\end{gathered} \tag{1.7}$$

where $\delta_j^{(k)} = 1$ if $j = k$ and $\delta_j^{(k)} = 0$ if $j \neq k$ are used freely in the paper. We denote the usual Lebesgue space by $\mathbf{L}^p(\mathbb{R}^2)$ with the norm $\|\phi\|_p = (\int_{\mathbb{R}^2}|\phi(x)|^p dx)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_\infty = \operatorname{ess\,sup}\{|\phi(x)|; x \in \mathbb{R}^2\}$ if $p = \infty$. For simplicity we write $\|\cdot\| = \|\cdot\|_2$. The weighted Sobolev space is defined by

$$\mathbf{H}_p^{m,k}(\mathbb{R}^2) = \{\phi \in \mathbf{L}^p(\mathbb{R}^2) : \|\langle x\rangle^k\langle i\nabla\rangle^m\phi\|_p < \infty\}, \tag{1.8}$$

where $m,k \in \mathbb{R}^+$, $1 \le p \le \infty$, $\langle x\rangle = \sqrt{1+x^2}$. We write for simplicity $\mathbf{H}^{m,k}(\mathbb{R}^2) = \mathbf{H}_2^{m,k}(\mathbb{R}^2)$ and the norm $\|\phi\|_{m,k} = \|\phi\|_{m,k,2}$. Now we define the analytic function space

$$\mathbf{A}_b^m = \left\{\phi \in \mathbf{L}^2(\mathbb{R}^2);\ \|\phi\|_{\mathbf{A}_b^m} = \sum_{|\beta|\le m}\sum_{\alpha}\frac{b^{|\alpha|}}{\alpha!}\|\Gamma^{\alpha+\beta}\phi\| < \infty\right\}, \tag{1.9}$$

where the vector $\Gamma = \Gamma(t) = (\mathcal{P},\Omega,\nabla)$, $b = b(t) = b_0 + (a-b_0)(\log(e+t))^{-\gamma}$, $0 < b_0 < a < 1$, $\gamma > 0$ is sufficiently small. Similarly, we write

$$\widetilde{\mathbf{A}}_b^m = \left\{\phi \in \mathbf{L}^2(\mathbb{R}^2);\ \|\phi\|_{\widetilde{\mathbf{A}}_b^m} = \sum_{|\beta|\le m}\sum_{\alpha}\frac{b^{|\alpha|}}{\alpha!}\|\widetilde{\Gamma}^{\alpha+\beta}\phi\| < \infty\right\}. \tag{1.10}$$

Here the summation is over all admissible multi-indices α. We often use the summations convention if it does not cause confusion. By $[s]$ we denote the largest integer less than or equal to s. Let $\mathbf{C}(\mathbf{I};\mathbf{B})$ be the space of continuous functions from a time interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. We denote different positive constants by the same letter C. We introduce the following functional spaces

$$\begin{aligned}
\mathbf{X}_b &= \{u \in \mathbf{C}(\mathbb{R};\mathbf{L}^2(\mathbb{R}^2));\ \|u\|_{\mathbf{X}_b} < \infty\}, \\
\mathbf{Y}_b &= \left\{u \in \mathbf{C}(\mathbb{R};\mathbf{L}^2(\mathbb{R}^2));\ \sup_{t>0}\|u(t)\|_{\mathbf{Y}_b} < \infty\right\},
\end{aligned} \tag{1.11}$$

where

$$\begin{aligned}\|u\|_{\mathbf{X}_b} &= \sup_{t>0}\|u(t)\|_{\mathbf{A}_b^3} + \sup_{t>0} t^{-1-\eta}\sum_{|\gamma|\le 1}\|\mathcal{J}^{\gamma}u(t)\|_{\mathbf{A}_b^2}\\ &\quad+\sum_{|\gamma|=1}\int_0^\infty \|\Theta^{\gamma}u\|_{\mathbf{A}_b^3}|b'|\,dt + \sum_{|\gamma|=1,|\sigma|\le 1}\int_1^\infty \|\Theta^{\gamma}\mathcal{J}^{\sigma}u\|_{\mathbf{A}_b^3}\frac{|b'|\,dt}{t^{1+\eta}}\\ &\quad+\sup_{t>0} t^{1-2\eta}\sum_{\alpha}\frac{b^{|\alpha|}}{\alpha!}\|\partial_t\mathcal{F}\mathcal{U}(-t)\Gamma^{\alpha}\nabla u(t)\|_\infty\\ &\quad+\sum_{|\delta|\le 3}\int_1^\infty\sum_{\alpha}\frac{b^{|\alpha|}}{\alpha!}\|\partial_t\mathcal{F}\mathcal{U}(-t)\Gamma^{\alpha+\delta}u(t)\|t^{2\eta-1/2}\,dt,\\ \|u(t)\|_{\mathbf{Y}_b} &= \|u(t)\|_{\mathbf{A}_b^2} + \sum_{|\beta|+|\gamma|\le 1} t^{-|\beta|-|\gamma|-\eta}\|\mathcal{J}^{\gamma}\Theta^{\beta}u(t)\|_{\mathbf{A}_b^1}\\ &\quad+t^{1-\eta}\sum_{|\gamma|+|\delta|\le 1}\sum_{\alpha}\frac{b^{|\alpha|}}{\alpha!}\|\partial_t\mathcal{F}\mathcal{U}(-t)\Gamma^{\alpha+\gamma}\Theta^{\delta}u(t)\|,\end{aligned} \tag{1.12}$$

where $\eta > 0$ is sufficiently small. We define the constants $\{b_n\}$ such that

$$0 < b_n < b_{n-1} < \cdots < b_1 < b_0 < a < 1. \tag{1.13}$$

Let $u_0(t) = \mathcal{U}(t)u^+$ with some final state $u^+ \in \mathbf{L}^2$ and $u_n(t)$, $n = 1,2,\ldots$, be the solution to the final problem for the linear Schrödinger equations

$$\mathcal{L}u_n = \sum_{m=0}^{n-1}\mathcal{N}(u_{n-1-m}, u_m), \tag{1.14}$$

such that $\lim_{t\to\infty} u_n(t) = 0$ in $\mathbf{L}^2$, where $\mathcal{L} = i\partial_t + (1/2)\Delta$ and

$$\mathcal{N}(\phi,\psi) = \sum_{j,k=1}^{2}\left(\lambda_{jk}(\partial_{x_j}\phi)(\partial_{x_k}\psi) + \mu_{jk}(\partial_{x_j}\bar{\phi})(\partial_{x_k}\bar{\psi})\right). \tag{1.15}$$

From [8] we see that if the initial data $\varphi \in \mathbf{A}_a^3$ are such that $x_j\varphi \in \mathbf{A}_a^2$ for $j = 1,2$ and the norm $\|\varphi\|_{\mathbf{A}_a^3} + \|x_1\varphi\|_{\mathbf{A}_a^2} + \|x_2\varphi\|_{\mathbf{A}_a^2} = \varepsilon$ is sufficiently small, then the final state $u^+ \in \mathbf{A}_{a_1}^2$, where $b_0 < a_1 < a$, hence $u_0 \in \mathbf{Y}_{b_0}$ and $\|u - u_0\|_{\mathbf{Y}_{b_0}} \le C\varepsilon^2 t^{-w}$ for all $t \ge 1$, where $w \in (0,1/2)$.

Now we state the main result in this paper.

Theorem 1.1. *We assume that the initial data $\varphi \in \mathbf{A}_a^3$ are such that $x_j\varphi \in \mathbf{A}_a^2$ for $j = 1,2$ and the norm $\|\varphi\|_{\mathbf{A}_a^3} + \|x_1\varphi\|_{\mathbf{A}_a^2} + \|x_2\varphi\|_{\mathbf{A}_a^2} = \varepsilon$ is sufficiently small. Then there exists a unique global solution $u(t,x) \in \mathbf{A}_{b(t)}^3$ of the Cauchy problem (1.1). Moreover, the estimates*

$$\|u_n(t)\|_{\mathbf{Y}_{b_n}} \le C_n\varepsilon^{n+1}t^{-nw}, \quad n = 0,1,2,\ldots \tag{1.16}$$

and the asymptotics

$$\left\|u(t) - \sum_{m=0}^{n-1}u_m(t)\right\|_{\mathbf{Y}_{b_n}} \le C_n\varepsilon^{n+1}t^{-nw}, \quad n = 1,2,\ldots \tag{1.17}$$

are valid for all $t \geq 1$*, where* $w \in (0, 1/2)$ *and*

$$C_n = C(n+1)^{2n} \left(\prod_{j=0}^{n} \left(\log \frac{b_j}{b_{j+1}} \right)^{-1} \right)^{2n}, \tag{1.18}$$

where C *is a positive constant independent of* n *and* b_j*.*

We assume in Theorem 1.1 that $0 < a < 1$. This ensures that the function space $\mathbf{A}_a^3$ for the initial data is not empty, as in [1, 2], we can see that our result is valid for the initial function ϕ, which has analytic continuation Φ to the domain

$$\prod = \{z \in \mathbb{C}^2;\ z_j = x_j + iy_j,\ x_j \in \mathbb{R},\ -C_1 - |x_j| \tan\vartheta < y_j < C_1 + |x_j| \tan\vartheta,\ j = 1, 2\}, \tag{1.19}$$

such that

$$\int\int_{\prod} |\Phi(z)|^2 dx\, dy < \infty, \tag{1.20}$$

where $\vartheta \in (0, \pi/2)$, $\sin\vartheta = C_2$, and $C_1, C_2 \in (a, 1)$. For example, we can take $1/(1+x^4)$, e^{-x^2} as the initial data for the Cauchy problem (1.1).

Denote $\widehat{u_0^+}(t,\xi) = \widehat{u^+}(\xi)$ and

$$\begin{aligned} \widehat{u_n^+}(t,\xi) = &-\frac{1}{4} \sum_{m=0}^{n-1} \widehat{u_{n-1-m}^+}\left(t, \frac{\xi}{2}\right) \widehat{u_m^+}\left(t, \frac{\xi}{2}\right) \sum_{j,k=1}^{2} \lambda_{jk} \xi_j \xi_k \int_\infty^t e^{it\xi^2/4} \frac{d\tau}{i\tau} \\ &-\frac{1}{4} \sum_{m=0}^{n-1} \overline{\widehat{u_{n-1-m}^+}}\left(t, -\frac{\xi}{2}\right) \overline{\widehat{u_m^+}}\left(t, -\frac{\xi}{2}\right) \sum_{j,k=1}^{2} \mu_{jk} \xi_j \xi_k \int_\infty^t e^{3it\xi^2/4} \frac{d\tau}{i\tau}. \end{aligned} \tag{1.21}$$

Corollary 1.2. *Let the conditions of Theorem 1.1 be fulfilled. Then the following asymptotics in* $\mathbf{L}^2$ *sense*

$$\mathcal{F}\mathcal{U}(-t)u(t) = \widehat{u^+}(\xi) + \sum_{j=1}^{n-1} \widehat{u_j^+}(t,\xi) + O(t^{-nw}) \tag{1.22}$$

are valid for large time $t \geq 1$*, where* $n = 1, 2, \ldots$*.*

For the convenience of the reader we now give the outline of the proof of Theorem 1.1. As in [8] we apply the operator $\mathcal{F}\mathcal{U}(-t)$ to (1.1) to get

$$i\partial_t \mathcal{F}\mathcal{U}(-t)u = I(t,\xi) + R(t,\xi), \tag{1.23}$$

where

$$\begin{aligned} I(t,\xi) = \frac{1}{it} \sum_{j,k=1}^{2} &\left(\lambda_{jk} \mathcal{D}_2 E^2 (\mathcal{F}\mathcal{U}(-t)\partial_{x_j} u)(\mathcal{F}\mathcal{U}(-t)\partial_{x_k} u)\right. \\ &\left. + \mu_{jk} \mathcal{D}_{-2} E^6 (\overline{\mathcal{F}\mathcal{U}(-t)\partial_{x_j} u})(\overline{\mathcal{F}\mathcal{U}(-t)\partial_{x_k} u})\right), \end{aligned} \tag{1.24}$$

$E = e^{it\xi^2/2}$, and R is a remainder term since in [8] we proved the estimate $\|R\| \le Ct^{-1-w}\sum_{|\alpha|\le 1}\|\Theta^\alpha u\|^2$, $\Theta = (Q,\Omega,\nabla)$, $0 < w < 1/2$. Then we show that the first term of the integral $\int I(t,\xi)dt$ is also convergent in view of the oscillating factor E. Roughly speaking, in [8] the following estimate was shown:

$$||u(t)|| \le ||u(1)|| + C\int_1^t \tau^{-1-w}\sum_{|\alpha|\le 1}||\Theta^\alpha u||^2 d\tau. \tag{1.25}$$

Similarly, we have

$$\sum_{|\beta|\le 1}||\Gamma^\beta u(t)|| \le \sum_{|\beta|\le 1}||(\Gamma^\beta u)(1)|| + C\int_1^t \tau^{-1-w}\sum_{|\alpha|\le 1,|\beta|\le 1}||\Theta^\alpha\Gamma^\beta u||^2 d\tau. \tag{1.26}$$

However, the right-hand sides of (1.25) and (1.26) contain an additional operator Θ (derivative loss with respect to derivative Θ). This is the reason why we used the analytic function spaces involving generalized derivative Γ, enabling us to get an additional regularity with respect to operator Γ, hence we obtain the estimate

$$\sum_{|\beta|\le 1}\frac{b(t)^{|\beta|}}{\beta!}||\Gamma^\beta u(t)|| < C\varepsilon. \tag{1.27}$$

Similarly, we have

$$\sum_{|\beta|}\le 1\frac{b_1^{|\beta|}}{\beta!}||\Gamma^\beta u(t) - \Gamma^\beta u(s)|| < C\varepsilon^2|t|^{-w} \tag{1.28}$$

for all $t > s > 0$. The last estimate implies existence of the usual scattering states u^+. Method of analytic function spaces involving usual derivatives was used by many authors (e.g., see [4, 9]) and analytic function spaces involving the generalized derivatives was used in [7]. By the definition of $u_n(t)$ we have with $\mathcal{L} = i\partial_t + (1/2)\Delta$

$$\begin{aligned}\mathcal{L}\left(u - \sum_{m=0}^{n-1}u_m\right) &= \mathcal{N}\left(u - \sum_{m=0}^{n-1}u_m, \sum_{m=0}^{n-1}u_m\right) + \mathcal{N}\left(\sum_{m=0}^{n-1}u_m, u - \sum_{m=0}^{n-1}u_m\right)\\ &\quad + \mathcal{N}\left(u - \sum_{m=0}^{n-1}u_m, u - \sum_{m=0}^{n-1}u_m\right) + R,\end{aligned} \tag{1.29}$$

where R is the remainder term since $\|R\|_{Y_{b_{n+1}}} \le C\varepsilon^{n+2}|t|^{-1-(n+1)w}$. In Section 3, we will prove that the other three terms are estimated by $C\varepsilon^{n+1}|t|^{-1-nw}$ in the norm $Y_{b_{n+1}}$. Then via the inequality

$$||\Gamma^\alpha u||_{Y_{b_{n+1}}} \le C\left(\prod_{j=1}^{n}\left(\log\frac{b_j}{b_{j+1}}\right)^{-1}\right)^{|\alpha|}\|u\|_{Y_{b_n}}, \tag{1.30}$$

for any $|\alpha|$ we obtain the desired result.

The rest of the paper is organized as follows. In Section 2, we state some preliminary estimates concerning the analytic functional spaces $\mathbf{A}_b^m$. Section 3 is devoted to the proof of Theorem 1.1 and Corollary 1.2.

2. Preliminary estimates. We summarize some lemmas proved in [7, 8], which are necessary to prove the theorem.

Lemma 2.1. *Let $\phi \in \mathbf{A}_b^{m,p}$, then*

$$\|\phi\|_{\tilde{\mathbf{A}}_b^{m,p}} \leq e^{2b}\|\phi\|_{\mathbf{A}_b^{m,p}}, \tag{2.1}$$

where

$$\begin{aligned}\mathbf{A}_b^{m,p} &= \left\{\phi \in \mathbf{L}^p(\mathbb{R}^2);\ \|\phi\|_{\mathbf{A}_b^{m,p}} = \sum_{|\beta|\leq m}\sum_{\alpha}\frac{b^{|\alpha|}}{\alpha!}\left\|\Gamma^{\alpha+\beta}\phi\right\|_p < \infty\right\},\\ \tilde{\mathbf{A}}_b^{m,p} &= \left\{\phi \in \mathbf{L}^p(\mathbb{R}^2);\|\phi\|_{\tilde{\mathbf{A}}_b^{m,p}} = \sum_{|\beta|\leq m}\sum_{\alpha}\frac{b^{|\alpha|}}{\alpha!}\left\|\tilde{\Gamma}^{\alpha+\beta}\phi\right\|_p < \infty\right\},\end{aligned} \tag{2.2}$$

and $2 \leq p \leq \infty$.

Lemma 2.2. *The following commutation relations are valid:*

$$\begin{aligned}&\left[\partial_{x_j}^l, \mathcal{J}_{x_j}\right] = l\partial_{x_j}^{l-1}, \qquad \left[\mathcal{J}_j^l, \partial_{x_j}\right] = -l\mathcal{J}_{x_j}^{l-1},\\ &\mathcal{P}^l\mathcal{J}_{x_j} = \sum_{0\leq m\leq l} C_l^m \mathcal{J}_{x_j}\mathcal{P}^{l-m}, \qquad \mathcal{P}^l\partial_{x_j} = \sum_{0\leq m\leq l} C_l^m(-1)^m\partial_{x_j}\mathcal{P}^{l-m}\\ &\mathcal{J}_{x_j}\mathcal{P}^l = \sum_{0\leq m\leq l} C_l^m(-1)^m\mathcal{P}^{l-m}\mathcal{J}_{x_j}, \qquad \partial_{x_j}\mathcal{P}^l = \sum_{0\leq m\leq l} C_l^m\mathcal{P}^{l-m}\partial_{x_j}\\ &\Omega_{x_jx_k}^l\partial_{x_j} = \sum_{0\leq 2m\leq l}(-1)^m C_l^{2m}\partial_{x_j}\Omega_{x_jx_k}^{l-2m} + \sum_{0\leq 2m+1\leq l}(-1)^{m+1}C_l^{2m+1}\partial_{x_k}\Omega_{x_jx_k}^{l-2m-1},\\ &\partial_{x_j}\Omega_{x_jx_k}^l = \sum_{0\leq 2m\leq l}(-1)^{m+1}C_l^{2m}\Omega_{x_jx_k}^{l-2m}\partial_{x_j} + \sum_{0\leq 2m+1\leq l}(-1)^m C_l^{2m+1}\Omega_{x_jx_k}^{l-2m-1}\partial_{x_k},\end{aligned} \tag{2.3}$$

where $C_l^m = l!/(l-m)!m!$ is the binomial coefficient.

Lemma 2.3. *The estimate*

$$\|\mathcal{J}\nabla\phi\|_{\mathbf{A}_b^{m,p}} \leq C\sum_{|\alpha|=1}\left\|\Theta^\alpha\phi\right\|_{\mathbf{A}_b^{m,p}} \tag{2.4}$$

is true.

Lemma 2.4. *The inequalities*

$$\begin{aligned}C_1\left\|\partial_{x_j}\phi\right\|_{\mathbf{A}_b^{m,p}} &\leq \sum_{|\beta|\leq m}\sum_{\alpha}\frac{b^{|\alpha|}}{\alpha!}\left\|\partial_{x_j}\Gamma^{\alpha+\beta}\phi\right\|_p \leq C_2\left\|\partial_{x_j}\phi\right\|_{\mathbf{A}_b^{m,p}},\\ C_1\left\|\mathcal{J}_{x_j}\phi\right\|_{\mathbf{A}_b^{m,p}} &\leq \sum_{|\beta|\leq m}\sum\frac{b^{|\alpha|}}{\alpha!}\left\|\mathcal{J}_{x_j}\Gamma^{\alpha+\beta}\phi\right\|_p + \|\phi\|_{\mathbf{A}_b^{m,p}}\\ &\leq C_2\left(\left\|\mathcal{J}_{x_j}\phi\right\|_{\mathbf{A}_b^{m,p}} + \|\phi\|_{\mathbf{A}_b^{m,p}}\right)\end{aligned} \tag{2.5}$$

are true for all $t > 0$, where $C_1, C_2 > 0$.

We define the evolution operator

$$\mathcal{V}(t)\phi=\mathcal{F}^{-1}e^{i\xi^2/2t}\mathcal{F}\phi=\frac{t}{2i\pi}\int e^{(-it/2)(\xi-y)^2}\phi(y)\,dy \tag{2.6}$$

and $\mathcal{K}=\mathcal{F}\mathcal{M}\mathcal{U}(-t)$. By a direct calculation we see that

$$\mathcal{V}(-t)\left(E^{\nu-1}\phi\right)=\mathcal{D}_\nu E^{\nu(\nu-1)}\mathcal{V}(-\nu t)\phi, \tag{2.7}$$

with $\mathcal{D}_\nu\phi=(1/\nu)\phi(\xi/\nu)$ and $E=e^{it\xi^2/2}$, where $\nu\neq 0$. We need the following lemma to get the decay estimates of the solution for large time.

Lemma 2.5. *The estimate*

$$\begin{aligned}&\|\mathcal{D}_\nu E^{\nu(\nu-1)}(\mathcal{V}(-\nu t)-1)(\mathcal{K}\phi)(\mathcal{K}\psi)\|\\&\quad+\|\mathcal{D}_\nu E^{\nu(\nu-1)}(\mathcal{K}\psi)(\mathcal{V}(-\nu t)-1)(\mathcal{K}\phi)\|\\&\le Ct^{\eta-1/2}\sum_{|\alpha|\le 1,|\beta|\le 1}\|\mathcal{J}^\alpha\phi\|\|\mathcal{J}^\beta\psi\|\end{aligned} \tag{2.8}$$

is valid for all $t>0$, *where* $\nu\neq 0$, $\eta>0$ *is sufficiently small.*

3. Proof of Theorem 1.1. We consider the linear Schrödinger equation

$$\mathcal{L}u_n=\sum_{m=0}^{n-1}\mathcal{N}(u_{n-1-m},u_m). \tag{3.1}$$

Since $u_0(t)$ is a solution of linear Schrödinger equation it is easy to see that $\|u_0(t)\|_{Y_{b_0}}\le C_0\varepsilon$. Then by induction we assume that

$$\|u_j(t)\|_{Y_{b_j}}\le C_j\varepsilon^{j+1}|t|^{-jw},\quad 0\le j\le n-1. \tag{3.2}$$

Multiplying both sides of (3.1) by $\mathcal{K}\Gamma^{\alpha+\delta}$, where $\mathcal{K}=\mathcal{F}\mathcal{M}\mathcal{U}(-t)$, we get

$$\begin{aligned}&\mathcal{L}_\xi\mathcal{K}\Gamma^{\alpha+\delta}u_n\\&=\frac{1}{it}\sum_{m=0}^{n-1}\sum_{\beta\le\alpha}\sum_{\gamma\le\delta}C_\alpha^\beta C_\delta^\gamma\sum_{j,k=1}^{2}\Big(\lambda_{jk}E(\mathcal{K}\tilde\Gamma^{\delta-\gamma}f)\mathcal{K}\Gamma^\gamma g+\mu_{jk}\bar E^3\Big(\overline{\mathcal{K}\tilde\Gamma^{\delta-\gamma}f}\Big)\overline{\mathcal{K}\Gamma^\gamma g}\Big),\end{aligned} \tag{3.3}$$

where $\mathcal{L}_\xi=i\partial_t+(1/2t^2)\Delta_\xi$, $f=\tilde\Gamma^{\alpha-\beta}\partial_{x_j}u_{n-1-m}$, $g=\Gamma^\beta\partial_{x_k}u_m$, $C_\alpha^\beta=\alpha!/(\alpha-\beta)!\beta!$, $|\delta|\le 2$. Applying the operator $\mathcal{V}(-t)=\overline{\mathcal{F}^{-1}\mathcal{M}(t)\mathcal{F}}$ to both sides of (3.3) we obtain by virtue of identity (2.7)

$$\begin{aligned}&i\partial_t\mathcal{V}(-t)\mathcal{K}\Gamma^{\alpha+\delta}u_n(t)\\&=\frac{1}{it}\sum_{m=0}^{n-1}\sum_{\gamma\le\delta}\sum_{\beta\le\alpha}C_\delta^\gamma C_\alpha^\beta\sum_{j,k=1}^{2}\mathcal{V}(-t)\Big(\lambda_{jk}E(\mathcal{K}\tilde\Gamma^{\delta-\gamma}f)\mathcal{K}\Gamma^\gamma g+\mu_{jk}\bar E^3\Big(\overline{\mathcal{K}\tilde\Gamma^{\delta-\gamma}f}\Big)\overline{\mathcal{K}\Gamma^\gamma g}\Big)\end{aligned}$$

$$= \frac{1}{it}\sum_{m=0}^{n-1}\sum_{\gamma\le\delta}\sum_{\beta\le\alpha} C_\delta^\gamma C_\alpha^\beta \sum_{j,k=1}^{2}\Big(\lambda_{jk}\mathcal{D}_2 E^2 \mathcal{V}(-2t)\big(\mathcal{K}\tilde{\Gamma}^{\delta-\gamma} f\big)\mathcal{K}\Gamma^\gamma g + \mu_{jk}\mathcal{D}_{-2}E^6\mathcal{V}(2t)\Big(\overline{\mathcal{K}\tilde{\Gamma}^{\delta-\gamma} f}\Big)\overline{\mathcal{K}\Gamma^\gamma g}\Big). \tag{3.4}$$

Then we write the identity

$$\begin{aligned}
&\mathcal{V}(-2t)\big(\mathcal{K}\tilde{\Gamma}^{\delta-\gamma} f\big)\mathcal{K}\Gamma^\gamma g \\
&\quad = \big(\mathcal{V}(-2t)\mathcal{K}\tilde{\Gamma}^{\delta-\gamma} f\big)\mathcal{V}(-2t)\mathcal{K}\Gamma^\gamma g - \big(\mathcal{K}\Gamma^\gamma g\big)\big(\mathcal{V}(-2t)-1\big)\big(\mathcal{K}\tilde{\Gamma}^{\delta-\gamma} f\big) \\
&\qquad - \big(\mathcal{V}(-2t)\mathcal{K}\tilde{\Gamma}^{\delta-\gamma} f\big)\big(\mathcal{V}(-2t)-1\big)\mathcal{K}\Gamma^\gamma g + \big(\mathcal{V}(-2t)-1\big)\big(\mathcal{K}\tilde{\Gamma}^{\delta-\gamma} f\big)\mathcal{K}\Gamma^\gamma g.
\end{aligned}\tag{3.5}$$

By Lemma 2.5, we have the estimate

$$\begin{aligned}
&\big\|\mathcal{D}_2 E^2\big(\mathcal{V}(-2t)-1\big)\big(\mathcal{K}\tilde{\Gamma}^{\delta-\gamma} f\big)\mathcal{K}\Gamma^\gamma g\big\| \\
&\qquad + \big\|\mathcal{D}_2 E^2\big(\mathcal{K}\Gamma^\gamma g\big)\big(\mathcal{V}(-2t)-1\big)\mathcal{K}\tilde{\Gamma}^{\delta-\gamma} f\big\| \\
&\quad \le C|t|^{\eta-1/2}\Bigg(\sum_{|\sigma|\le 1}\Big\|\mathcal{J}^\sigma\tilde{\Gamma}^{\sigma-\gamma} f\Big\|\Bigg)\Bigg(\sum_{|\sigma|\le 1}\Big\|\mathcal{J}^\sigma\Gamma^\gamma g\Big\|\Bigg).
\end{aligned}\tag{3.6}$$

Thus we can rewrite (3.4) in the form

$$\begin{aligned}
&i\partial_t\mathcal{V}(-t)\mathcal{K}\Gamma^{\alpha+\delta}u_n(t) \\
&\quad = \frac{1}{it}\sum_{m=0}^{n-1}\sum_{\gamma\le\delta}\sum_{\beta\le\alpha} C_\delta^\gamma C_\alpha^\beta \sum_{j,k=1}^{2}\Big(\lambda_{jk}\mathcal{D}_2 E^2\big(\mathcal{V}(-t)\mathcal{K}\tilde{\Gamma}^{\delta-\gamma} f\big)\mathcal{V}(-t)\mathcal{K}\Gamma^\gamma g \\
&\qquad + \mu_{jk}\mathcal{D}_{-2}E^6\Big(\overline{\mathcal{V}(-t)\mathcal{K}\tilde{\Gamma}^{\delta-\gamma} f}\Big)\overline{\mathcal{V}(-t)\mathcal{K}\Gamma^\gamma g}\Big) + R_1(t),
\end{aligned}\tag{3.7}$$

where the remainder term $R_1(t)$ can be estimated by virtue of (3.6) and Lemmas 2.1, 2.2, 2.3, and 2.4 as follows:

$$\begin{aligned}
&\sum_\alpha \frac{(b_n)^{|\alpha|}}{\alpha!}\big\|R_1(t)\big\| \\
&\quad \le C|t|^{\eta-3/2}\sum_{m=0}^{n-1}\sum_{\gamma\le\delta}\sum_{\alpha}\sum_{\beta\le\alpha} C_\delta^\gamma C_\alpha^\beta\frac{(b_n)^{|\alpha|}}{\alpha!}\Bigg(\sum_{|\sigma|\le 1}\big\|\mathcal{J}^\sigma\tilde{\Gamma}^{\delta-\gamma} f\big\|\Bigg)\sum_{|\sigma|\le 1}\big\|\mathcal{J}^\sigma\Gamma^\gamma g\big\| \\
&\quad \le C|t|^{\eta-3/2}\sum_{m=0}^{n-1}\sum_{|\sigma|\le 1}\big\|\Theta^\sigma u_{n-1-m}(t)\big\|_{\mathbf{A}^2_{b_n}}\sum_{|\sigma|\le 1}\big\|\Theta^\sigma u_m(t)\big\|_{\mathbf{A}^2_{b_n}} \\
&\quad \le C|t|^{\eta-3/2}\Bigg(\prod_{j=0}^{n}\Big(\log\frac{b_j}{b_{j+1}}\Big)^{-1}\Bigg)^2\sum_{m=0}^{n-1}\big\|u_{n-1-m}(t)\big\|_{\mathbf{A}^2_{b_{n-1-m}}}\big\|u_m(t)\big\|_{\mathbf{A}^2_{b_m}} \\
&\quad \le C\Bigg(\prod_{j=0}^{n}\Big(\log\frac{b_j}{b_{j+1}}\Big)^{-1}\Bigg)^2\Bigg(\sum_{m=0}^{n-1} C_{n-1-m}C_m\Bigg)\varepsilon^{n+1}|t|^{\eta-3/2-(n-1)w}.
\end{aligned}\tag{3.8}$$

Since

$$\begin{aligned}\sum_{m=0}^{n-1} C_{n-1-m}C_m &\le \sum_{m=0}^{n-1}(n-m)^{2(n-1-m)}\left(\prod_{j=0}^{n-1-m}\left(\log\frac{b_j}{b_{j+1}}\right)^{-1}\right)^{2(n-1-m)} \\ &\quad\times(m+1)^{2m}\left(\prod_{j=0}^{m}\left(\log\frac{b_j}{b_{j+1}}\right)^{-1}\right)^{2m} \\ &\le\left(\prod_{j=0}^{n-1}\left(\log\frac{b_j}{b_{j+1}}\right)^{-1}\right)^{2(n-1)}(n+1)^{2n},\end{aligned}\tag{3.9}$$

we have

$$\begin{aligned}\sum_{\alpha}\frac{(b_n)^{|\alpha|}}{\alpha!}\|R_1(t)\| &\le C\left(\prod_{j=0}^{n}\left(\log\frac{b_j}{b_{j+1}}\right)^{-1}\right)^{2n}(n+1)^{2n}\varepsilon^{n+1}|t|^{\eta-3/2-(n-1)w} \\ &\le C_n\varepsilon^{n+1}|t|^{\eta-3/2-(n-1)w}.\end{aligned}\tag{3.10}$$

By virtue of the identity $\mathcal{V}(-t)\mathcal{K}=\overline{\mathcal{F}^{-1}\mathcal{M}\mathcal{F}}\mathcal{F}\mathcal{M}\mathcal{U}(-t)=\mathcal{F}\mathcal{U}(-t)$, we have $i\xi_j\mathcal{V}(-t)\mathcal{K}=\mathcal{V}(-t)\mathcal{K}\partial_{x_j}$. Hence by (3.7) we get

$$\begin{aligned}&i\partial_t\mathcal{F}\mathcal{U}(-t)\Gamma^{\alpha+\delta}u_n(t) \\ &\quad=\frac{1}{t}\sum_{m=0}^{n-1}\sum_{\gamma\le\delta}\sum_{\beta\le\alpha}C_\delta^\gamma C_\alpha^\beta\sum_{j,k=1}^{2}\Big(\lambda_{jk}\mathcal{D}_2E^2\xi_j\big(\mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1}\widetilde{\Gamma}^{\delta-\gamma}f\big)\mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g \\ &\qquad-\mu_{jk}D_{-2}E^6\xi_j\Big(\overline{\mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1}\widetilde{\Gamma}^{\delta-\gamma}f}\Big)\overline{\mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g}\Big)+R_1(t).\end{aligned}\tag{3.11}$$

If $|\delta-\gamma|<|\gamma|$ we exchange f and g in the right-hand side of (3.11). By virtue of the equality $E^\nu=(1+(it/2)\nu\xi^2)^{-1}\partial_t(tE^\nu)$ we obtain the identity

$$\frac{\phi}{t}E^\nu=\partial_t\left(\frac{\phi E^\nu}{1+(it/2)\nu\xi^2}\right)-\frac{E^\nu\partial_t\phi}{1+(it/2)\nu\xi^2}+\frac{1+it\nu\xi^2}{t(1+(it/2)\nu\xi^2)^2}\phi E^\nu.\tag{3.12}$$

Therefore, we get from (3.11)

$$i\partial_t\Psi=R_2,\tag{3.13}$$

where

$$\begin{aligned}\Psi&=\mathcal{F}\mathcal{U}(-t)\Gamma^{\alpha+\delta}u_n(t) \\ &\quad+\sum_{m=0}^{n-1}\sum_{\gamma\le\delta}\sum_{\beta\le\alpha}C_\delta^\gamma C_\alpha^\beta \\ &\quad\times\sum_{j,k=1}^{2}\Bigg(\lambda_{jk}\mathcal{D}_2\frac{\xi_jE^2}{1+it\xi^2}\Big(\mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1}\widetilde{\Gamma}^{\delta-\gamma}f\Big)\mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g \\ &\qquad+\mu_{jk}\mathcal{D}_{-2}\frac{\xi_jE^6}{1+3it\xi^2}\Big(\overline{\mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1}\widetilde{\Gamma}^{\delta-\gamma}f}\Big)\overline{\mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g}\Bigg),\end{aligned}\tag{3.14}$$

$R_2 = R_1 + \sum_{j=1}^{3} I_j$, and

$$
\begin{aligned}
I_1 &= \frac{1}{it} \sum_{m=0}^{n-1} \sum_{\gamma \le \delta} \sum_{\beta \le \alpha} C_\delta^\gamma C_\alpha^\beta \\
&\quad \times \sum_{j,k=1}^{2} \Bigg(\lambda_{jk} \mathcal{D}_2 \frac{(1+2it\xi^2)E^2}{(1+it\xi^2)^2} \xi_j \Big(\mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1} \widetilde{\Gamma}^{\delta-\gamma} f \Big) \mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g \\
&\qquad - \mu_{jk} \mathcal{D}_{-2} \frac{(1+6it\xi^2)E^6}{(1+3it\xi^2)^2} \xi_j \Big(\overline{\mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1} \widetilde{\Gamma}^{\delta-\gamma} f} \Big) \overline{\mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g} \Bigg), \\
I_2 &= -\frac{1}{it} \sum_{m=0}^{n-1} \sum_{\gamma \le \delta} \sum_{\beta \le \alpha} C_\delta^\gamma C_\alpha^\beta \sum_{j,k=1}^{2} \lambda_{jk} \mathcal{D}_2 \frac{\xi_j E^2}{1+it\xi^2} \\
&\qquad \times \Big(\big(\partial_t \mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1} \widetilde{\Gamma}^{\delta-\gamma} f \big) \mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g \\
&\qquad + \big(\mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1} \widetilde{\Gamma}^{\delta-\gamma} f \big) \partial_t \mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g \Big), \\
I_3 &= -\frac{1}{it} \sum_{m=0}^{n-1} \sum_{\gamma \le \delta} \sum_{\beta \le \alpha} C_\delta^\gamma C_\alpha^\beta \sum_{j,k=1}^{2} \mu_{jk} \mathcal{D}_{-2} \frac{\xi_j E^6}{1+3it\xi^2} \\
&\qquad \times \Big(\Big(\overline{\partial_t \mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1} \widetilde{\Gamma}^{\delta-\gamma} f} \Big) \overline{\mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g} \\
&\qquad + \Big(\overline{\mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1} \widetilde{\Gamma}^{\delta-\gamma} f} \Big) \overline{\partial_t \mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g} \Big).
\end{aligned}
\tag{3.15}
$$

By Hölder's inequality, the identity $\mathcal{J}_j = \mathcal{U}(t)x_j\mathcal{U}(-t)$, and Lemmas 2.1, 2.2, 2.3, and 2.4 we get the estimates

$$
\begin{aligned}
&\sum_\alpha \frac{(b_n)^{|\alpha|}}{\alpha!} \| I_1(t) \| \\
&\quad \le C|t|^{-3/2} \sum_{m=0}^{n-1} \sum_\alpha \sum_{\beta \le \alpha} C_\alpha^\beta \frac{(b_n)^{|\alpha|}}{\alpha!} \sum_{j,k=1}^{2} \Bigg(\sum_{|\delta| \le 2} \left\| \partial_{x_j}^{-1} \widetilde{\Gamma}^\delta f \right\| \Bigg) \\
&\qquad \times \sum_{|\gamma| \le 1} \left\| \mathcal{V}(-t)\mathcal{K}\Gamma^\gamma g \right\|_\infty \\
&\quad \le C|t|^{-3/2} \sum_{m=0}^{n-1} \sum_\alpha \sum_{\beta \le \alpha} C_\alpha^\beta \frac{(b_n)^{|\alpha|}}{\alpha!} \Bigg(\sum_{|\delta| \le 2} \left\| \partial_{x_j}^{-1} \widetilde{\Gamma}^\delta f \right\| \Bigg) \\
&\qquad \times \sum_{|\gamma| \le 1} \Bigg(\sum_{|\sigma| \le 1} \| \mathcal{J}^\sigma \Gamma^\gamma g \|^{1-\eta} \Bigg) \sum_{|\sigma| \le 2} \| \mathcal{J}^\sigma \Gamma^\gamma g \|^\eta \\
&\quad \le C_n \varepsilon^{n+1} |t|^{-3/2-(n-1-m)w-mw+(1-mw)\eta} \\
&\quad \le C_n \varepsilon^{n+1} |t|^{\eta-3/2-(n-1-m\eta)w}
\end{aligned}
\tag{3.16}
$$

for $|\delta| \leq 3$. In the same way we obtain

$$\begin{aligned}
&\sum_{\alpha} \frac{(b_n)^{|\alpha|}}{\alpha!} \left(\|I_2(t)\| + \|I_3(t)\|\right) \\
&\quad \leq C|t|^{-1/2} \sum_{m=0}^{n-1} \sum_{\alpha} \sum_{\beta \leq \alpha} \sum_{|\gamma| \leq 1} C_{\delta}^{\gamma} C_{\alpha}^{\beta} \frac{(b_n)^{|\alpha|}}{\alpha!} \\
&\qquad \times \sum_{j,k=1}^{2} \left(\|\partial_t \mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1} \tilde{\Gamma}^{\delta-\gamma} f\| \|\mathcal{F}\mathcal{U}(-t)\Gamma^{\gamma} g\|_{\infty}\right. \\
&\qquad\qquad \left. + \|\partial_t \mathcal{F}\mathcal{U}(-t)\Gamma^{\gamma} g\|_{\infty} \|\mathcal{F}\mathcal{U}(-t)\partial_{x_j}^{-1} \tilde{\Gamma}^{\delta-\gamma} f\|\right) \\
&\quad \leq C|t|^{-1/2} \sum_{m=0}^{n-1} \sum_{\alpha} \frac{(b_n)^{|\alpha|}}{\alpha!} \\
&\qquad\qquad \times \left(\varepsilon^{m+1} |t|^{-mw} \sum_{|\delta| \leq 3} \left\|\partial_t \mathcal{F}\mathcal{U}(-t)\tilde{\Gamma}^{\alpha+\delta} u_{n-1-m}(t)\right\|\right. \\
&\qquad\qquad\qquad \left. + \varepsilon^{n-m} |t|^{-(n-m-1)w} \sum_{\gamma \leq 1} \left\|\partial_t \mathcal{F}\mathcal{U}(-t)\Gamma^{\alpha+\gamma} \nabla u_m(t)\right\|\right) \\
&\quad \leq C_n \varepsilon^{n+1} |t|^{\eta-3/2-(n-1)w}.
\end{aligned} \tag{3.17}$$

In view of (3.10), (3.16), and (3.17) we have the estimate

$$\sum_{\alpha} \frac{(b_n)^{|\alpha|}}{\alpha!} \|R_2(t)\| \leq C_n \varepsilon^{n+1} |t|^{-1-nw}. \tag{3.18}$$

Multiplying both sides of (3.13) by $\overline{\Psi(t)}$, integrating with respect to the space variables, and taking the imaginary part of the result, we obtain the inequality $(d/dt)\|\Psi(t)\| \leq \|R_2(t)\|$, hence

$$\frac{d}{dt} \sum_{\alpha} \frac{(b_n)^{|\alpha|}}{\alpha!} \|\Psi(t)\| \leq \sum_{\alpha} \frac{(b_n)^{|\alpha|}}{\alpha!} \|R_2(t)\| \leq C_n \varepsilon^{n+1} |t|^{-1-nw}. \tag{3.19}$$

Then integration with respect to t in view of (3.18) yields

$$\|u_n(t)\|_{\mathbf{A}^2_{b_n}} \leq C_n \varepsilon^{n+1} |t|^{-nw}. \tag{3.20}$$

Applying the operator $\mathcal{J}_{x_l}$ to both sides of (3.1), we get

$$\begin{aligned}
\mathcal{L}\mathcal{J}_{x_l} \Gamma^{\alpha+\delta} u_n = \sum_{\gamma \leq \delta} \sum_{\beta \leq \alpha} C_{\delta}^{\gamma} C_{\alpha}^{\beta} \sum_{j,k=1}^{2} &\left(\lambda_{jk} (\tilde{\Gamma}^{\delta-\gamma} f) \mathcal{J}_{x_l} \Gamma^{\gamma} g + it\lambda_{jk} (\partial_{x_l} \tilde{\Gamma}^{\delta-\gamma} f) \Gamma^{\gamma} g\right. \\
&\left. + \mu_{jk} \overline{(\tilde{\Gamma}^{\delta-\gamma} f) \mathcal{J}_{x_l} \Gamma^{\gamma} g} - it\mu_{jk} \overline{(\partial_{x_l} \tilde{\Gamma}^{\delta-\gamma} f) \Gamma^{\gamma} g}\right),
\end{aligned} \tag{3.21}$$

hence by the classical energy method, via the inequality

$$\|(\partial_{x_j} \phi)(\partial_{x_k} \psi)\| \leq \frac{C}{t} \|\Theta\phi\| \|\Theta\psi\|, \tag{3.22}$$

and Lemmas 2.1, 2.2, 2.3, and 2.4 we obtain

$$\sum_{\alpha} \frac{(b_n)^{|\alpha|}}{\alpha!} \frac{d}{dt} \|\mathcal{J}_{x_l} \Gamma^{\alpha+\delta} u_n(t)\| \leq C_n \varepsilon^{n+1} |t|^{-1/2-(n-1)w+\eta} \tag{3.23}$$

for $|\delta| \leq 1$. Multiplying both sides of the last inequality by $t^{-1-\eta}$ and integrating with respect to t, we have

$$t^{-1-\eta} \sum_{|\gamma|=1} \|\mathcal{J}^{\gamma} u_n(t)\|_{\mathbf{A}^2_{b_n}} \leq C_n \varepsilon^{n+1} |t|^{-nw}. \tag{3.24}$$

By the identity $\mathcal{P}u = \mathcal{Q}u + 2it\mathcal{N}(v)$, and Lemmas 2.1, 2.2, 2.3, and 2.4 we see that

$$\begin{aligned} \|\mathcal{Q}u_n\|_{\mathbf{A}^1_{b_n}} &\leq \|\mathcal{P}u_n\|_{\mathbf{A}^1_{b_n}} + C \sum_{m=0}^{n-1} \|\mathcal{Q}u_{n-1-m}\|_{\mathbf{A}^1_{b_n}} \|\mathcal{Q}u_m\|_{\mathbf{A}^1_{b_n}} \\ &\leq \|\mathcal{P}u_n\|_{\mathbf{A}^1_{b_n}} + C_n \varepsilon^{n+1} |t|^{-nw}. \end{aligned} \tag{3.25}$$

Therefore, by virtue of (3.24) and (3.25) we get

$$t^{-1-\eta} \sum_{|\gamma|+|\delta|=1} \|\mathcal{J}^{\gamma} \Theta^{\delta} u_n(t)\|_{\mathbf{A}^2_{b_n}} \leq C_n \varepsilon^{n+1} |t|^{-nw}. \tag{3.26}$$

In the same way as above by virtue of (3.11) and (3.26) we obtain

$$t^{1-\eta} \sum_{|\gamma|+|\delta|\leq 1} \sum_{\alpha} \frac{(b_n)^{|\alpha|}}{\alpha!} \|\partial_t \mathcal{F}\mathcal{U}(-t) \Gamma^{\alpha+\gamma} \Theta^{\delta} u_n(t)\| \leq C_n \varepsilon^{n+1} |t|^{-nw}. \tag{3.27}$$

By (3.20), (3.26), and (3.27) we have the first part of the theorem

$$\|u_n(t)\|_{\mathbf{Y}_{b_n}} \leq C_n \varepsilon^{n+1} |t|^{-nw} \tag{3.28}$$

for any $n \in \mathbb{N}$. We next prove the last part of the theorem by induction. By the definition of $u_n(t)$ we have with $\mathcal{L} = i\partial_t + (1/2)\Delta$

$$\mathcal{L}\left(u - \sum_{m=1}^{n} u_m\right) = \mathcal{N}(u,u) - \sum_{k=1}^{n} \sum_{m=0}^{k-1} \mathcal{N}(u_{k-1-m}, u_m) = I + R, \tag{3.29}$$

where

$$\begin{aligned} I = {} & \mathcal{N}\left(u - \sum_{m=0}^{n-1} u_m, \sum_{m=0}^{n-1} u_m\right) + \mathcal{N}\left(\sum_{m=0}^{n-1} u_m, u - \sum_{m=0}^{n-1} u_m\right) \\ & + \mathcal{N}\left(u - \sum_{m=0}^{n-1} u_m, u - \sum_{m=0}^{n-1} u_m\right) \end{aligned} \tag{3.30}$$

and R consists of quadratic nonlinearities involving $u_k u_l$ and $\bar{u}_k \bar{u}_l$ with $k + l \geq n$. We have

$$i\partial_t \mathcal{F}\mathcal{U}(-t) \Gamma^{\alpha+\delta}\left(u - \sum_{m=0}^{n} u_m\right) = \mathcal{F}\mathcal{U}(-t) \Gamma^{\alpha+\delta}(I + R). \tag{3.31}$$

In the same way as in the proof of (3.7) we estimate

$$\begin{aligned}&\mathcal{F}\mathcal{U}(-t)\Gamma^{\alpha+\delta}\mathcal{N}(\phi,\psi)\\&\quad=\frac{1}{it}\sum_{m=0}^{n-1}\sum_{\gamma\le\delta}\sum_{\beta\le\alpha}C_\delta^\gamma C_\alpha^\beta\sum_{j,k=1}^{2}\left(\lambda_{jk}\mathcal{D}_2E^2\left(\mathcal{F}\mathcal{U}(-t)\tilde{\Gamma}^{\delta-\gamma}f\right)\mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g\right.\\&\qquad\left.+\mu_{jk}\mathcal{D}_{-2}E^6\left(\overline{\mathcal{F}\mathcal{U}(-t)\tilde{\Gamma}^{\delta-\gamma}f}\right)\overline{\mathcal{F}\mathcal{U}(-t)\Gamma^\gamma g}\right)+R_3(t),\end{aligned}\tag{3.32}$$

$f=\tilde{\Gamma}^{\alpha-\beta}\partial_{x_j}\phi$, $g=\Gamma^\beta\partial_{x_k}\psi$. We have by (3.10) and the fact that $b_n<b_{n-1}$

$$\sum_\alpha\frac{(b_n)^{|\alpha|}}{\alpha!}\|R_3(t)\|\le C|t|^{\eta-3/2}\sum_{|\sigma|\le1}\|\Theta^\sigma\phi(t)\|_{\mathbf{A}^3_{b_n}}\sum_{|\sigma|\le1}\|\Theta^\sigma\psi(t)\|_{\mathbf{A}^3_{b_n}}.\tag{3.33}$$

Hence by the assumption and (3.28)

$$\sum_\alpha\frac{(b_{n+1})^{|\alpha|}}{\alpha!}\|\Gamma^{\alpha+\delta}I(t)\|\le C_n\varepsilon^{n+2}|t|^{-1-(n+1)w}.\tag{3.34}$$

We also have by (3.28)

$$\sum_\alpha\frac{(b_{n+1})^{|\alpha|}}{\alpha!}\|\Gamma^{\alpha+\delta}R(t)\|\le C\|u_k(t)u_l(t)\|_{\mathbf{Y}_{b_{n+1}}}\le C_n\varepsilon^{n+2}|t|^{-1-(n+1)w}.\tag{3.35}$$

Thus in view of (3.28), (3.31), (3.32), (3.34), and (3.35) we obtain

$$\left\|u-\sum_{m=0}^{n}u_m\right\|_{\mathbf{Y}_{b_{n+1}}}\le C_n\varepsilon^{n+2}|t|^{-1-(n+1)w}\tag{3.36}$$

which yields the second part of the theorem. Theorem 1.1 is proved.

Proof of Corollary 1.2. By (1.1) we have

$$\begin{aligned}i\partial_t\mathcal{F}\mathcal{U}(-t)u(t)=\frac{1}{t}\sum_{j,k=1}^{2}&\left(\lambda_{jk}\mathcal{D}_2E^2\xi_j\left(\mathcal{F}\mathcal{U}(-t)u\right)\mathcal{F}\mathcal{U}(-t)\partial_{x_k}u\right.\\&\left.-\mu_{jk}\mathcal{D}_{-2}E^6\xi_j\left(\overline{\mathcal{F}\mathcal{U}(-t)u}\right)\overline{\mathcal{F}\mathcal{U}(-t)\partial_{x_k}u}\right)+R(t)\end{aligned}\tag{3.37}$$

and the property of the solution of (1.1) we see that $\|R\|\le C\varepsilon^2|t|^{-1-wt}$. In the same way as in the proof of Theorem 1.1 we have

$$\|\mathcal{F}\mathcal{U}(-t)u(t)-\widehat{u^+}(\xi)\|\le C_1\varepsilon^2|t|^{-wt}.\tag{3.38}$$

By the definition of u we see that

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t)u(t)-\widehat{u^+}(\xi)=&-\frac{1}{4}\sum_{j,k=1}^{2}\int_\infty^t\frac{\lambda_{jk}}{i\tau}E^{1/2}d\tau\,\xi_j\xi_k\widehat{u^+}\left(\frac{\xi}{2}\right)\widehat{u^+}\left(\frac{\xi}{2}\right)\\&-\frac{1}{4}\sum_{j,k=1}^{2}\int_\infty^t\frac{\mu_{jk}}{i\tau}E^{3/2}d\tau\,\xi_j\xi_k\overline{\widehat{u^+}}\left(-\frac{\xi}{2}\right)\overline{\widehat{u^+}}\left(-\frac{\xi}{2}\right)+I,\end{aligned}\tag{3.39}$$

where

$$\begin{aligned} I = &-\frac{1}{4}\sum_{j,k=1}^{2}\int_{\infty}^{t}\frac{\lambda_{jk}}{i\tau}E^{1/2}(\mathcal{F}\mathcal{U}(-t)u)^{2}\left(t,\frac{\xi}{2}\right)d\tau\,\xi_j\xi_k \\ &-\frac{1}{4}\sum_{j,k=1}^{2}\int_{\infty}^{t}\frac{\mu_{jk}}{i\tau}E^{3/2}(\overline{\mathcal{F}\mathcal{U}(-t)u})^{2}\left(t,-\frac{\xi}{2}\right)d\tau\,\xi_j\xi_k \\ &+\frac{1}{4}\sum_{j,k=1}^{2}\int_{\infty}^{t}\frac{\lambda_{jk}}{i\tau}E^{1/2}d\tau\,\xi_j\xi_k\widehat{u^{+}}\left(\frac{\xi}{2}\right)\widehat{u^{+}}\left(\frac{\xi}{2}\right) \\ &+\frac{1}{4}\sum_{j,k=1}^{2}\int_{\infty}^{t}\frac{\mu_{jk}}{i\tau}E^{3/2}d\tau\,\xi_j\xi_k\overline{\widehat{u^{+}}}\left(-\frac{\xi}{2}\right)\overline{\widehat{u^{+}}}\left(-\frac{\xi}{2}\right). \end{aligned} \tag{3.40}$$

From (3.38) the $\mathbf{L}^2$ norm of I is estimated as $\|I(t)\| \le C\varepsilon^3|t|^{-2w}$. Therefore we get

$$\begin{aligned} &\Bigg\|\mathcal{F}\mathcal{U}(-t)u(t)-\widehat{u^{+}}(\xi)+\frac{1}{4}\sum_{j,k=1}^{2}\int_{\infty}^{t}\frac{\lambda_{jk}}{i\tau}E^{1/2}d\tau\,\xi_j\xi_k\widehat{u^{+}}\left(\frac{\xi}{2}\right)\widehat{u^{+}}\left(\frac{\xi}{2}\right) \\ &+\frac{1}{4}\sum_{j,k=1}^{2}\int_{\infty}^{t}\frac{\mu_{jk}}{i\tau}E^{3/2}d\tau\,\xi_j\xi_k\overline{\widehat{u^{+}}}\left(-\frac{\xi}{2}\right)\overline{\widehat{u^{+}}}\left(-\frac{\xi}{2}\right)\Bigg\| \le C_3\varepsilon^3|t|^{-2w}. \end{aligned} \tag{3.41}$$

We iterate this procedure to get

$$\begin{aligned} &\Bigg\|\mathcal{F}\mathcal{U}(-t)u(t)-\widehat{u^{+}}(\xi) \\ &\quad+\frac{1}{4}\sum_{l=1}^{n}\sum_{m=0}^{l-1}\sum_{j,k=1}^{2}\int_{\infty}^{t}\frac{\lambda_{jk}}{i\tau}E^{1/2}d\tau\,\xi_j\xi_k\widehat{u_{l-1-m}^{+}}\left(t,\frac{\xi}{2}\right)\widehat{u_m^{+}}\left(t,\frac{\xi}{2}\right) \\ &\quad+\frac{1}{4}\sum_{l=1}^{n}\sum_{m=0}^{l-1}\sum_{j,k=1}^{2}\int_{\infty}^{t}\frac{\mu_{jk}}{i\tau}E^{3/2}d\tau\,\xi_j\xi_k\overline{\widehat{u_{l-1-m}^{+}}}\left(t,-\frac{\xi}{2}\right)\overline{\widehat{u_m^{+}}}\left(t,-\frac{\xi}{2}\right)\Bigg\| \\ &\le C_{n+1}\varepsilon^{n+2}|t|^{-(n+1)w} \end{aligned} \tag{3.42}$$

with $\widehat{u_0^{+}}(t,\xi) = \widehat{u^{+}}(\xi)$. This completes the proof of Corollary 1.2. □

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Nakao Hayashi: Department of Mathematics, Graduate School of Science, Osaka University, Osaka 560-0043, Japan
E-mail address: nhayashi@math.wani.osaka-u.ac.jp

Pavel I. Naumkin: Instituto de Matemáticas, UNAM Campus Morelia, AP 61-3 (Xangari), Morelia, CP 58089, Michoacán, Mexico
E-mail address: pavelni@matmor.unam.mx