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GAUSSIAN INTEGERS WITH SMALL PRIME FACTORS

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ABSTRACT. Let $\Psi_G(x^t,x)$ denote the number of Gaussian integers with norm not exceeding x^{2t} whose Gaussian prime factors have norm not exceeding x^2. Previous estimates have required restrictions on the parameter t with respect to x. The purpose of this note is to present asymptotic estimates for $\Psi_G(x^t,x)$ for all ranges of the parameter t with respect to x.

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1. <u>INTRODUCTION</u>. Let α denote a Gaussian integer, ρ a Gaussian prime, $N\alpha = \alpha \cdot \bar{\alpha}$ the norm of α, and δ, ε arbitrary positive constants. Throughout the discussion the constants implied by the use of the $\mathcal{O}$-notation will be absolute unless otherwise indicated.

For real numbers $x \geqq 1$ and $t \geqq 0$, J. H. Jordan [3] and the author [2] gave asymptotic estimates for the number of Gaussian integers with norms not exceeding x^{2t} having only Gaussian prime factors with norm not exceeding x^2. However, Jordan's estimate fixed the parameter t and our estimate had t bounded with respect to x. The purpose of this note is to present an asymptotic estimate for all ranges of the parameter t with respect to x.

2. <u>MAIN RESULTS</u>.

THEOREM. Let $\Psi_G(x^t,x)$ denote the number of Gaussian integers with norm not exceeding x^{2t} having only Gaussian prime factors with norm not exceeding x^2. Then:

i) If $t \leqq (\log x)^{3/5-\delta}$, then

$$\Psi_G(x^t,x) = \pi x^{2t}\{Z(t) + \mathcal{O}_\varepsilon(\frac{tZ(t)}{\log x})\} \tag{2.1}$$

for t outside the interval $(1, 1+\varepsilon)$ where $Z(t)$ (the well-known Dickman function) satisfies the equation

$$t\, Z'(t) = -Z(t-1)$$

with initial condition $Z(t) = 1$ for $0 \leqq t \leqq 1$. Further as $t \to \infty$

$$Z(t) = \exp\{-t(\log t + \log\log t - 1 + \frac{\log\log t}{\log t}) + \mathcal{O}(\frac{t}{\log t})\}$$

ii) If $(\log x)^{3/5-\delta} < t \leqq x/\log x$, then

$$\Psi_G(x^t,x) = x^{2t} \exp\{-t(\log t + \log\log t - 1) + \mathcal{O}(\frac{t \log\log t}{\log t})\} \tag{2.2}$$

iii) If $x/\log x < t \leqq x^2/(e\log x)$, then

$$\Psi_G(x^t,x) = x^{2t}\exp\{-t\,(\log t + \log\log t + \mathcal{O}(1))\} \tag{2.3}$$

iv) If $t > x^2/(e\log x)$, then

$$\Psi_G(x^t,x) = \exp\{\tfrac{1}{4}\pi_G(x)\log t - x^2 + \frac{x^2\log\log x}{2\log x} + \mathcal{O}(\frac{x^2}{\log x})\} \tag{2.4}$$

where $\pi_G(x)$ denotes the number of Gaussian primes with norm not exceeding x^2.

PROOF of the Theorem. Now Case i) follows from Theorem 5 of [2] and the behavior of $Z(t)$. To derive (2.2) and (2.3) we will first write a lower estimate and then an upper estimate for $\Psi_G(x^t,x)$ that are of the same order.

For the lower estimate we follow the manner of A. S. Fainleib [1] (for rational integers) and consider the sum

$$\sum_{\substack{N\alpha \leqq x^{2t} \\ \rho|\alpha \Rightarrow N\rho \leqq x^2}} \log N\alpha \tag{2.5}$$

We easily see that (2.5) does not exceed

$$2t\log x\ \ \Psi_G(x^t,x),$$

and after some routine calculation, (2.5) is at least as large as

$$2(1-\varepsilon)\,x^{2t}\log x\int_{t-1}^{t-\delta} x^{-2u}\,\Psi_G(x^u,x)\,du$$

for $0 < \delta < 1$ (fixed) and $\varepsilon = \varepsilon(x) \ll \exp(-a(\log x)^{3/5})$ for a an absolute positive constant.

Thus we have

$$x^{-2t}\,\Psi_G(x^t,x) > \frac{(1-\varepsilon)}{t}\int_{t-1}^{t-\delta} x^{-2u}\,\Psi_G(x^u,x)\,du \tag{2.6}$$

Now we let $Z_1(t)$ be defined by the equation

$$t\, Z_1'(t) = Z_1(t) + (1-\varepsilon)\, Z_1(t-\delta) - (1-\varepsilon)\, Z_1(t-1)$$

with initial condition $Z_1(t) = t$ for $0 \leqq t \leqq \delta$. By Lemma 1 of Fainleib [1], as $t \to \infty$

$$Z_1(t) = b_0 t + b_1 + \exp\{-t(\log t + \log\log t - 1) + O(\frac{t \log\log t}{\log t}) + O(t\varepsilon)\}$$

where b_0 and b_1 are real numbers. It is easy to see that for $t \geqq 1$

$$Z_1''(t) = \frac{(1-\varepsilon)}{t} \int_{t-1}^{t-\delta} Z_1''(u)\,du\,.$$

Now we let

$$K(t,x) = x^{-2t}\, \Psi_G(x^t,x) - \lambda Z_1''(t)$$

where λ is a sufficiently small positive real number. Then for $0 \leqq t \leqq 1$, $K(t,x) \geqq 0$, and for $t \geqq 1$

$$K(t,x) \geqq \frac{(1-\varepsilon)}{t} \int_{t-1}^{t-\delta} K(u,x)\,du\,.$$

Therefore by Lemma 2 of Fainleib [1], $K(t,x) \geqq 0$ for all $t \geqq 0$ so that

$$\Psi_G(x^t,x) \geq \lambda\, x^{2t}\, Z_1''(t)$$

which in turn, implies that

$$\Psi_G(x^t,x) \geqq x^{2t}\exp\{-t(\log t + \log\log t - 1) + O(\frac{t \log\log t}{\log t}) + O(t\varepsilon)\}, \quad (2.7)$$

for $t \geqq 1$ which is the lower estimate that we need.

Now we follow the manner of B. V. Levin and A. S. Fainleib [4] to obtain the following general results which as a special case of Lemma 2 give the required upper estimate for $\Psi_G(x^t,x)$.

LEMMA 1. Let f be a completely multiplicative non-negative function satisfying for $x \le 0$ and $y \le 0$

$$\sum_{\substack{N\rho^r \le x^2 \\ N\rho \le y^2}} \lambda_f(N\rho^r) = \tau \log(\min(x,y)) + D + h(\min(x^2,y^2)) + R(x^2,y^2)$$

where $\lambda_f(N\rho^r) = f(N\rho^r)\cdot\log N\rho$, τ is a real number, D is an absolute constant, $h(u) = O((\log u))^{-1}$, and

$$\int_2^\infty |R(u,y^2)| \quad (u^\delta \log u)^{-1}\, du < \infty$$

for all $\delta > 0$. Further assume for every Gaussian prime ρ and $s > -1$ that $\sum_{r=1}^\infty f(N\rho^r)N\rho^{-rs}$ converges. Then, if $0 < \delta = \delta(y) \le 1 - 1/\log y$,

$$\log P(\delta-1,y) = \frac{\tau}{8}\frac{y^{2(1-\delta)}}{(1-\delta)\log y} + \frac{\tau}{4}\log(\frac{1}{1-\delta}) + O(\frac{y^{2(1-\delta)}}{(1-\delta)^2 \log y}) + $$

$$+O(\int_2^\infty |R(u,y^2)| \quad (u^\delta \log u)^{-1}\, du) \quad , \quad (2.9)$$

with

$$P(s,y) = \prod_{N\rho \le y^2}{}' (1 + \sum_{r=1}^\infty f(N\rho^r)\ N\rho^{-rs})$$

where the ′ implies the index is over only those Gaussian primes ρ in the first quadrant of the complex plane or on the positive real axis.

PROOF. Let $x \ge y$. Applying Abel's summation, we find that

$$\sum_{\substack{N\rho^r \le x^2 \\ N\rho \le y^2}}{}' \frac{\lambda_f(N\rho^r)}{N\rho^{rs} \log N\rho^r} = \frac{F(y^2)}{y^{2s} \log y^2} - \int_2^{y^2} F(u)d(\frac{1}{u^\delta \log u}) + \frac{R(x^2,y^2)}{x^{2s} \log x^2} +$$

$$+\int_2^{x^2} R(u,y^2)\frac{s \log u+1}{u^{s+1} \log^2 u}\, du$$

where $F(u) = \frac{\tau}{4} \log u + D + h(u)$.

Assume s > 0, and letting x tend to ∞ , we obtain

$$\sum_{N\rho\leqq y^2}{}' \sum_{r=1}^{\infty} \frac{\lambda_f(N\rho^r)}{N\rho^{rs}\log N\rho^r} = \frac{F(y^2)}{y^{2s}\log y^2} - \int_2^{y^2} F(u)d\left(\frac{1}{u^s\log u}\right) + \int_2^{\infty} R(u,y^2)\,\frac{s\log u+1}{u^{s+1}\log^2 u}\,du.$$

Now since $\sum_{r=1}^{\infty} f(N\rho^r)\, N\rho^{-rs} < 1$ for all ρ such that $N\rho \leqq y^2$ if s is sufficiently large, we have

$$\log P(s,y) = \sum_{N\rho\leqq y^2}{}' \sum_{r=1}^{\infty} \frac{\lambda_f(N\rho^r)}{N\rho^{rs}\log N\rho^r} .$$

Hence by the uniqueness of analytic continuation

$$\log P(\delta-1,y) = \frac{F(y^2)}{y^{2(\delta-1)}\log y^2} - \int_2^{y^2} F(u)d\left(\frac{1}{u^{\delta-1}\log u}\right) + \int_2^{\infty} R(u,y^2)\,\frac{(\delta-1)\log u+1}{u^{\delta}\log^2 u}\,du, \tag{2.10}$$

for all δ > 0.

Substituting $F_1(u) + h(u)$ for $F(u)$ where $F_1(u) = \frac{\tau}{4}\log u + D$, we see that the right hand side of (2.10) is equal to

$$\frac{\tau}{4}\int_2^{y^2} (u^{\delta}\log u)^{-1}du + O(1) + O(y^{2(1-\delta)}\log^{-2}y) + O\left(\int_2^{y^2} (u^{\delta}\log^2 u)^{-1}du\right) + $$
$$+ O\left(\int_2^{\infty} |R(u,y^2)|\,(u^{\delta}\log u)^{-1}du\right)$$

with

$$\int_2^{y^2} (u^{\delta}\log u)^{-1}du = \frac{y^{2(1-\delta)}}{(1-\delta)\log y^2} + O\left(\frac{y^{2(1-\delta)}}{(1-\delta)^2\log^2 y}\right) .$$

Therefore substituting these results into (2.10) we get (2.9) to prove Lemma 1.

In the following lemma, we shall require that $R(x^2,y^2)$ satisfy the condition

$$R(x^2,y^2) = O\left(x^{-2 + \frac{2}{\left[\frac{\log x}{\log \min(x,y)}\right] + 1}}\right) \qquad (2.11)$$

which is generally satisfied when

$$f(N\rho^r) \log N\rho = O\left(\frac{\log N\rho}{N\rho^r}\right) .$$

LEMMA 2. Let f be a completely multiplicative non-negative function satisfying (2.8) where $R(x^2,y^2)$ satisfies (2.11) and let

$$F(x^t,x) = \sum_{\substack{N\alpha \le x^{2t} \\ \rho|\alpha \Rightarrow N\rho \le x^2}} N\alpha\, f(N\alpha)$$

Then for every t such that $\frac{\tau e}{8} < t < \frac{\tau}{8e} \cdot \frac{x^2}{\log x}$

$$F(x^t,x) \le x^{2t} \exp\{-t(\log t + \log\log t - (1 + \log\tfrac{\tau}{8}) + \frac{\log\log t}{\log t}) + O(\frac{t}{\log t}) + O(\log\log x) + O(\frac{t^2\log^2 t}{x^2\log x})\} , \qquad (2.12)$$

PROOF. Let $0 < \delta \le 1 - 1/(2\log x)$, then

$$F(x^t,x) \le x^{2t\delta} \sum_{\substack{N\alpha \le x^{2t} \\ \rho|\alpha \Rightarrow N\rho \le x^2}} (N\alpha)^{1-\delta} f(N\alpha) \le x^{2t\delta} P(1-\delta, x) .$$

Using Lemma 1, then

$$F(x^t,x) \le x^{2t\delta} \exp\{\frac{\tau}{8} \frac{x^{2(1-\delta)}}{(1-\delta)\log x} + \frac{\tau}{4} \log(\frac{1}{1-\delta}) + O(\frac{x^{2(1-\delta)}}{(1-\delta)^2\log^2 x}) + O(\int_2^\infty |R(u,x^2)|(u^\delta \log u)^{-1} du) + \log C_f\}$$

where C_f is an absolute constant depending on f.

Now

$$\int_2^\infty |R(u,x^2)| (u^\delta \log u)^{-1} du = O\left(\frac{x^{2(1-2\delta)} - 1}{2(1-2\delta)\log x} + \log\log x + \int_2^\infty \exp(-\delta uz + \frac{uz}{[u]+1})\frac{du}{u}\right)$$

where $z = 2\log x$. Further

$$\int_2^\infty \exp(-\delta uz + \frac{uz}{[u]+1})\frac{du}{u} = O\left(\frac{x^{2(1-2\delta)}}{\log x}\right) .$$

Hence,

$$F(x^t,x) \le x^{2t\delta}\exp\{\frac{\tau}{8}\frac{x^{2(1-\delta)}}{(1-\delta)\log x} + \frac{\tau}{4}\log(\frac{1}{1-\delta}) + O\left(\frac{x^{2(1-\delta)}}{(1-\delta)^2\log^2 x}\right) + O\left(\frac{x^{2(1-2\delta)}}{\log x}\right) + \log C_f\} \tag{2.13}$$

Now if we let

$$\delta = 1 - \frac{1}{2\log x}(\log t + \log\log t - \log\frac{\tau}{8} + \frac{\log\log t}{\log t})$$

in (2.13) we get (2.12) to comlpete the proof of Lemma 2.

For the special case $f(N\alpha) = N\alpha^{-1}$ we see that $\tau = 8$ so that by Lemma 2, if $e < t < \frac{1}{e}\frac{x^2}{\log x}$, then

$$\Psi_G(x^t,x) \le x^{2t}\exp\{-t(\log t + \log\log t - 1 + \frac{\log\log t}{\log t}) + O(\frac{t}{\log t}) + O(\log\log x) + O(\frac{t^2\log^2 t}{x^2\log x})\} \tag{2.14}$$

which is the required upper estimate for $\Psi_G(x^t,x)$.

Combining (2.7) and (2.14), we derive Cases ii) and iii) of the Theorem.

Finally, for Case iv), fix $y = x^t$ where $t > \frac{1}{e}\frac{x^2}{\log x}$. We again follow the manner of Levin and Fainleib [4] by letting $F_k(y)$ denote the number of Gaussian integers with norm not exceeding y^2 whose prime factors are "among" the first k Gaussian primes, i.e., we can arrange the Gaussian primes lying either in the first quadrant of the complex plane or on the positive real axis by the relation $i < j$ if $N\rho_i \leqq N\rho_j$.

We see that

$$F_k(y) = \sum_{0 \leqq n \leqq \frac{2\log x}{\log N\rho_k}} F_{k-1}\left(\frac{y}{N\rho_k^{\,n/2}}\right)$$

where ρ_k denotes the k-th Gaussian prime lying either in the first quadrant of the complex plane or on the positive real axis.

Now $F_0(y) = 4$ and

$$F_1(y) = 4\left(1 + \left[\frac{2\log y}{\log 2}\right]\right) \leqq 4 \cdot \frac{2}{1!\,\log 2} \cdot 2\log y .$$

Therefore proceeding by induction on $k \geqq 1$, since $t > \frac{1}{e}\frac{x^2}{\log x}$, we see that

$$F_k(y) \leqq 4 \cdot \frac{2^k}{k!\,\prod_k} \cdot 2^k(\log y)^k , \qquad (2.15)$$

Where $\prod_k = \prod_{v=1}^{k}{}' \log N\rho_v$.

Similarily,

$$F_k(y) \geqq 4 \frac{1}{k! \prod k} 2^k (\log y)^k , \tag{2.16}$$

Now we let $k = \frac{1}{4}\pi_G(x)$, then

$$\Psi_G(x^t,x) = F_k(y) \leqq \exp\{\frac{1}{4}\pi_G(x) \log 2t - x^2 + \frac{x^2 \log\log x}{2 \log x} + O(\frac{x^2}{\log x})\}$$

and

$$\Psi_G(x^t,x) = F_k(y) \geqq \exp\{\frac{1}{4}\pi_G(x) \log 2t - x^2 + \frac{x^2 \log\log x}{2 \log x} + O(\frac{x^2}{\log x})\}$$

which implies (2.4) to conclude the proof of the Theorem.

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