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PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. The extreme points for prestarlike functions having negative coefficients are determined. Coefficient, distortion and radii of univalence, starlikeness, and convexity theorems are also obtained.

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1. INTRODUCTION.

Let S denote the class of functions normalized by $f(0) = f'(0) - 1 = 0$ that are analytic and univalent in the unit disk U. Given α, $0 \leq \alpha \leq 1$, a function $f \in S$ is said to be in the class of functions starlike of order α, denoted by $S^*(\alpha)$, if

$$\mathrm{Re}\{zf'(z)/f(z)\} \geq \alpha \qquad (z \in U),$$

and is said to be in the class of functions <u>convex of order</u> α, denoted by $K(\alpha)$, if

$$\mathrm{Re}\{1 + zf''(z)/f'(z)\} \geq \alpha \qquad (z \in U).$$

Further, let T and $T^*[\alpha]$ denote the subclasses of S and $S^*(\alpha)$, respectively, whose elements can be expressed in the form $f(z) = z - \Sigma_{n=2}^{\infty} |a_n| z^n$.

The <u>convolution</u> or <u>Hadamard product</u> of two power series $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ and $g(z) = \Sigma_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f*g)(z) = \Sigma_{n=0}^{\infty} a_n b_n z^n$. An analytic function normalized by $f(0) = f'(0) - 1 = 0$ is said to be in the class of functions <u>prestarlike of order</u> α, $0 \leq \alpha \leq 1$, denoted by R_α, if $f*s_\alpha \in S^*(\alpha)$ where $s_\alpha(z) = z/(1-z)^{2(1-\alpha)}$. The function s_α is the well-known extremal function for the class $S^*(\alpha)$. In the sequel, we let

$$C(\alpha,n) = \frac{\Pi_{k=2}^{n}(k-2\alpha)}{(n-1)!} \qquad (n=2,\ 3,\ \ldots.), \tag{1.1}$$

so that s_α can be written in the form

$$s_\alpha(z) = z + \Sigma_{n=2}^{\infty} C(\alpha,n) z^n.$$

Note that $C(\alpha,n)$ is a decreasing function of α with

$$\lim_{n \to \infty} C(\alpha,n) = \left\{ \begin{array}{ll} \infty, & \alpha < 1/2 \\ 1, & \alpha = 1/2 \\ 0, & \alpha > 1/2 \end{array} \right\} \tag{1.2}$$

The class R_α was introduced by Ruscheweyh [3], who showed that a necessary and sufficient condition for f to be in R_α is that the functional

$$G(\alpha,z) = \frac{f(z) * \frac{s_\alpha(z)}{1-z}}{f(z) * s_\alpha(z)}$$

satisfy

$$\text{Re } G(\alpha,z) > 1/2 \qquad (z \in U). \tag{1.3}$$

Since $s_1(z) = z$, we say that f is prestarlike of order 1 if and only if $\text{Re}\{f(z)/z\} > 1/2$ $(z \in U)$. Note that $R_0 = K(0)$ and $R_{1/2} = S^*(1/2)$. In [3] it was shown that

$$R_\alpha \subset R_\beta \qquad \text{for } 0 \le \alpha < \beta \le 1,$$

which generalizes the well-known result that $K(0) \subset S^*(1/2)$.

In Section 2, we obtain a sufficient condition in terms of the modulus of the coefficients for a function to be in R_α and show that this condition is also necessary for the subclass

$$R[\alpha] = R_\alpha \cap T. \tag{1.4}$$

In Section 3, we obtain the extreme points for the closed convex hull of $R[\alpha]$ and use them to prove distortion and covering theorems. In Section 4, we determine the radii of univalence, starlikeness, and convexity for $R[\alpha]$. Finally, we find the smallest $\beta = \beta(\alpha)$ for which $T^*[\alpha] \subset R[\beta]$, $0 \le \alpha < 1$.

2. COEFFICIENT INEQUALITIES FOR THE CLASS $R[\alpha]$.

We first obtain a relationship between the order of prestarlikeness of a function and the modulus of its coefficients.

THEOREM 1. Let $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$. If

$$\Sigma_{n=2}^{\infty} \frac{(n-\alpha)C(\alpha,n)}{1-\alpha} |a_n| \le 1, \qquad 0 \le \alpha \le 1, \tag{2.1}$$

then $f \in R_\alpha$.

REMARK. For the case $\alpha = 1$ in the theorem and all subsequent results, the expression $(n-\alpha)C(\alpha,n)/(1-\alpha)$ is taken to be 2, its limit as $\alpha \to 1^-$.

PROOF. An equivalent formulation of (1.3) is $|1/G(\alpha,z) - 1| < 1$, $0 \le \alpha \le 1$. For $s_\alpha(z) = z + \Sigma_{n=2}^{\infty} C(\alpha,n)z^n$, we have $s_\alpha(z)/(1-z) = z + \Sigma_{n=2}^{\infty} \sigma(\alpha,n)z^n$ where $\sigma(\alpha,n) = 1 + \Sigma_{k=2}^{n} C(\alpha,k)$. Thus

$$\left|\frac{1}{G(\alpha,z)} - 1\right| = \left|\frac{\Sigma_{n=2}^{\infty} \sigma(\alpha,n-1)a_n z^n}{z + \Sigma_{n=2}^{\infty} \sigma(\alpha,n)a_n z^n}\right|$$

$$\le \frac{\Sigma_{n=2}^{\infty} \sigma(\alpha,n-1)|a_n|}{1 - \Sigma_{n=2}^{\infty} \sigma(\alpha,n)|a_n|},$$

which is bounded by 1 whenever (2.1) is satisfied.

The converse of Theorem 1 is also true for the class $R[\alpha]$, defined by (1.4). Since a necessary and sufficient condition [4] for $f(z) = z - \Sigma_{n=2}^{\infty} |b_n| z^n$ to be in $T^*[\alpha]$ is that

$$\Sigma_{n=2}^{\infty}(n-\alpha) |b_n| \le 1 - \alpha, \tag{2.2}$$

we obtain

THEOREM 2. A function $f(z) = z - \Sigma_{n=2}^{\infty} |a_n| z^n$ is in $R[\alpha]$ if and only if

$$\Sigma_{n=2}^{\infty} \frac{(n-\alpha)C(\alpha,n)}{1 - \alpha} |a_n| \le 1, \qquad 0 \le \alpha \le 1.$$

COROLLARY. If $f(z) = z - \Sigma_{n=2}^{\infty} |a_n| z^n \in R[\alpha]$, $0 \le \alpha \le 1$, then $|a_n| \le (1-\alpha)/(n-\alpha)C(\alpha,n)$, with equality only for functions of the form $z - (1-\alpha)z^n/(n-\alpha)C(\alpha,n)$.

We now determine the extreme points of this class.

3. EXTREME POINTS OF R[α].

For any compact family of analytic functions $\mathfrak{F}$, it is well known that the real part of any continuous linear functional over $\mathfrak{F}$ is maximized (minimized) at one of the extreme points of the closed convex hull of $\mathfrak{F}$. The solutions to several extremal problems in R[α] follow easily from the extreme points for this class. In view of Theorem 2, we see that R[α] is a closed convex family. Thus, the extreme points are obtained in

THEOREM 3. Set $f_1(z) = z$ and $f_n(z) = z - (1-\alpha)z^n/(n-\alpha)C(\alpha,n)$, $n=2, 3, \ldots$. Then $f \in R[\alpha]$, $0 \le \alpha \le 1$, if and only if it can be expressed in the form $f(z) = \Sigma_{n=1}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \geq 0$ and $\Sigma_{n=1}^{\infty} \lambda_n = 1$.

PROOF. Suppose $f(z) = \Sigma_{n=1}^{\infty} \lambda_n f_n(z)$. Then

$$\Sigma_{n=2}^{\infty} \frac{(n-\alpha)C(\alpha,n)}{1-\alpha} \cdot \frac{\lambda_n(1-\alpha)}{(n-\alpha)C(\alpha,n)} = \Sigma_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1.$$

Therefore, $f \in R[\alpha]$ by Theorem 2.

Conversely, suppose $f(z) = z - \Sigma_{n=2}^{\infty}|a_n|z^n \in R[\alpha]$, $0 \le \alpha \le 1$. Then $|a_n| \leq (1-\alpha)/(n-\alpha)C(\alpha,n)$, $n=2, 3, \ldots$. Set $\lambda_n = (n-\alpha)C(\alpha,n)|a_n|/(1-\alpha)$ and $\lambda_1 = 1 - \Sigma_{n=2}^{\infty} \lambda_n$. From Theorem 2, it follows that $\lambda_1 \geq 0$. Since $f(z) = \Sigma_{n=1}^{\infty} \lambda_n f_n(z)$, the proof is complete.

As an immediate consequence of Theorem 3, we obtain distortion theorems for the class R[α].

THEOREM 4. If $f(z) = z - \Sigma_{n=2}^{\infty}|a_n|z^n \in R[\alpha]$, $0 \leq \alpha \leq 1$, then

$$r - \frac{1}{2(2-\alpha)}r^2 \leq |f(z)| \leq r + \frac{1}{2(2-\alpha)}r^2 \qquad (|z| = r),$$

with equality only for $f_2(z) = z - \frac{1}{4-2\alpha}z^2$, $z = \pm r$.

PROOF. From Theorem 3, we have

$$r - \max_n \frac{1-\alpha}{(n-\alpha)C(\alpha,n)} r^n \leq |f(z)| \leq r + \max_n \frac{1-\alpha}{(n-\alpha)C(\alpha,n)} r^n .$$

It suffices to show that $A(\alpha,n) = (1-\alpha)/(n-\alpha)C(\alpha,n)$ is a decreasing function of n. From (1.1), it follows that

$$C(\alpha,n+1) = \frac{n+1-2\alpha}{n} \cdot C(\alpha,n). \tag{3.1}$$

Therefore, we have $A(\alpha,n+1) \leq A(\alpha,n)$, $n=2, 3, \ldots$, whenever $(n+1-\alpha)(n+1-2\alpha) \geq n(n-\alpha)$ This is equivalent to $(1-\alpha)[1+2(n-\alpha)] \geq 0$, which proves the result.

COROLLARY. The disk $|z| < 1$ is mapped onto a domain that contains the disk $|w| < (3-2\alpha)/(4-2\alpha)$ for any $f \in R[\alpha]$, $0 \leq \alpha \leq 1$. The result is sharp, with extremal function

$$f_2(z) = z - \frac{1}{2(2-\alpha)} z^2 \in R[\alpha].$$

PROOF. Let $r \to 1^-$ in Theorem 4.

As a second application of Theorem 3, we have

THEOREM 5. If $f \in R[\alpha]$, $0 \leq \alpha \leq 1$, then $1 - M(\alpha,r) \leq |f'(z)| \leq 1 + M(\alpha,r)$ $(|z| = r)$, where

$$M(\alpha,r) = \max_n \frac{(1-\alpha)n}{(n-\alpha)C(\alpha,n)} r^{n-1} .$$

COROLLARY. If $f \in R[\alpha]$ with either $r \leq 2/3$ or $\alpha \leq 1/2$, then

$$1 - \frac{1}{2-\alpha} r \leq |f'(z)| \leq 1 + \frac{1}{2-\alpha} r \qquad (|z| = r).$$

PROOF. It suffices to show that

$$g(\alpha,r,n) = \frac{(1-\alpha)nr^{n-1}}{(n-\alpha)C(\alpha,n)}$$

is a decreasing function of n. In view of (3.1), the inequality $g(\alpha,r,n+1) \leq g(\alpha,r,n)$ is equivalent to

$$h(\alpha,r,n) = (1-r)n^2 + [2-3\alpha-(1-\alpha)r]n + (1-\alpha)(1-2\alpha) + ar \geq 0.$$

Since, for n fixed, h is a decreasing function of r, we have

$$h(\alpha,r,n) \geq h(\alpha,1,n) = (1-2\alpha)n + (1-\alpha)(1-2\alpha) + \alpha \geq 0$$

for $\alpha \leq 1/2$. Since h is also a decreasing function of α, it follows for $r \leq 2/3$ that

$$h(\alpha,r,n) \geq h(1,r,n) = (1-r)n^2 - n + r \geq h(1,2/3,n) \geq h(1,2/3,2) = 0.$$

REMARKS. 1. Since $h(1,r,2) = 2 - 3r < 0$ for $r > 2/3$, we have $g(1,r,2) < g(1,r,3)$. Thus the corollary will not be true for all α when $r > 2/3$.

2. We next show that the corollary will not be true for all r when $\alpha > 1/2$. For each α, $1/2 < \alpha < 1$, we must find an $r = r(\alpha)$ such that $M(\alpha,r) > (1/\ (2-\alpha))r$. It suffices to show for $n = n(\alpha)$ sufficiently large that $(1-\alpha)n/(n-\alpha)C(\alpha,n) > 1/(2-\alpha)$, which is equivalent to

$$C(\alpha,n) < \frac{(1-\alpha)(2-\alpha)n}{n-\alpha}. \qquad (3.2)$$

Since $C(\alpha,n) \to 0$ for $\alpha > 1/2$ and the right hand side of (3.2) is bounded below by $(1-\alpha)(2-\alpha) > 0$, the result follows.

4. RADII OF UNIVALENCE, STARLIKENESS, AND CONVEXITY.

The functions in $R[\alpha]$ for $0 \leq \alpha \leq 1/2$ are starlike of a positive order. The bound is given in

THEOREM 6. If $f \in R[\alpha]$, $0 \le \alpha \le 1/2$, then $f \in T^*[2(1-\alpha)/(3-2\alpha)]$. The result is sharp, with extremal function

$$f_2(z) = z - \frac{1}{2(2-\alpha)} z^2 .$$

PROOF. From (2.2), it suffices to show that

$$\Sigma_{n=2}^{\infty} \frac{(n-\alpha)C(\alpha,n)}{1-\alpha} |a_n| \le 1$$

implies

$$\Sigma_{n=2}^{\infty} \frac{n - \frac{2(1-\alpha)}{3-2\alpha}}{1 - \frac{2(1-\alpha)}{3-2\alpha}} | a_n | \le 1 .$$

This will follow if

$$(3-2\alpha)n - 2(1-\alpha) \le \frac{n-\alpha}{1-\alpha} C(\alpha,n)$$

or, equivalently, when

$$g(\alpha,n) = \frac{[(3-2\alpha)n - 2(1-\alpha)](1-\alpha)}{(n-\alpha)C(\alpha,n)} \le 1 .$$

Since $g(\alpha,2) = 1$, it suffices to show that $g(\alpha,n)$ is a decreasing function of n. In view of (3.1), the inequality $g(\alpha,n+1) \le g(\alpha,n)$, $n=2, 3, \dots$, is equivalent to

$$\frac{(3-2\alpha)n^2+n}{(n+1-\alpha)(n+1-2\alpha)} \le \frac{(3-2\alpha)n-2(1-\alpha)}{n-\alpha} .$$

This holds if and only if, for each fixed α, we have

$$h(n) = (4\alpha^2-8\alpha+3)n^2 + (6\alpha^2-4\alpha^3-1)n + 4\alpha^3 - 10\alpha^2 + 8\alpha - 2 \ge 0.$$

Note that $h(1) = 0$ and

$$h(n+1) - h(n) = 2(4\alpha^2-8\alpha+3)n + 2(1-4\alpha+5\alpha^2-2\alpha^3) = A(\alpha)n + B(\alpha) .$$

Since $A(\alpha) \geq 0$ and $B(\alpha) \geq 0$ for $0 \leq \alpha \leq 1/2$, the result follows.

REMARK. For $\alpha = 0$, Theorem 6 reduces to the known result [4] that $R[0] = K[0] \subset T^*[2/3]$.

When $\alpha > 1/2$, $R[\alpha] \not\subset S$. We will show that $g_n(z) = z - 2z^n/n \in R[\alpha] - S$ for $n = n(\alpha)$ sufficiently large. If $n \geq 4$, then $g_n \in R[1]$. If $1/2 < \alpha < 1$, then $g_n * s_\alpha = z - 2C(\alpha,n)z^n/n$. Taking (1.2) into account, we have $2C(\alpha,n)/n \leq (1-\alpha)/(n-\alpha)$ for n sufficiently large, so that $g_n * s_\alpha \in T^*[\alpha]$. Therefore, $g_n \in R[\alpha]$.

We next determine the largest disk in which $R[\alpha]$ is univalent.

THEOREM 7. The radius of univalence and starlikeness for $R[\alpha]$, $1/2 < \alpha \leq 1$, is

$$r(\alpha) = \min_n \left\{\frac{(n-\alpha)C(\alpha,n)}{n(1-\alpha)}\right\}^{1/(n-1)} .$$

PROOF. For $f(z) = z - \Sigma_{n=2}^{\infty}|a_n|z^n$ in $R[\alpha]$, the inequality $|zf'/f - 1| \lesssim 1$ is valid for $|z| \leq r$ whenever $\Sigma_{n=2}^{\infty} n|a_n|r^{n-1} \leq 1$. In view of Theorem 2, this is true if

$$r \leq \left[\frac{(n-\alpha)C(\alpha,n)}{n(1-\alpha)}\right]^{1/(n-1)} .$$

Hence, f is starlike for $|z| \leq r(\alpha)$. On the other hand, for some n we have

$$f_n'(z) = 1 - \frac{n(1-\alpha)}{(n-\alpha)C(\alpha,n)} z^{n-1} = 0 \quad \text{when } z = r(\alpha).$$

Thus f is not univalent (or starlike) for $|z| \leq r$, $r > r(\alpha)$.

COROLLARY. The radius of univalence and starlikeness for $R[1]$ is $1/\sqrt[3]{2} \approx .794$.

PROOF. We must show that

$$r(1) = \min_n \left(\frac{2}{n}\right)^{1/(n-1)} = \frac{1}{\sqrt[3]{2}} .$$

This can be found by differentiating $g(x) = (2/x)^{1/(x-1)}$ and observing that g is decreasing for $2 \le x \le x_0$ and increasing for $x > x_0$, where x_0 satisfies $4 < x_0 < 5$. Since $g(4) < g(5)$, the result follows.

Thus, for all α, f in $R[\alpha]$ is univalent and starlike when $|z| < 1/\sqrt[3]{2}$.

REMARK. MacGregor showed [2] that the radius of univalence and starlikeness for R_1 is $1/\sqrt{2} \approx .707$.

We will now obtain the radius of convexity for $R[\alpha]$.

THEOREM 8. If $f \in R[\alpha]$, $0 \le \alpha \le 1$, then f is convex in the disk

$$|z| < r(\alpha) = \inf_n \left\{\frac{(n-\alpha)C(\alpha,n)}{n^2(1-\alpha)}\right\}^{1/(n-1)} .$$

The result is sharp, with the extremal function of the form

$$f_n(z) = z - \frac{1-\alpha}{(n-\alpha)C(\alpha,n)} z^n$$

for some n.

PROOF. For $f(z) = z - \Sigma_{n=2}^{\infty}|a_n|z^n \in R[\alpha]$, it suffices to show that $|zf''(z)/f'(z)| \le 1$ for $|z| \le r(\alpha)$. We have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{-\Sigma_{n=2}^{\infty} n(n-1)|a_n|z^{n-1}}{1 - \Sigma_{n=2}^{\infty} n|a_n|z^{n-1}}\right| \le \frac{\Sigma_{n=2}^{\infty} n(n-1)|a_n|\ |z|^{n-1}}{1 - \Sigma_{n=2}^{\infty} n|a_n|\ |z|^{n-1}}$$

which is bounded by 1 whenever $\Sigma_{n=2}^{\infty} n^2|a_n|\ |z|^{n-1} \le 1$. From Theorem 2, this will hold whenever

$$n^2|z|^{n-1} \leq \frac{(n-\alpha)C(\alpha,n)}{1-\alpha}, \qquad n=2,\ 3,\ \ldots,$$

or

$$|z| \leq \left\{\frac{(n-\alpha)C(\alpha,n)}{n^2(1-\alpha)}\right\}^{1/(n-1)} \tag{4.1}$$

The result follows upon setting $|z| = r(\alpha)$ in (4.1).

Using arguments simiiliar to those in the corollary to Theorem 7, we have the following

COROLLARY. The radius of convexity for R[1] is $\sqrt{2}/3 \approx .471$.

REMARK. The radius of convexity r.c. for R_1 is known to satisfy $.395 \leq$ r.c. $\leq .40$. See [1].

We conclude with the determination of the smallest $\beta = \beta(\alpha)$ for which $T^*[\alpha] \subset R[\beta]$.

THEOREM 9. If $f \in T^*[\alpha]$ then $f \in R[(2-3\alpha)/2(1-\alpha)]$ for $0 \leq \alpha \leq 1/2$, and $f \in R[1/2]$ for $1/2 < \alpha < 1$. The result is sharp, with extremal function

$$z - \frac{1-\alpha}{2-\alpha}z^2 \qquad \text{for } 0 \leq \alpha \leq 1/2$$

and of the form

$$z - \frac{1-\alpha}{n-\alpha}\,z^n \qquad \text{for } 1/2 < \alpha < 1\ .$$

PROOF. From Theorem 2 it suffices to show, for $0 \leq \alpha \leq 1/2$, that $\Sigma_{n=2}^{\infty}(n-\alpha)|a_n|/(1-\alpha) \leq 1$ implies $\Sigma_{n=2}^{\infty}(n-\beta)C(\beta,n)|a_n|/(1-\beta) \leq 1$, where $\beta = \beta(\alpha) = (2-3\alpha)/(2-2\alpha)$. This will follow if we can show that

$$g(\alpha,\beta,n) = \frac{1-\alpha}{n-\alpha}\,\frac{n-\beta}{1-\beta}\ C(\beta,n) \leq 1\ .$$

Since $g(\alpha,\beta,2) = 1$, it is sufficient to show that $g(\alpha,\beta,n)$ is a decreasing function of n. In view of (3.1), $g(\alpha,\beta,n+1) \le g(\alpha,\beta,n)$ whenever

$$\frac{(n+1-\beta)(n+1-2\beta)}{n(n+1-\alpha)} \le \frac{n-\beta}{n-\alpha} ,$$

which is equivalent to

$$p(\alpha,n) = \frac{1-2\alpha}{2(1-\alpha)^2} \left(2(1-\alpha)n^2 + (\alpha^2+2\alpha-2)n - \alpha^2\right) \ge 0 .$$

However, since

$$p(\alpha,n+1) - p(\alpha,n) = \frac{1-2\alpha}{2(1-\alpha)^2} \left(4(1-\alpha)n+\alpha^2\right)$$

is nonnegative for $0 \le \alpha \le 1/2$, the first inclusion is proved.

The inclusion relation $T^*[\alpha_1] \subset T^*[\alpha_2]$ for $\alpha_1 \ge \alpha_2$ shows that $T^*[\alpha] \subset R[1/2]$ for $\alpha > 1/2$. But for any $\alpha < 1$ and $\gamma < 1/2$, we will show that

$$f(z) = z - \frac{1-\alpha}{n-\alpha} z^n \in T^*[\alpha] - R[\gamma]$$

for $n = n(\alpha,\gamma)$ sufficiently large. If $f \in R[\gamma]$, then

$$f*s_\gamma = z - \frac{1-\alpha}{n-\alpha} C(\gamma,n)z^n \in T^*[\gamma].$$

This is true if and only if

$$\frac{1-\alpha}{n-\alpha} C(\gamma,n) \le \frac{1-\gamma}{n-\gamma}$$

or, equivalently, if

$$C(\gamma,n) \le \left(\frac{n-\alpha}{n-\gamma}\right)\left(\frac{1-\gamma}{1-\alpha}\right) . \tag{4.2}$$

Since $C(\gamma,n) \to \infty$ for $\gamma < 1/2$ and the right hand side of (4.2) is bounded, the inequality is not true for n sufficiently large.

REMARK. $T^*[1] = \{z\} \subset R[0]$.

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