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# ON HAUSDORFF COMPACTIFICATIONS OF NON-LOCALLY COMPACT SPACES

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ABSTRACT. Let X be a completely regular, Hausdorff space and let R be the set of points in X which do not possess compact neighborhoods. Assume R is compact. If X has a compactification with a countable remainder, then so does the quotient X/R, and a countable compactification of X/R implies one for X-R. A characterization of when X/R has a compactification with a countable remainder is obtained. Examples show that the above implications cannot be reversed.

*KEY WORDS AND PHRASES. Countable remainders, compactifications, non-locally compact spaces, components of* $\beta X - X$.

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## 1. INTRODUCTION.

Let X be a completely regular, Hausdorff topological space. The question of characterizing when X has a Hausdorff compactification $\alpha X$, where $\alpha X - X$ is countably infinite, has been answered for the locally compact case by Magill [2] and for the case when $\alpha X = \beta X$ by Okuyama [4] (where $\beta X$ is the Stone-Cech compactification of X). In case X is an arbitrary completely regular space, no such characterization has been given. The purpose of this paper is to contribute results toward such a characterization.

Let R be the set of points in X which do not possess compact neighborhoods. Then for all compactifications $\alpha X$ of X, $R = Cl_{\alpha X}(\alpha X - X) \cap X$. (See [5].) Herein we observe that for compact R, a necessary condition for X to have a countable compactification is that X/R have one. The main theorem of this paper characterizes when X/R has a countable compactification.

## 2. CHARACTERIZATION OF $\alpha(X/R)$.

Throughout this paper all compactifications are Hausdorff compactifications. Let N denote the natural numbers. If R is a compact, non-empty subset of a completely regular space X and if X has a coutable compactification $\gamma X$, then a countable compactification of X/R can be obtained from $\gamma X$ by identifying R to a single point. It is readily verified that the resulting space is Hausdorff.

If $\alpha(X/R)$ is a countable compactification of X/R, then $\alpha(X/R)$ is also a countable compactification of X - R. Thus, we have the following:

THEOREM 1. If X is completely regular and R is compact, then each of the following conditions implies the next:

(A) X has a countable compactification;

(B) X/R has a countable compactification;

(C) $X - R$ has a countable compactification.

Examples will be provided to show that none of these implications can be reversed.

If $R$ is non-compact, then (A) no longer implies (C) as in Theorem 1. Let $X$ be the unit disc in the standard plane with a countable dense subset removed from the boundary. The remaining boundary points constitute $R$. Then, clearly, $X$ has a countable compactification but $X - R$, the open disc, has no countable compactification.

Let $Y = (\beta X - X) \cup R$.

THEOREM 2. Let $X$ be a completely regular Hausdorff space with $R$ compact and non-empty. Then the following are equivalent:

(A) $X/R$ has a countable compactification.

(B) $R$ is a $G_\delta$-set in $Y$ and components of $R$ are components of $Y$.

PROOF. (A) implies (B). Take $\{p_n | n \in N\} = \gamma(X/R) - X/R$, where $\gamma(X/R)$ is a countable compactification of $X/R$, and let $t_0$ be the canonical mapping of $X$ into $\gamma(X/R)$. Then $t_0$ has an extension $t$ which maps $\beta X$ onto $\gamma(X/R)$. We first show that $t$ carries $\beta X - X$ onto $\gamma(X/R) - X/R$. Since the restriction of $t$ to $X - R$ is a homeomorphism and $X - R$ is dense in $\beta X$ and in $\gamma(X/R)$, $t$ carries $Y$ onto $[\gamma(X/R) - X/R] \cup \{r\}$, where $r = t[R]$ (cf. Lemma 6.11 [1]). If $x \in R$ and $y \in \beta X - X$, then since $R$ is compact there exists a compact neighborhood $N_R$ of $R$ in $\beta X$ such that $y \notin N_R$. Set $N = N_R \cap X$. Since $R \subseteq N$, $t_0[N]$ is a neighborhood of $t(x) = r$ in $X/R$. Thus, there is a neighborhood $G$ in $\gamma(X/R)$ for which $t_0[N] = G \cap X/R$. If $N_y$ is any neighborhood of $y$ in $\beta X$, choose $z \in N_y \cap (X - N)$. Then $t(z) \notin G$ and it follows from the continuity of $t$ that $t(x) \neq t(y)$. Hence $t[\beta X - X] = \gamma(X/R) - X/R$.

Next, let $K_n = t^{-1}(p_n)$, for each $n \in N$. Evidently, $\beta X - X = \bigcup\{K_n | n \in N\}$. Since each $K_n$ is compact, the sets $Y - K_n$ are open in $Y$ and $R = \bigcap\{Y - K_n | n \in N\}$. Thus $R$ is a $G_\delta$-set in $Y$.

Let $C$ be a component of $R$ and let $C_1$ be a component of $Y$, where $C \subseteq C_1$. If $C \neq C_1$, choose $x \in C_1 - C$. Now there exists a continuous injection $f$ of $\{p_n \in N\}$ $\{r\}$ into the real numbers. (See [3]). But $f \circ t\, C_1$ must be connected and not a singleton, since $t[R] \neq t(x)$. This contradicts the fact that the image of $f$ is countable. Thus, $C = C_1$, so that components of $R$ are components of $Y$.

(B) implies (A). First we show that there exist sets $\{U_n | n \in N\}$ which are clopen in $Y$ such that $\bigcap\{U_n | n \in N\} = R$. Note that $Y$ is compact. Let $\{V_n | n \in N\}$ be open subsets of $Y$ satisfying $\bigcap\{V_n | n \in N\} = R$. For each $n \in N$, set $K_n = Y - V_n$. We assume that each $K_n \neq \emptyset$. Let $(x,r) \in K_n \times R$. Since $x$ and $r$ are in distinct quasi-components of $Y$, there exists a clopen neighborhood $W_n(x,r)$ of $r$ in $Y$, where $x \notin W_n(x,r)$. Now $\{W_n(x,r) | r \in R\}$ is an open covering of $R$ so that a finite subfamily $\{W_n(x,r_i) | i = 1,\ldots,p(x)\}$ covers $R$. Take $W_n(x) = \bigcup\{W_n(x,r_i) | i = 1,\ldots,p(x)\}$. Thus $W_n(x)$ is a clopen subset of $Y$. $R \subseteq W_n(x)$, and $x \notin W_n(x)$. Since $\{Y - W_n(x) | x \in K_n\}$ is an open cover of $K_n$, there is a finite subcover $\{Y - W_n(x_j) | j = 1,\ldots,q(n)\}$.

For each $n \in N$, let $U_n = \bigcap\{W_n(x_j) | j = 1,\ldots,q(n)\}$. Then each $U_n$ is a clopen subset of $Y$, $R \subseteq U_n$ and $K_n \subseteq Y - U_n$. Hence $R = \bigcap\{U_n | n \in N\}$.

Let $C_1 = Y - U_1$, and for $n > 1$, take $C_n = [Y - \bigcap\{U_i | i = 1,\ldots,n\}] - \bigcup\{C_i | i = 1,\ldots,n-1\}$. Then each $C_n$ is a clopen subset of $Y$ and $\beta X - X = \bigcup\{C_n | n \in N\}$.

Let $\sim$ be the equivalence relation in $\beta X$ which identifies each $C_n$ to a point and $R$ to a point. The projection of $\beta X$ onto $\beta X/\sim$ is denoted by $\Pi$.

For each $n \in N$, consider the point $\Pi[C_n]$ in $\beta X/\sim$. Now $\{C_n, Y - C_n\}$ is a partition of $Y$ into disjoint open sets. Thus, $C_n$ and $Y - C_n$ can be separated by open sets $U$ and $V$ in $\beta X$. Evidently, $\Pi[U]$ and $\Pi[V]$ are disjoint open subsets of $\beta X/\sim$. This shows that $\Pi[C_n]$ can be separated from any other point of $\beta X/\sim$. Since points of $\beta X - Y$ have compact $\beta X$ - neighborhoods in $\beta X - Y$, it follows that $\beta X/\sim$ is a compact Hausdorff space.

It remains to show that $X/R$ can be embedded in $\beta X/\sim$ in the desired manner. Let $i$ be the natural embedding of $X$ in $\beta X$ and let $p$ be the projection of $X$ onto $X/R$. Since $i$ is relation preserving, a continuous mapping $j$ of $X/R$ into $\beta X/\sim$ is induced such that $j \circ p = \Pi \circ i$. It follows that $j$ is also a closed mapping, hence an embedding of $X/R$ into $\beta X/\sim$ as desired. This completes the proof.

In [2] Magill shows that a locally compact space $X$ has a countable compactification if and only if $\beta X - X$ has infinitely many components. As an application of the proof of Theorem 2, the following is proven.

COROLLARY 3. Let $X$ be completely regular with $R$ compact. If $X$ has a countable compactification, then $\beta X - X$ has infinitely many components.

PROOF. Let $t$ be a continuous mapping of $\beta X$ onto $\alpha(X/R)$ which carries $\beta X - X$ onto $\alpha(X/R) - X/R$. Since the subspace $K = (\alpha(X/R) - X/R) \cup \{t(R)\}$ is compact and countable, it contains an open countable discrete subspace. Since $\alpha(X/R) - X/R$ contains infinitely many components of $K$, $Y$ must contain infinitely many components.

The converse of Corollary 3 is false when $X$ is not locally compact. Example (A) shows that $X/R$ can have a countable compactification, so that $\beta X - X$ has infinitely many components, but $X$ has no countable compactification. Example (A) also shows that condition (B) of Theorem 1 is not sufficient to insure that $X$ has a countable compactification when $R$ is compact.

EXAMPLE (A). Let S be the closed unit square in $R^2$, I be the unit interval, $L_0 = I \times \{0\}$, and, for $n \in N$, $L_n = I \times \{\frac{1}{n+1}\}$. For $X = S - \bigcup_{n \in N} L_n$, it is clear that X is not rim compact, and hence does not have a countable compactification (cf. [6]). Furthermore, $R = L_0$ and S is a compactification of X. The existence of a continuous surjection from $\beta X$ onto S which leaves X fixed and which carries $\beta X - X$ onto $S - X$ guarantees that condition (B) of Theorem 2 is satisfied. Hence X/R has a countable compactification.

The following example shows that for R non-empty and compact the implication of (C) by (B) of Theorem 1 cannot be reversed. It suffices to exhibit X, with R a singleton, where X - R has a countable compactification but X does not.

EXAMPLE (B). In the plane $R^2$ take
$X = [\{(x,y) | -1 < x < 1; -1 < y < 1\} \cup \{(1,0)\}] - \{(\frac{-n}{n+1}, 0) | n \in N\}$. Then $R = \{(1,0)\}$. Since X is not rim compact, it has no countable compactification. However, a countable compactification for X - R is obtained by adjoining the points $(\frac{-n}{n+1}, 0)$, for each $n \in N$, and taking the one-point compactification of the resulting space.

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