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# CHARACTERIZING SYMMETRIC DIAMETRICAL GRAPHS OF ORDER 12 AND DIAMETER 4

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A diametrical graph $G$ is said to be symmetric if $d(u,v)+d(v,\bar{u})=d(G)$ for all $u,v\in V(G)$, where $\bar{u}$ is the buddy of $u$. If moreover, $G$ is bipartite, then it is called an $S$-graph. It would be shown that the Cartesian product $K_2\times C_6$ is not only the unique $S$-graph of order 12 and diameter 4, but also the unique symmetric diametrical graph of order 12 and diameter 4.

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**1. Introduction.** Diametrical graphs are an interesting class of graphs. They have been investigated by quite many authors under different names. Some of them studied the properties of these graphs, see Mulder [5, 6], Parthasarathy and Nandakumar [7], and Göbel and Veldman [3]. Certain special classes of diametrical graphs have been classified and studied by others, see Göbel and Veldman [3] and Berman and Kotzig [2].

In that direction, Al-Addasi [1] has studied some properties of bipartite diametrical graphs of diameter 4 and constructed an $S$-graph of diameter 4 and order $4k$ for any $k\ge 2$. (Recall that the Cartesian product $G_1\times G_2$ of two graphs $G_1$ and $G_2$ is the graph whose vertex set consists of all ordered pairs $(x_1,x_2)$ where $x_1$ is a vertex of $G_1$ and $x_2$ is a vertex of $G_2$ such that two vertices $(x_1,x_2)$ and $(y_1,y_2)$ are adjacent exactly when either $x_1=y_1$ and $x_2y_2$ is an edge of $G_2$, or $x_1y_1$ is an edge of $G_1$ and $x_2=y_2$. Also recall that $K_2$, $C_6$ denote the complete graph with two vertices and the cycle of length 6, respectively.) For $k=3$, this $S$-graph is isomorphic to $K_2\times C_6$. In this paper, we show that up to isomorphism the graph $K_2\times C_6$ is not only the unique $S$-graph of order 12 and diameter 4 but also the unique symmetric diametrical graph of such an order and diameter.

For undefined notions and terminology, the reader is referred to Harary [4]. We consider only finite simple connected graphs with no loops or multiple edges. We would use $V(G)$, $E(G)$ to denote the vertex set and edge set of the graph $G$, respectively. The distance $d_G(u,v)$ (or simply $d(u,v)$) between two vertices $u$, $v$ in $G$, is the length of a shortest $(u,v)$-path in $G$, where the length of a path is the number of its edges. The diameter $d(G)$ of a graph $G$ is the maximal possible distance between two vertices in $G$. For any two vertices $u$, $v$ in $G$, the interval $I_G(u,v)$ is the set of vertices $\{w\in V(G): w$ lies on a shortest $(u,v)$-path in $G\}$, when no confusion can arise, we write $I(u,v)$, see Mulder [5]. The order of a graph $G$ is the number of vertices of $G$. The set of all vertices in a graph $G$, which are at distance $k$ from a vertex $v$ in $G$, is denoted by $N_k(v)$; the set of all neighbors $N_1(v)$ of $v$ is also denoted by $N(v)$. The degree of a vertex $v$ in a graph $G$, denoted by $\deg_G v$, is the number of vertices in

$N(v)$. If $A$ is a subset of the vertex set of a graph $G$, then $\langle A\rangle$ denotes the subgraph of $G$ induced by $A$. A subgraph of $G$ containing all vertices of $G$ is called a spanning subgraph of $G$. If $S$ is a subset of the vertex set of the graph $G$, then $G-S$ is the subgraph of $G$ induced by $V(G)-S$. If $G$ is connected while $G-S$ is not, then $S$ is called a vertex cut of $G$. If $B$ is a set of edges joining vertices from $G$ where $B\cap E(G)=\emptyset$, then the graph $G+B$ is obtained from $G$ by adding all edges in $B$.

Two vertices $u$ and $v$ of a nontrivial connected graph $G$ are said to be diametrical if $d(u,v)=d(G)$. A nontrivial connected graph $G$ is called diametrical if each vertex $v$ of $G$ has a unique diametrical vertex $\bar{v}$, the vertex $\bar{v}$ is called the buddy of $v$, see Mulder [5, 6]. A diametrical graph $G$ is called symmetric if $d(u,v)+d(v,\bar{u})=d(G)$ for all $u,v\in V(G)$, that is, $V(G)=I(u,\bar{u})$ for any $u\in V(G)$, see Göbel and Veldman [3]. A bipartite symmetric diametrical graph is called an $S$-graph, see Berman and Kotzig [2].

## 2. Symmetric diametrical graphs.

In this section, we introduce some properties of symmetric diametrical graphs that we will use in the sequel. The following two results are proved in Göbel and Veldman [3].

**Theorem 2.1.** *If $S$ is a vertex cut of a diametrical graph $G$, then no vertex of $S$ has degree $|S|-1$ in the induced subgraph $\langle S\rangle$ of $G$.*

The previous theorem implies that no vertex cut of a diametrical graph induces a complete subgraph. In particular, a diametrical graph has no cut vertex.

**Corollary 2.2.** *Every diametrical graph $G$ other than $K_2$ has no vertex of degree 1.*

**Proposition 2.3.** *Let $G$ be a diametrical graph of diameter $d$. Then $G$ is symmetric if and only if for each pair $u,v\in V(G)$ with $v\in N_i(u)$, we have $\bar{v}\in N_{d-i}(u)$.*

**Proof.** Let $G$ be symmetric and let $u,v\in V(G)$. Then $d(v,u)+d(u,\bar{v})=d$. If $v\in N_1(u)$, then $d(u,\bar{v})=d-i$, that is, $\bar{v}\in N_{d-i}(u)$.

Conversely, assume that $\bar{v}\in N_{d-i}(u)$ whenever $u,v\in V(G)$ with $v\in N_1(u)$. Let $x,y\in V(G)$. Then $x\in N_i(u)$ for some $i\in\{0,1,\dots,d\}$ and, by assumption, $\bar{x}\in N_{d-i}(y)$. Hence $d(x,y)+d(y,\bar{x})=i+d-i=d$. Thus $G$ is symmetric. □

**Corollary 2.4.** *If $G$ is a symmetric diametrical graph of diameter $d$ and $u\in V(G)$, then for each $0\le i\le d$, $N_{d-i}(u)=\{\bar{v}:v\in N_i(u)\}$. And hence $|N_{d-i}(u)|=|N_i(u)|$.*

**Proof.** Since $G$ is symmetric, $v\in N_i(u)$ if and only if $\bar{v}\in N_{d-i}(u)$. Hence $N_{d-i}(u)=\{\bar{v}:v\in N_i(u)\}$. Also, since the buddy is unique, $|N_{d-i}(u)|=|N_i(u)|$. □

A diametrical graph is called harmonic if $\bar{u}\bar{v}\in E(G)$ whenever $uv\in E(G)$. The result of Theorem 2.5 is shown in Göbel and Veldman [3].

**Theorem 2.5.** *Every symmetric diametrical graph is harmonic.*

## 3. Symmetric diametrical graphs of order 12 and diameter 4

**Theorem 3.1.** *In a symmetric diametrical graph $G$ of order 12 and diameter 4, there is no vertex of degree 2.*

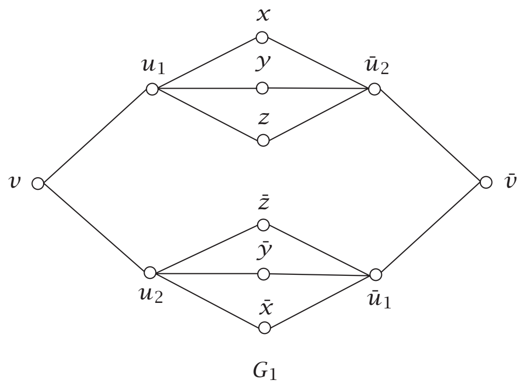


FIGURE 3.1

**PROOF.** Assume to the contrary that $G$ has a vertex $v$ of degree 2. Let $N(v) = \{u_1, u_2\}$. By Corollary 2.4, $N_3(v) = \{\bar{u}_1, \bar{u}_2\}$. Hence $N_2(v)$ contains exactly six vertices. Since $N(v)$ is a vertex cut of $G$, by Theorem 2.1, the vertices $u_1$ and $u_2$ are nonadjacent. The same holds for $\bar{u}_1$ and $\bar{u}_2$. Clearly, $\bar{x} \in N_2(v)$ whenever $x \in N_2(v)$. So $N_2(v)$ consists of three pairs of diametrical vertices. Since $d(G) = 4$, each of the two vertices $u_1$ and $u_2$ cannot be adjacent to more than three vertices of $N_2(v)$. But every vertex of $N_2(v)$ is adjacent to at least one of the two vertices $u_1$ and $u_2$, so $u_1$ is adjacent to exactly three vertices of $N_2(v)$ and $u_2$ is adjacent to the other three. If $x$, $y$, and $z$ are the vertices from $N_2(v)$ adjacent to $u_1$, then $\bar{x}$, $\bar{y}$, and $\bar{z}$ are those adjacent to $u_2$. By Theorem 2.5, the vertex $\bar{u}_1$ is adjacent to $\bar{x}$, $\bar{y}$, and $\bar{z}$; while $\bar{u}_2$ is adjacent to $x$, $y$, and $z$. So we get the spanning subgraph $G_1$ of $G$ depicted in Figure 3.1. For all $u \in \{x, y, z\}$ and all $w \in \{\bar{x}, \bar{y}, \bar{z}\}$, the vertices $u$ and $w$ are not adjacent; for otherwise, $d(u_1, \bar{u}_1) \le 3$. Hence $G_1 \subseteq G \subseteq G_1 + \{xy, xz, yz, \bar{x}\bar{y}, \bar{x}\bar{z}, \bar{y}\bar{z}\}$. This implies that $d(x, \bar{z}) = d(x, \bar{y}) = d(x, \bar{x}) = 4$, a contradiction. □

**THEOREM 3.2.** *A symmetric diametrical graph $G$ of order 12 and diameter 4 contains no vertex of degree 4.*

**PROOF.** Assume to the contrary that $G$ has a vertex $v$ of degree 4, and let $N(v) = \{u_1, u_2, u_3, u_4\}$. By Corollary 2.4, $|N_3(v)| = 4$ and hence $|N_2(v)| = 2$. Clearly, $N_2(v)$ consists of a vertex and its buddy, say $N_2(v) = \{x, \bar{x}\}$. Then any vertex of $N(v)$ is adjacent to at most one of the two vertices $x$, $\bar{x}$. But, by Corollary 2.2 and Theorem 3.1, each vertex of $N(v)$ has degree at least 3. Then $\deg_{\langle N(v)\rangle} z \ge 1$ for any $z \in N(v)$. Now, since $x_2 \in N_2(v)$, there is a vertex, say $u_1$, of $N(v)$ adjacent to $x$. But $u_1$ has a neighbor in $N(v)$, say $u_2$. By Theorem 2.5, the vertex $\bar{u}_1$ is adjacent to both $\bar{x}$ and $\bar{u}_2$. Thus $u_2$ is not adjacent to $\bar{x}$, because $d(G) = 4$. So, since $G$ is symmetric, that is, $V(G) = I(v, \bar{v})$, the vertex $u_2$ is adjacent to $x$, and hence $\bar{u}_2$ is adjacent to $\bar{x}$. Since $\bar{x} \in N_2(v)$, then at least one of $u_3$, $u_4$, say $u_4$, is adjacent to $\bar{x}$. Then $\bar{u}_4$ is adjacent to $x$. The vertex $u_3$ is adjacent to exactly one of the two vertices $x$ and $\bar{x}$, so we distinguish two cases.

**CASE 1.** If $u_3$ is adjacent to $\bar{x}$. Then $\bar{u}_3$ is adjacent to $x$. Since $d(x, \bar{x}) = 4$, the vertex $u_3$ is not adjacent to any of $u_1$, $u_2$. So, by Theorem 3.1, the vertex $u_3$ must be adjacent to $u_4$, and hence $\bar{u}_3$ is adjacent to $\bar{u}_4$. It is obvious that any additional

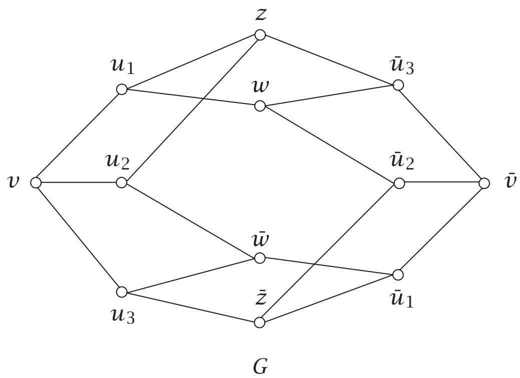


FIGURE 3.2

edge in $N(v)$ or $N_3(v)$ would decrease the distance between $x$ and $\bar{x}$ to 3. Then $d(u_1,\bar{u}_1) = d(u_1,\bar{u}_2) = 4$, contradicting $G$ is diametrical.

**CASE 2.** If $u_3$ is adjacent to $x$. Then $\bar{u}_3$ is adjacent to $\bar{x}$. By Theorem 3.1, the vertex $u_4$ has at least one neighbor in $N(v)$. But then $d(x,\bar{x}) = 3$, a contradiction.

Therefore, $G$ cannot contain a vertex of degree 4. □

**THEOREM 3.3.** *A symmetric diametrical graph $G$ of order* 12 *and diameter* 4 *is isomorphic to $K_2 \times C_6$.*

**PROOF.** If $G$ has a vertex $v$ of degree greater than 4, then, by Corollary 2.4, $|N_3(v)| > 4$ and hence $|V(G)| = 1 + |N(v)| + |N_2(v)| + |N_3(v)| + 1 > 12$, a contradiction. So $G$ has no vertex of degree greater than 4. Then, by the previous theorem, every vertex of $G$ has degree at most 3. But from Corollary 2.2 and Theorem 3.1, every vertex of $G$ has degree at least 3. Hence $G$ is 3-regular. Pick a vertex $v$ from $V(G)$ and let $N(v) = \{u_1,u_2,u_3\}$. Then $|N_2(v)| = 4$. Since $N(v)$ is a vertex cut of $G$, then by Theorem 2.1, each vertex of $N(v)$ has at most one neighbor in $N(v)$. Hence $\langle N(v)\rangle$ has at most one edge. We proceed by contradiction to show that $E(\langle N(v)\rangle) = \emptyset$. So, assume that there is an edge, say $u_1u_2$, in $\langle N(v)\rangle$ and hence $\bar{u}_1\bar{u}_2 \in E(G)$. Then $u_1$ has exactly one neighbor, say $x$, in $N_2(v)$. Then, by Theorem 2.5, the vertex $\bar{u}_1$ is adjacent to $\bar{x}$. Similarly, $u_2$ has exactly one neighbor $y$ in $N_2(v)$. The vertex $y$ is different from $\bar{x}$ because otherwise $d(x,\bar{x}) \le 3$, which is impossible. Also $y$ is different from $x$ because $G$ is 3-regular and each of the four vertices in $N_2(v)$ has at least one neighbor in $N(v)$. Thus $N_2(v) = \{x,\bar{x},y,\bar{y}\}$. By Theorem 2.5, $\bar{u}_2\bar{y} \in E(G)$. Since $G$ is 3-regular and each of $u_1$, $u_2$ has already three neighbors, the neighbor of each of $\bar{x}$, $\bar{y}$ from $N(v)$ is $u_3$. Then, again by Theorem 2.5, the edges $x\bar{u}_3$, $y\bar{u}_3$ belong to $E(G)$. Then, the 3-regularity of $G$ and Theorem 2.5 imply that either $xy, \bar{x}\bar{y} \in E(G)$ or $x\bar{y}, \bar{x}y \in E(G)$. But now we have either $d(x,\bar{y}) = d(x,\bar{x})$ or $d(x,\bar{x}) = 3$, respectively, a contradiction in any case. Therefore, we deduce that $\langle N(v)\rangle$ has no edges. Then each of $u_1$, $u_2$, $u_3$ has two neighbors from $N_2(v)$. If we let $N(u_i,v)$ denote the set of neighbors of $u_i$ from $N_2(v)$, (for $i = 1,2,3$), then $\{N(u_1,v),N(u_2,v),N(u_3,v)\} \subseteq \{\{x,y\},\{x,\bar{y}\},\{\bar{x},y\},\{\bar{x},\bar{y}\}\}$. Then there exist $i,j \in \{1,2,3\}$ with $i \ne j$ such that $N(u_i,v) \cap N(u_j,v) = \emptyset$, and hence $|N(u_k,v) \cap N(u_i,v)| = |N(u_k,v) \cap N(u_j,v)| =$

1, where $\{k\} = \{1,2,3\} - \{i,j\}$. Then $G$ is the graph depicted in Figure 3.2 where $\{z,\bar{z},w,\bar{w}\} = \{x,\bar{x},y,\bar{y}\}$. Now it is obvious that $G$ is isomorphic to the Cartesian product $K_2 \times C_6$. □

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