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STRICT POSITIVE DEFINITENESS ON SPHERES VIA DISK POLYNOMIALS

V. A. MENEGATTO and A. P. PERON

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We characterize complex strictly positive definite functions on spheres in two cases, the unit sphere of $\mathbb{C}^q$, $q \geq 3$, and the unit sphere of the complex ℓ^2. The results depend upon the Fourier-like expansion of the functions in terms of disk polynomials and, among other things, they enlarge the classes of strictly positive definite functions on real spheres studied in many recent papers.

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1. Introduction. This paper is concerned with positive definite kernels that are useful to perform interpolation on spheres. Its main purpose is to study strictly positive definite kernels on the unit sphere Ω_{2q} of $\mathbb{C}^q$, $q \geq 2$, and the unit sphere Ω_∞ of ℓ^2. Specifically, we will provide an elementary description of strict positive definiteness of kernels which are inner-product dependent, that is, of the form

$$(z,w) \in \Omega_{2q} \times \Omega_{2q} \longmapsto f(\langle z,w\rangle), \tag{1.1}$$

for some continuous complex function f (the *shape function*) defined at least on the disk $B_2 := \{z \in \mathbb{C} : |z| \leq 1\}$. Hereafter, $\langle \cdot,\cdot \rangle$ will denote the usual inner product in $\mathbb{C}^q$ or ℓ^2.

A shape function whose associated kernel is positive definite on Ω_{2q} will be termed a *positive definite function* on Ω_{2q}. Thus, a positive definite function f on Ω_{2q} is a continuous function such that

$$\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} f(\langle \xi_\mu, \xi_\nu \rangle) \geq 0 \tag{1.2}$$

for $N \geq 1$, $\{\xi_1,\ldots,\xi_N\} \subset \Omega_{2q}$, and $\{c_1,\ldots,c_N\} \subset \mathbb{C}$. The reader can refer to [1] for the basic facts about positive definite and related kernels. If N has been fixed, we say that the positive definite function f is *strictly positive definite* of order N on Ω_{2q} if the inequality above is strict when the points ξ_μ are pairwise distinct and $\sum_{\mu=1}^{N} |c_\mu| > 0$. Finally, f is *strictly positive definite* on Ω_{2q} if it is strictly positive definite of order N on Ω_{2q}, $N = 1,2,\ldots$.

If we are given distinct points $\xi_1,\ldots,\xi_N$ on Ω_{2q}, complex numbers $\lambda_1,\ldots,\lambda_N$, and a strictly positive definite function f of order at least N on Ω_{2q}, then there is a unique function of the form

$$\xi \in \Omega_{2q} \longmapsto \sum_{\nu=1}^{N} \overline{c_\nu} f(\langle \xi, \xi_\nu \rangle) \tag{1.3}$$

that interpolates the data $\{(\xi_\mu, \lambda_\mu) : \mu = 1, \ldots, N\}$. This interpolation procedure is a complex version of what is often called *spherical radial basis function interpolation* (see [4, 6, 8, 12]). In the latter, the ξ_μ belong to a sphere in some $\mathbb{R}^m$ and the function f used in the generation of solutions is usually real valued. The method is well established and has many applications in the handling of problems of global nature that emerge in several areas such as meteorology, oceanography, satellite-related sciences, and so forth, where the unit sphere in $\mathbb{R}^3$ is the standard model.

A strictly positive definite function on Ω_{2q} is also strictly positive definite on every real sphere embedded in it, in the sense considered in the references quoted above. The results in this paper will then provide a concise way of constructing complex strictly positive definite functions on the real spheres. As a matter of fact, since the definitions of positive definiteness used here encompass those used in the real setting, our results will amplify the classes of positive definite functions that can be used to perform interpolation on the real spheres.

The outline of the paper is as follows. In Section 2, we collect basic results about positive definite functions on spheres and disk polynomials. This is crucial in the paper, because the analysis of strict positive definiteness is based on the expansion of positive definite functions on Ω_{2q} in terms of certain disk polynomials. Section 3 contains two technical results, one of them being an alternative to the concept of strict positive definiteness on Ω_{2q}. Section 4 contains the main part of the paper, including characterizations of strict positive definiteness on Ω_{2q}, $q \geq 3$.

2. Prerequisites.

The strict positive definiteness of a positive definite function on Ω_{2q}, $q < \infty$, can be detected by looking at the series expansion of the function in terms of certain disk polynomials. This section contains the basic facts about them along with some basic properties needed in the sequel.

Positive definite functions on Ω_{2q} are representable in the form

$$f(z) = \sum_{(m,n)\in\mathbb{Z}_+^2} a_{m,n}(f) R_{m,n}^{q-2}(z), \quad z \in B_2, \tag{2.1}$$

in which $a_{m,n}(f) \geq 0$ for all m and n and $\sum_{(m,n)\in\mathbb{Z}_+^2} a_{m,n}(f) < \infty$. The symbol $R_{m,n}^{q-2}$ stands for the disk polynomial of degree $m+n$ associated to the integer $q-2$ when q is finite while $R_{m,n}^{\infty}(z) = z^m \overline{z}^n$ (see [7, 11]). The *disk polynomial* of degree $m+n$ in x and y associated with a nonnegative real number α is the polynomial $R_{m,n}^{\alpha}$ given by

$$R_{m,n}^{\alpha}(z) := r^{|m-n|} e^{i(m-n)\theta} R_{m\wedge n}^{(\alpha,|m-n|)}(2r^2-1), \quad z = re^{i\theta} = x+iy, \tag{2.2}$$

where $R_{m\wedge n}^{(\alpha,|m-n|)}$ is the usual Jacobi polynomial of degree $m \wedge n := \min\{m,n\}$ associated with the numbers α and $|m-n|$. The normalization for the Jacobi polynomials adopted here is $R_{m\wedge n}^{(\alpha,|m-n|)}(1) = 1$. The expansion (2.1) is then possible due to the following two facts (see [2, 9]): $R_{m,n}^{q-2}$ is a zonal surface harmonic of degree $m+n$ in $2q$ real variables in the classical sense and the set $\{R_{m,n}^{\alpha} : 0 \leq m,\ n < \infty\}$ is a complete orthogonal set in $L^2(B_2, dw_\alpha)$, where dw_α is the measure on B_2 given by

$$dw_\alpha(z) = \frac{\alpha+1}{\pi}(1-x^2-y^2)^\alpha\,dx\,dy, \quad z = x+iy. \tag{2.3}$$

The following properties of disk polynomials are straightforward [9].

Lemma 2.1. *Let m and n be nonnegative integers and α a nonnegative real. The following properties hold:*

(i) $R^{\alpha}_{m,n}(e^{i\varphi}z) = e^{i(m-n)\varphi}R^{\alpha}_{m,n}(z)$, $\varphi \in [0,2\pi)$, $z \in B_2$;

(ii) $R^{\alpha}_{m,n}(1) = 1$;

(iii) $|R^{\alpha}_{m,n}(z)| \le 1$, $z \in B_2$;

(iv) $R^{\alpha}_{m,n}(\overline{z}) = \overline{R^{\alpha}_{m,n}(z)} = R^{\alpha}_{n,m}(z)$, $z \in B_2$;

(v) $R^{\alpha}_{m,n}$ *is an even function if $m-n$ is even, and an odd function otherwise.*

Next, we list the less elementary properties of disk polynomials to be used in this work. The first one is the so-called addition formula (see [9, 14]), which is quoted here in a version adapted to our purposes.

Lemma 2.2. *If m and n are nonnegative integers, q is an integer at least 3, $\theta_1,\theta_2 \in [0,\pi/2]$, $\phi_1,\phi_2 \in [0,2\pi)$, and $w \in B_2$, then*

$$\begin{aligned} &R^{q-2}_{m,n}(\cos\theta_1\cos\theta_2 e^{i(\phi_1-\phi_2)} + \sin\theta_1\sin\theta_2 w) \\ &\qquad = \sum_{k=0}^{m}\sum_{l=0}^{n} b^{k,l}_{m,n,q-2} Q^{k,l}_{m,n}(\theta_1,\phi_1)\overline{Q^{k,l}_{m,n}(\theta_2,\phi_2)}R^{q-3}_{k,l}(w), \end{aligned} \tag{2.4}$$

in which

$$Q^{k,l}_{m,n}(\theta,\phi) := (\sin\theta)^{k+l}R^{q-2+k+l}_{m-k,n-l}(\cos\theta e^{i\phi}), \quad (\theta,\phi) \in [0,\pi/2]\times[0,2\pi), \tag{2.5}$$

and the constants $b^{k,l}_{m,n,q-2}$ are all positive.

Before stating the next lemma, we need some additional notation. We need to deal with elements from the space Π^q composed of complex polynomials in the independent complex variables $z \in \mathbb{C}^q$ and $\overline{z}$. A typical element of Π^q is of the form

$$p(z,\overline{z}) = \sum_{|r|\le m}\sum_{|s|\le n} a_{r,s}z^r\overline{z}^s, \quad z \in \mathbb{C}^q,\ a_{r,s} \in \mathbb{C},\ r,s \in \mathbb{Z}^q_+,\ m,n \in \mathbb{Z}_+, \tag{2.6}$$

where standard multi-index notation is in force. The subset of Π^q, composed of polynomials with the property that

$$p(\lambda z,\lambda\overline{z}) = \lambda^m\overline{\lambda}^n p(z,\overline{z}), \quad \lambda \in \mathbb{C}, \tag{2.7}$$

becomes a subspace. The elements of this subspace which are in the kernel of the complex Laplacian also form a subspace of Π^q. The set of restrictions of elements of this subspace to Ω_{2q} will be denoted by $\mathcal{H}^q_{m,n}$. It is also a vector space over $\mathbb{C}$ and its dimension will be written as $N(q;m,n)$.

Lemma 2.3 is the summation formula for disk polynomials (see [9, 14]).

Lemma 2.3. *Let q be an integer at least 2 and $\{Y^q_1,\dots,Y^q_{N(q;m,n)}\}$ an orthonormal basis of $\mathcal{H}^q_{m,n}$. Then*

$$R^{q-2}_{m,n}(\langle\xi,\zeta\rangle) = \frac{\omega_{2q}}{N(q;m,n)}\sum_{k=1}^{N(q;m,n)} Y^q_k(\xi)\overline{Y^q_k(\zeta)}, \quad \xi,\zeta \in \Omega_{2q}, \tag{2.8}$$

where ω_{2q} denotes the surface area of Ω_{2q}.

For further information on disk polynomials we suggest [2, 3, 13, 14, 16] and the references therein.

3. Basic technical results. In this section, we present a crucial formulation of strict positive definiteness (SPD) on Ω_{2q} to be used in the proofs of the main results of the paper. Before that, we formulate two elementary technical lemmas.

Lemma 3.1. *Let q be an integer at least 2 and ζ an element of Ω_{2q}. Then, every element ξ of Ω_{2q} is uniquely representable in the form*

$$\xi = te^{i\varphi}\zeta + \sqrt{1-t^2}\,\xi', \quad 0 \le t \le 1,\ \varphi \in [0,2\pi), \tag{3.1}$$

in which ξ' is an element of Ω_{2q} in the hyperplane of $\mathbb{C}^q$ orthogonal to ζ. The coefficient $te^{i\varphi}$ is precisely $\langle \xi, \zeta\rangle$.

Lemma 3.2. *If $\lambda_1,\dots,\lambda_n$ are n distinct complex numbers and A is a subset of $\mathbb{C}$ with an accumulation point, then the functions*

$$z \in A \longmapsto \exp(\lambda_k z), \quad k = 1,\dots,n \tag{3.2}$$

form a linearly independent set.

Proof. See [5, page 87]. □

Before stating our first result, we introduce a new terminology. It comes from the fact that the strict positive definiteness of a positive definite function f on Ω_{2q} does not depend on the actual values of the coefficients $a_{m,n}(f)$ (see [10]). We say that a subset K of $\mathbb{Z}_+^2$ *induces SPD* of order N on Ω_{2q} if every positive definite function f for which

$$K_q(f) := \{(m,n) : a_{m,n}(f) > 0\} = K \tag{3.3}$$

is strictly positive definite of order N on Ω_{2q}.

Theorem 3.3 below is the major result in this section and it yields different formulations for the concept of SPD on Ω_{2q}. The results in Section 4 will only require the fact that condition (i) implies condition (iv). The other implications in Theorem 3.3 are of independent interest at this moment.

Theorem 3.3. *Let K be a subset of $\mathbb{Z}_+^2$, q a positive integer at least 3, f a positive definite function on Ω_{2q} satisfying $K_q(f) = K$, $\xi_1,\dots,\xi_N$ distinct points on Ω_{2q}, and $c_1,\dots,c_N$ complex numbers. The following assertions are equivalent:*

(i) $\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} f(\langle \xi_\mu, \xi_\nu\rangle) = 0$;

(ii) $\sum_{\mu=1}^{N} c_\mu Y^q(\xi_\mu) = 0$, $Y^q \in \mathcal{H}_{m,n}^q$, $(m,n) \in K$;

(iii) $\sum_{\mu=1}^{N} c_\mu R_{m,n}^{q-2}(\langle \xi_\mu, \eta\rangle) = 0$, $(m,n) \in K$, $\eta \in \Omega_{2q}$;

(iv) $\sum_{\mu=1}^{N} c_\mu Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu) R_{k,l}^{q-3}(\langle \xi'_\mu, \eta'\rangle) = 0$, $(m,n) \in K$, $0 \le k \le m$, $0 \le l \le n$, $\eta' \in \Omega_{2q-2}$.

Proof. The equivalence among (i), (ii), and (iii) follows directly from the definitions and Lemma 2.3. A direct application of Lemma 2.2 reveals that (iv) implies (iii). Next, we use induction to prove that (iii) implies (iv). We use Lemma 3.1 to write

$$\xi_\mu = \cos\theta_\mu e^{i\varphi_\mu}\varepsilon_1 + \sin\theta_\mu \xi'_\mu, \quad \theta_\mu \in [0,\pi/2],\ \varphi_\mu \in [0,2\pi),\ \mu = 1,\dots,N. \tag{3.4}$$

Given $\eta \in \Omega_{2q}$ written as $\eta = \cos\theta e^{i\psi}\varepsilon_1 + \sin\theta\eta'$, $\theta \in [0,\pi/2]$, $\psi \in [0,2\pi)$, Lemma 2.2 yields

$$\begin{aligned}\sum_{\mu=1}^{N} c_\mu R_{m,n}^{q-2}(\langle \xi_\mu,\eta\rangle) &= \sum_{\mu=1}^{N} c_\mu \sum_{k=0}^{m}\sum_{l=0}^{n} b_{m,n,q-2}^{k,l} Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu)\overline{Q_{m,n}^{k,l}(\theta,\psi)} R_{k,l}^{q-3}(\langle \xi'_\mu,\eta'\rangle)\\ &= \sum_{k=0}^{m}\sum_{l=0}^{n} b_{m,n,q-2}^{k,l}\overline{Q_{m,n}^{k,l}(\theta,\psi)}\sum_{\mu=1}^{N} c_\mu Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu) R_{k,l}^{q-3}(\langle \xi'_\mu,\eta'\rangle)\\ &= b_{m,n,q-2}^{0,0}\overline{R_{m,n}^{q-2}(\cos\theta e^{i\psi})}\sum_{\mu=1}^{N} c_\mu R_{m,n}^{q-2}(\cos\theta_\mu e^{i\varphi_\mu}) R_{0,0}^{q-3}(\langle \xi'_\mu,\eta'\rangle)\\ &\quad+\sum_{k=0}^{m}\sum_{\substack{l=0\\ l+k\neq 0}}^{n} b_{m,n,q-2}^{k,l}\overline{Q_{m,n}^{k,l}(\theta,\psi)}\sum_{\mu=1}^{N} c_\mu Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu) R_{k,l}^{q-3}(\langle \xi'_\mu,\eta'\rangle).\end{aligned} \tag{3.5}$$

Hence, if condition (iii) holds,

$$\begin{aligned}&b_{m,n,q-2}^{0,0}\overline{R_{m,n}^{q-2}(\cos\theta e^{i\psi})}\sum_{\mu=1}^{N} c_\mu R_{m,n}^{q-2}(\cos\theta_\mu e^{i\varphi_\mu}) R_{0,0}^{q-3}(\langle \xi'_\mu,\eta'\rangle)\\ &\quad+\sum_{k=0}^{m}\sum_{\substack{l=0\\ l+k\neq 0}}^{n} b_{m,n,q-2}^{k,l}\overline{Q_{m,n}^{k,l}(\theta,\psi)}\sum_{\mu=1}^{N} c_\mu Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu) R_{k,l}^{q-3}(\langle \xi'_\mu,\eta'\rangle)=0,\end{aligned} \tag{3.6}$$

for $(m,n) \in K$, $\theta \in [0,\pi/2]$, $\psi \in [0,2\pi)$, and $\eta' \in \Omega_{2q-2}$. Choosing $\theta = \psi = 0$, we obtain

$$b_{m,n,q-2}^{0,0}\overline{R_{m,n}^{q-2}(1)}\sum_{\mu=1}^{N} c_\mu R_{m,n}^{q-2}(\cos\theta_\mu e^{i\varphi_\mu}) R_{0,0}^{q-3}(\langle \xi'_\mu,\eta'\rangle) = 0, \quad (m,n)\in K,\ \eta'\in\Omega_{2q-2}. \tag{3.7}$$

Since each coefficient $b_{m,n,q-2}^{0,0}$ is positive, this reduces to

$$\sum_{\mu=1}^{N} c_\mu R_{m,n}^{q-2}(\cos\theta_\mu e^{i\varphi_\mu}) = 0, \quad (m,n)\in K. \tag{3.8}$$

This equation corresponds to condition (iv) in the case $k = l = 0$. Next, we assume that condition (iv) holds when $k+l < p$ (>0) and show it holds when $k+l = p$. Recalling (3.6) and using the induction hypotheses, we see that

$$\begin{aligned}0 &= \sum_{k=0}^{m}\sum_{\substack{l=0\\ l+k\geq p}}^{n} b_{m,n,q-2}^{k,l}\overline{Q_{m,n}^{k,l}(\theta,\psi)}\sum_{\mu=1}^{N} c_\mu Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu) R_{k,l}^{q-3}(\langle \xi'_\mu,\eta'\rangle)\\ &= (\sin\theta)^p \sum_{k=0}^{m}\sum_{\substack{l=0\\ l+k\geq p}}^{n} b_{m,n,q-2}^{k,l}(\sin\theta)^{k+l-p} R_{m-k,n-l}^{q-2+k+l}(\cos\theta e^{-i\psi})\\ &\quad\times\sum_{\mu=1}^{N} c_\mu Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu) R_{k,l}^{q-3}(\langle \xi'_\mu,\eta'\rangle),\end{aligned} \tag{3.9}$$

for $(m,n)\in K$, $\theta\in[0,\pi/2]$, $\psi\in[0,2\pi)$, and $\eta'\in\Omega_{2q-2}$. Hence,

$$\begin{aligned}&\sum_{k=0}^{m}\sum_{\substack{l=0\\ l+k\geq p}}^{n} b_{m,n,q-2}^{k,l}(\sin\theta)^{k+l-p}R_{m-k,n-l}^{q-2+k+l}(\cos\theta e^{-i\psi})\\ &\qquad\times\sum_{\mu=1}^{N}c_\mu Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu)R_{k,l}^{q-3}(\langle\xi'_\mu,\eta'\rangle)=0,\end{aligned}\tag{3.10}$$

for $(m,n)\in K$, $\theta\in(0,\pi/2]$, $\psi\in[0,2\pi)$, and $\eta'\in\Omega_{2q-2}$. Due to Lemma 2.1, this reduces to

$$\begin{aligned}&\sum_{k=0}^{m}\sum_{\substack{l=0\\ k+l\geq p}}^{n} b_{m,n,q-2}^{k,l}(\sin\theta)^{k+l-p}e^{-i(m-n-k+l)\psi}R_{m-k,n-l}^{q-2+k+l}(\cos\theta)\\ &\qquad\times\sum_{\mu=1}^{N}c_\mu Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu)R_{k,l}^{q-3}(\langle\xi'_\mu,\eta'\rangle)=0\end{aligned}\tag{3.11}$$

for $(m,n)\in K$, $\theta\in(0,\pi/2]$, $\psi\in[0,2\pi)$, and $\eta'\in\Omega_{2q-2}$. Letting $\theta\to0^+$, we obtain

$$\sum_{k=0}^{m}\sum_{\substack{l=0\\ l+k=p}}^{n} b_{m,n,q-2}^{k,l}e^{-i(m-n-k+l)\psi}R_{m-k,n-l}^{q-2+k+l}(1)\sum_{\mu=1}^{N}c_\mu Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu)R_{k,l}^{q-3}(\langle\xi'_\mu,\eta'\rangle)=0,\tag{3.12}$$

for $(m,n)\in K$, $\psi\in[0,2\pi)$, and $\eta'\in\Omega_{2q-2}$. Next, we use Lemma 3.2 to see that, for each $(m,n)\in K$, the functions

$$\psi\longmapsto e^{-i(m-n-k+l)\psi},\quad k=0,\dots,m,\ l=0,\dots,n,\ k+l=p\tag{3.13}$$

form a linearly independent set over $[0,2\pi)$. It follows that

$$b_{m,n,q-2}^{k,l}R_{m-k,n-l}^{q-2+k+l}(1)\sum_{\mu=1}^{N}c_\mu Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu)R_{k,l}^{q-3}(\langle\xi'_\mu,\eta'\rangle)=0,\quad (m,n)\in K,\ k+l=p,\tag{3.14}$$

that is,

$$\sum_{\mu=1}^{N}c_\mu Q_{m,n}^{k,l}(\theta_\mu,\varphi_\mu)R_{k,l}^{q-3}(\langle\xi'_\mu,\eta'\rangle)=0,\quad (m,n)\in K,\ k+l=p.\tag{3.15}$$

The proof is now complete. □

4. Main results. In this section, we use Theorem 3.3 to obtain an elementary characterization of SPD of all orders on Ω_{2q}, $q\geq3$. We begin with a necessary condition for SPD proved in [10].

Lemma 4.1. *Let q be an element of $\{2,3,\dots\}\cup\{\infty\}$, K a subset of $\mathbb{Z}_+^2$, and N a positive integer. If K induces SPD of order N on Ω_{2q} then $\{m-n:(m,n)\in K\}$ intersects the sets $N\mathbb{Z}+j$, $j=0,1,\dots,N-1$.*

Theorem 4.2. *Let q be an integer at least 3 and K a subset of $\mathbb{Z}_+^2$. Then K induces SPD of all orders on Ω_{2q} if and only if the set $\{m-n : (m,n) \in K\}$ contains infinitely many even and infinitely many odd integers.*

Proof. The condition is certainly necessary, by Lemma 4.1. Conversely, let N be a positive integer, $\xi_1,\dots,\xi_N$ distinct points on Ω_{2q}, and f a positive definite function on Ω_{2q} such that $K_q(f) = K$. We will show that the matrix with entries $(f(\langle \xi_\mu, \xi_\nu\rangle))$ is positive definite under the hypothesis that $\{m-n : (m,n) \in K\}$ contains infinitely many even and infinitely many odd integers. To do that we will assume that

$$\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} f(\langle \xi_\mu, \xi_\nu\rangle) = 0, \tag{4.1}$$

and will show that $c_\mu = 0$ for all μ. Let $j \in \{1,\dots,N\}$ and choose an orthogonal transformation such that $T_j(\xi_j) = (0,1,0,\dots,0)$. Next, use Lemma 3.1 to write

$$T_j(\xi_\mu) = \cos\theta_\mu e^{i\psi_\mu}\varepsilon_1 + \sin\theta_\mu \xi'_\mu, \quad \theta_\mu \in [0,\pi/2],\ \psi_\mu \in [0,2\pi),\ \mu = 1,\dots,N, \tag{4.2}$$

where $\xi'_\mu \in \Omega_{2q-2}$, $\mu = 1,\dots,N$. Due to our choice of T_j, $\theta_j = \pi/2$ and $\xi'_j = T_j(\xi_j)$. Since

$$\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} f(\langle T_j(\xi_\mu), T_j(\xi_\nu)\rangle) = \sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} f(\langle \xi_\mu, \xi_\nu\rangle) = 0, \tag{4.3}$$

we may use Theorem 3.3 to conclude that

$$\sum_{\mu=1}^{N} c_\mu Q_{m,n}^{k,l}(\theta_\mu, \varphi_\mu) R_{k,l}^{q-3}(\langle \xi'_\mu, \eta'\rangle) = 0, \quad (m,n) \in K,\ 0 \le k \le m,\ 0 \le l \le n,\ \eta' \in \Omega_{2q-2}. \tag{4.4}$$

In particular,

$$\sum_{\mu=1}^{N} c_\mu (\sin\theta_\mu)^{m+n} R_{m,n}^{q-3}(\langle \xi'_\mu, \eta'\rangle) = 0, \quad (m,n) \in K,\ \eta' \in \Omega_{2q-2}. \tag{4.5}$$

We now split the proof into two cases. If $\langle T_j(\xi_\mu), T_j(\xi_j)\rangle \ne -1$, $\mu \ne j$, we use our hypothesis to select a sequence $\{(m_\nu, n_\nu)\}$ from $\{(m,n) \in K : m-n \text{ is even}\}$ such that $\{m_\nu + n_\nu\}$ is increasing. The inequality

$$\left| (\sin\theta_\mu)^{m_\nu+n_\nu} R_{m_\nu,n_\nu}^{q-3}(\langle \xi'_\mu, \eta'\rangle) \right| \le |\sin\theta_\mu|^{m_\nu+n_\nu}, \quad \mu \ne j,\ \nu = 1,2,\dots,\ \eta' \in \Omega_{2q-2} \tag{4.6}$$

and the fact that $|\sin\theta_\mu| < 1$, $\mu \ne j$, imply that

$$\lim_{\nu\to\infty} (\sin\theta_\mu)^{m_\nu+n_\nu} R_{m_\nu,n_\nu}^{q-3}(\langle \xi'_\mu, \eta'\rangle) = 0, \quad \mu \ne j,\ \eta' \in \Omega_{2q-2}. \tag{4.7}$$

Using this information with $\eta' = \xi_j'$, we obtain

$$\begin{aligned} 0 &= \lim_{\nu\to\infty}\sum_{\mu=1}^{N} c_\mu(\sin\theta_\mu)^{m_\nu+n_\nu}R^{q-3}_{m_\nu,n_\nu}(\langle \xi_\mu',\xi_j'\rangle) \\ &= c_j+\lim_{\nu\to\infty}\sum_{\substack{\mu=1\\ \mu\neq j}}^{N} c_\mu(\sin\theta_\mu)^{m_\nu+n_\nu}R^{q-3}_{m_\nu,n_\nu}(\langle \xi_\mu',\xi_j'\rangle) \\ &= c_j. \end{aligned} \tag{4.8}$$

If $\langle T_j(\xi_j),T_j(\xi_s)\rangle = -1$ for some s then $\theta_s = \pi/2$ and $\xi_s' = (0,-1,0,\dots,0)$. Repeating the procedure used in the previous case and recalling that $R^{q-3}_{m,n}$ is an even function when $m-n$ is even, we obtain

$$0 = \lim_{\nu\to\infty}\sum_{\mu=1}^{N} c_\mu(\sin\theta_\mu)^{m_\nu+n_\nu}R^{q-3}_{m_\nu,n_\nu}(\langle \xi_\mu',\xi_j'\rangle) = c_j+c_s. \tag{4.9}$$

To complete the argument, we extract a sequence $\{(r_\nu,s_\nu)\}$ from the set $\{(m,n)\in K : m-n \text{ is odd}\}$ such that $\{r_\nu+s_\nu\}$ is increasing. Since $R^{q-3}_{m,n}$ is an odd function when $m-n$ is odd, we arrive at

$$0 = \lim_{\nu\to\infty}\sum_{\mu=1}^{N} c_\mu(\sin\theta_\mu)^{r_\nu+s_\nu}R^{q-3}_{r_\nu,s_\nu}(\langle \xi_\mu',\xi_j'\rangle) = c_j-c_s. \tag{4.10}$$

It is now clear that $c_j = 0$. □

We conclude the paper by proving a version of Theorem 4.2 for the case $q=\infty$. A result of this sort was the main goal intended, but not reached, in [15]. The following lemma is the only additional result needed.

Lemma 4.3. *If q is a positive integer at least 2 and m and n are nonnegative integers, then there are positive constants $c^j_{q,m,n}$, $j=0,\dots,m\wedge n$, such that*

$$z^m\bar{z}^n = \sum_{j=0}^{m\wedge n} c^j_{q,m,n}R^{q-2}_{m-j,n-j}(z), \quad z\in B_2. \tag{4.11}$$

Proof. A proof, including exact values of the constants $c^j_{q,m,n}$, can be found in [2, page 28]. □

Theorem 4.4. *A subset K of $\mathbb{Z}^2_+$ induces SPD of all orders on Ω_∞ if and only if the set $\{m-n : (m,n)\in K\}$ contains infinitely many even and infinitely many odd integers.*

Proof. One half follows from Lemma 4.1. For the other, let K be a subset of $\mathbb{Z}^2_+$ such that $\{m-n:(m,n)\in K\}$ contains infinitely many even and infinitely many odd integers. We will show that K induces SPD of order $N\geq 3$ on Ω_∞. Let $\xi_1,\dots,\xi_N$ be distinct points on Ω_∞ and f a positive definite function on Ω_∞ such that $K_\infty(f) = K$.

Without loss of generality, we can assume that $\{\xi_1,\dots,\xi_N\} \subset \Omega_{2N}$. By Lemma 4.3,

$$\begin{aligned}
f(\langle \xi_\mu,\xi_\nu\rangle) &= \sum_{(m,n)\in K} a_{m,n}(f)\langle \xi_\mu,\xi_\nu\rangle^m \overline{\langle \xi_\mu,\xi_\nu\rangle}^n \\
&= \sum_{(m,n)\in K} a_{m,n}(f)\sum_{j=0}^{m\wedge n} c^j_{N,m,n} R^{N-2}_{m-j,n-j}(\langle \xi_\mu,\xi_\nu\rangle) \\
&= \sum_{(\alpha,\beta)\in L} d^N_{\alpha,\beta}(f) R^{N-2}_{\alpha,\beta}(\langle \xi_\mu,\xi_\nu\rangle) \\
&= g(\langle \xi_\mu,\xi_\nu\rangle),
\end{aligned} \tag{4.12}$$

where $L := \bigcup_{(m,n)\in K}\{(m-j,n-j) : j = 0,\dots,m\wedge n\}$ and g is a positive definite function on Ω_{2N} satisfying $K_N(g) = L$. Since

$$\{\alpha-\beta : (\alpha,\beta)\in L\} = \{m-n : (m,n)\in K\}, \tag{4.13}$$

L contains infinitely many even and infinitely many odd integers. Thus, Theorem 4.2 implies that L induces SPD of all orders on Ω_{2N}. This guarantees that $(g(\langle \xi_\mu,\xi_\nu\rangle))$ is positive definite and, consequently, so is $(f(\langle \xi_\mu,\xi_\nu\rangle))$. Therefore, K induces SPD of order N on Ω_∞, completing the proof. □

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V. A. Menegatto: Departamento de Matemática, ICMC-USP - São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil

E-mail address: menegatt@icmc.sc.usp.br

A. P. Peron: Departamento de Matemática, CCE - Universidade Estadual de Maringá, Avenida Colombo 5790, 87020-900 Maringá PR, Brazil

E-mail address: apperon@uem.br