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CR-SUBMANIFOLDS OF A NEARLY TRANS-SASAKIAN MANIFOLD

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This paper considers the study of CR-submanifolds of a nearly trans-Sasakian manifold, generalizing the results of trans-Sasakian manifolds and thus those of Sasakian manifolds.

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1. Introduction. In 1978, Bejancu introduced the notion of CR-submanifold of a Kaehler manifold [1]. Since then several papers on CR-submanifolds of Kaehler manifold have been published. On the other hand, CR-submanifolds of a Sasakian manifold have been studied by Kobayashi [6], Shahid et al. [10], Yano and Kon [11], and others. Bejancu and Papaghuic [2] studied CR-submanifolds of a Kenmotsu manifold. In 1985, Oubina introduced a new class of almost contact metric manifold known as trans-Sasakian manifold [7]. This class contains α-Sasakian and β-Kenmotsu manifold [5]. Geometry of CR-submanifold of trans-Sasakian manifold was studied by Shahid [8, 9]. A nearly trans-Sasakian manifold [4] is a more general concept.

The results of this paper are the generalization of the results obtained earlier by several authors, namely [6, 8, 9] and others.

2. Preliminaries. Let $\overline{M}$ be an n-dimensional almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g) satisfying [3]

$$\begin{gathered}\phi^2X=-X+\eta(X)\xi,\quad \eta(\xi)=1,\ \phi\circ\xi=0,\\ g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\quad g(X,\xi)=\eta(X),\end{gathered}\tag{2.1}$$

where X and Y are vector fields tangent to $\overline{M}$.

An almost contact metric structure (ϕ,ξ,η,g) on $\overline{M}$ is called trans-Sasakian if [7]

$$(\overline{\nabla}_X\phi)(Y)=\alpha\{g(X,Y)\xi-\eta(Y)X\}+\beta\{g(\phi X,Y)\xi-\eta(Y)\phi X\},\tag{2.2}$$

where α and β are nonzero constants, $\overline{\nabla}$ denotes the Riemannian connection of g on $\overline{M}$, and we say that the trans-Sasakian structure is of type (α,β).

Further, an almost contact metric manifold $\overline{M}(\phi,\xi,\eta,g)$ is called nearly trans-Sasakian if [4]

$$\begin{aligned}(\overline{\nabla}_X\phi)(Y)+(\overline{\nabla}_Y\phi)(X)=\alpha\{&2g(X,Y)\xi-\eta(Y)X-\eta(X)Y\}\\ &-\beta\{\eta(Y)\phi X+\eta(X)\phi Y\}.\end{aligned}\tag{2.3}$$

It is clear that any trans-Sasakian manifold, and thus any Sasakian manifold, satisfies the above relation.

Let M be an m-dimensional isometrically immersed submanifold of a nearly trans-Sasakian manifold $\overline{M}$ and denote by the same g the Riemannian metric tensor field induced on M from that of $\overline{M}$.

Definition 2.1. An m-dimensional Riemannian submanifold M of a nearly trans-Sasakian manifold $\overline{M}$ is called a CR-submanifold if ξ is tangent to M and there exists a differentiable distribution $D : x \in M \to D_x \subset T_xM$ such that

(1) the distribution D_x is invariant under ϕ, that is, $\phi D_x \subset D_x$ for each $x \in M$;

(2) the complementary orthogonal distribution $D^\perp : x \in M \to D_x^\perp \subset T_xM$ of D is anti-invariant under ϕ, that is, $\phi D_x^\perp \subset T_x^\perp M$ for all $x \in M$, where T_xM and $T_x^\perp M$ are the tangent space and the normal space of M at x, respectively.

If $\dim D_x^\perp = 0$ (resp., $\dim D_x = 0$), then the CR-submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution D (resp., $D^\perp$) is called the horizontal (resp., vertical) distribution. Also, the pair $(D, D^\perp)$ is called ξ-horizontal (resp., vertical) if $\xi_x \in D_x$ (resp., $\xi_x \in D_x^\perp$) [6].

For any vector field X tangent to M, we put [6]

$$X = PX + QX, \tag{2.4}$$

where PX and QX belong to the distribution D and $D^\perp$.

For any vector field N normal to M, we put [6]

$$\phi N = BN + CN, \tag{2.5}$$

where BN (resp., CN) denotes the tangential (resp., normal) component of ϕN.

Let $\overline{\nabla}$ (resp., ∇) be the covariant differentiation with respect to the Levi-Civita connection on $\overline{M}$ (resp., M). The Gauss and Weingarten formulas for M are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y); \qquad \overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{2.6}$$

for $X, Y \in TM$ and $N \in T^\perp M$, where h (resp., A) is the second fundamental form (resp., tensor) of M in $\overline{M}$, and $\nabla^\perp$ denotes the normal connection. Moreover, we have

$$g(h(X,Y),N) = g(A_N X, Y). \tag{2.7}$$

3. Some basic lemmas.

First we prove the following lemma.

Lemma 3.1. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then*

$$\begin{aligned} P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_{\phi QX} Y) - P(A_{\phi QY} X) \\ = \phi P \nabla_X Y + \phi P \nabla_Y X + 2\alpha g(X,Y) P\xi - \alpha\eta(Y) PX \\ - \alpha\eta(X) PY - \beta\eta(Y)\phi PX + \beta\eta(X)\phi PY, \end{aligned} \tag{3.1}$$

$$\begin{aligned} Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi QX} Y) - Q(A_{\phi QY} X) \\ = 2Bh(X,Y) - \alpha\eta(Y) QX - \alpha\eta(X) QY + 2\alpha g(X,Y) Q\xi, \end{aligned} \tag{3.2}$$

$$\begin{aligned} h(X,\phi PY) + h(Y,\phi PX) + \nabla_X^\perp \phi QY + \nabla_Y^\perp \phi QX \\ = \phi Q \nabla_Y X + \phi Q \nabla_X Y + 2Ch(X,Y) - \beta\eta(Y)\phi QX - \beta\eta(X)\phi QY \end{aligned} \tag{3.3}$$

for $X, Y \in TM$.

Proof. From the definition of the nearly trans-Sasakian manifold and using (2.4), (2.5), and (2.6), we get

$$\begin{aligned} &\nabla_X \phi PY + h(X,\phi PY) - A_{\phi QY}X + \nabla^{\perp}_X \phi QY - \phi(\nabla_X Y + h(X,Y)) \\ &\quad + \nabla_Y \phi PX + h(Y,\phi PX) - A_{\phi QX}Y + \nabla^{\perp}_Y \phi QX - \phi(\nabla_Y X + h(X,Y)) \\ &= \alpha\{2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} \end{aligned} \tag{3.4}$$

for any $X,Y \in TM$.

Now using (2.4) and equaling horizontal, vertical, and normal components in (3.4), we get the result. □

Lemma 3.2. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then*

$$\begin{aligned} 2(\overline{\nabla}_X \phi)(Y) &= \nabla_X \phi Y - \nabla_Y \phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y] \\ &\quad + \alpha\{2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} \end{aligned} \tag{3.5}$$

for any $X,Y \in D$.

Proof. By Gauss formula (2.6), we get

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = \nabla_X \phi Y + h(X,\phi Y) - \nabla_Y \phi X - h(Y,\phi X). \tag{3.6}$$

Also, we have

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = (\overline{\nabla}_X \phi)(Y) - (\overline{\nabla}_Y \phi)(X) + \phi[X,Y]. \tag{3.7}$$

From (3.6) and (3.7), we get

$$(\overline{\nabla}_X \phi)(Y) - (\overline{\nabla}_Y \phi)(X) = \nabla_X \phi Y + h(X,\phi Y) - \nabla_Y \phi X - h(Y,\phi X) - \phi[X,Y]. \tag{3.8}$$

Also for nearly trans-Sasakian manifolds, we have

$$(\overline{\nabla}_X \phi)(Y) + (\overline{\nabla}_Y \phi)(X) = \alpha\{2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\}. \tag{3.9}$$

Combining (3.8) and (3.9), the lemma follows. □

In particular, we have the following corollary.

Corollary 3.3. *Let M be a ξ-vertical CR-submanifold of a nearly trans-Sasakian manifold, then*

$$2(\overline{\nabla}_X \phi)(Y) = \nabla_X \phi Y - \nabla_Y \phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y] + 2\alpha g(X,Y)\xi \tag{3.10}$$

for any $X,Y \in D$.

Similarly, by Weingarten formula (2.6), we get the following lemma.

Lemma 3.4. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$, then*

$$\begin{aligned}2(\overline{\nabla}_Y\phi)(Z) &= A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \phi[Y,Z] \\ &\quad + \alpha\{2g(Y,Z)\xi - \eta(Y)Z - \eta(Z)Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\}\end{aligned} \tag{3.11}$$

for any $Y,Z \in D^{\perp}$.

Corollary 3.5. *Let M be a ξ-horizontal CR-submanifold of a nearly trans-Sasakian manifold, then*

$$2(\overline{\nabla}_Y\phi)(Z) = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \phi[Y,Z] + 2\alpha g(Y,Z)\xi \tag{3.12}$$

for any $Y,Z \in D^{\perp}$.

Lemma 3.6. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$, then*

$$\begin{aligned}2(\overline{\nabla}_X\phi)(Y) &= \alpha\{2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} \\ &\quad - A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]\end{aligned} \tag{3.13}$$

for any $X \in D$ and $Y \in D^{\perp}$.

4. Parallel distributions

Definition 4.1. The horizontal (resp., vertical) distribution D (resp., $D^{\perp}$) is said to be parallel [1] with respect to the connection ∇ on M if $\nabla_X Y \in D$ (resp., $\nabla_Z W \in D^{\perp}$) for any vector field $X,Y \in D$ (resp., $W,Z \in D^{\perp}$).

Now we prove the following proposition.

Proposition 4.2. *Let M be a ξ-vertical CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. If the horizontal distribution D is parallel, then*

$$h(X,\phi Y) = h(Y,\phi X) \tag{4.1}$$

for all $X,Y \in D$.

Proof. Using parallelism of horizontal distribution D, we have

$$\nabla_X\phi Y \in D, \quad \nabla_Y\phi X \in D \quad \text{for any } X,Y \in D. \tag{4.2}$$

Thus using the fact that $QX = QY = 0$ for $Y \in D$, (3.2) gives

$$Bh(X,Y) = g(X,Y)Q\xi \quad \text{for any } X,Y \in D. \tag{4.3}$$

Also, since

$$\phi h(X,Y) = Bh(X,Y) + Ch(X,Y), \tag{4.4}$$

then

$$\phi h(X,Y) = g(X,Y)Q\xi + Ch(X,Y) \quad \text{for any } X,Y \in D. \tag{4.5}$$

Next from (3.3), we have

$$h(X,\phi Y) + h(Y,\phi X) = 2Ch(X,Y) = 2\phi h(X,Y) - 2g(X,Y)Q\xi, \tag{4.6}$$

for any $X, Y \in D$. Putting $X = \phi X \in D$ in (4.6), we get

$$h(\phi X, \phi Y) + h(Y, \phi^2 X) = 2\phi h(\phi X, Y) - 2g(\phi X, Y)Q\xi \tag{4.7}$$

or

$$h(\phi X, \phi Y) - h(Y, X) = 2\phi h(\phi X, Y) - 2g(\phi X, Y)Q\xi. \tag{4.8}$$

Similarly, putting $Y = \phi Y \in D$ in (4.6), we get

$$h(\phi Y, \phi X) - h(X, Y) = 2\phi h(X, \phi Y) - 2g(X, \phi Y)Q\xi. \tag{4.9}$$

Hence from (4.8) and (4.9), we have

$$\phi h(X, \phi Y) - \phi h(Y, \phi X) = g(X, \phi Y)Q\xi - g(\phi X, Y)Q\xi. \tag{4.10}$$

Operating ϕ on both sides of (4.10) and using $\phi\xi = 0$, we get

$$h(X, \phi Y) = h(Y, \phi X) \tag{4.11}$$

for all $X, Y \in D$. □

Now, for the distribution $D^\perp$, we prove the following proposition.

Proposition 4.3. *Let M be a ξ-vertical CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. If the distribution $D^\perp$ is parallel with respect to the connection on M, then*

$$(A_{\phi Y}Z + A_{\phi Z}Y) \in D^\perp \quad \textit{for any } Y, Z \in D^\perp. \tag{4.12}$$

Proof. Let $Y, Z \in D^\perp$, then using Gauss and Weingarten formula (2.6), we obtain

$$\begin{aligned} &-A_{\phi Z}Y + \nabla_Y^\perp \phi Z - A_{\phi Y}Z + \nabla_Z^\perp \phi Y \\ &\quad = \phi\nabla_Y Z + \phi\nabla_Z Y + 2\phi h(Y, Z) \\ &\qquad + \alpha\{2g(X, Y)\xi - \eta(Y)Z - \eta(Z)Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} \end{aligned} \tag{4.13}$$

for any $Y, Z \in D^\perp$. Taking inner product with $X \in D$ in (4.13), we get

$$g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = g(\nabla_Y Z, \phi X) + g(\nabla_Z Y, \phi X). \tag{4.14}$$

If the distribution $D^\perp$ is parallel, then $\nabla_Y Z \in D^\perp$ and $\nabla_Z Y \in D^\perp$ for any $Y, Z \in D^\perp$. So from (4.14) we get

$$g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = 0 \qquad \text{or} \qquad g(A_{\phi Y}Z + A_{\phi Z}Y, X) = 0 \tag{4.15}$$

which is equivalent to

$$(A_{\phi Y}Z + A_{\phi Z}Y) \in D^\perp \quad \text{for any } Y, Z \in D^\perp, \tag{4.16}$$

and this completes the proof. □

Definition 4.4 [6]**.** A CR-submanifold is said to be mixed totally geodesic if $h(X, Z) = 0$ for all $X \in D$ and $Z \in D^\perp$.

The following lemma is an easy consequence of (2.7).

Lemma 4.5. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then M is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$.*

Definition 4.6 [6]. A normal vector field $N \neq 0$ is called D-parallel normal section if $\nabla^{\perp}_X N = 0$ for all $X \in D$.

Now we have the following proposition.

Proposition 4.7. *Let M be a mixed totally geodesic ξ-vertical CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then the normal section $N \in \phi D^{\perp}$ is D-parallel if and only if $\nabla_X \phi N \in D$ for all $X \in D$.*

Proof. Let $N \in \phi D^{\perp}$. Then from (3.2) we have

$$Q(\nabla_Y \phi X) = 0 \quad \text{for any } X \in D,\ Y \in D^{\perp}. \tag{4.17}$$

In particular, we have $Q(\nabla_Y X) = 0$. By using it in (3.3), we get

$$\nabla^{\perp}_X \phi Q Y = \phi Q \nabla_X Y \qquad \text{or} \qquad \nabla^{\perp}_X N = -\phi Q \nabla_X \phi N. \tag{4.18}$$

Thus, if the normal section $N \neq 0$ is D-parallel, then using Definition 4.6 and (4.18), we get

$$\phi Q(\nabla_X \phi N) = 0 \tag{4.19}$$

which is equivalent to $\nabla_X \phi N \in D$ for all $X \in D$. The converse part easily follows from (4.18). □

5. Integrability conditions of distributions. First we calculate the Nijenhuis tensor $N_\phi(X,Y)$ on a nearly trans-Sasakian manifold $\overline{M}$. For this, first we prove the following lemma.

Lemma 5.1. *Let $\overline{M}$ be a nearly trans-Sasakian manifold, then*

$$\begin{aligned}(\overline{\nabla}_{\phi X}\phi)(Y) = {} & 2\alpha g(\phi X, Y)\xi - \eta(Y)\phi X + \beta\eta(Y)X - \beta\eta(X)\eta(Y)\xi \\ & - \eta(X)\overline{\nabla}_Y\xi + \phi(\overline{\nabla}_Y\phi)(X) + \eta(\overline{\nabla}_Y X)\xi\end{aligned} \tag{5.1}$$

for any $X, Y \in T\overline{M}$.

Proof. From the definition of nearly trans-Sasakian manifold $\overline{M}$, we have

$$\begin{aligned}(\overline{\nabla}_{\phi X}\phi)(Y) = {} & 2\alpha g(\phi X, Y)\xi - \eta(Y)\phi X + \beta\eta(Y)X \\ & - \beta\eta(Y)\eta(X)\xi - (\overline{\nabla}_Y\phi)(\phi X).\end{aligned} \tag{5.2}$$

Also, we have

$$\begin{aligned}(\overline{\nabla}_Y\phi)(\phi X) &= \overline{\nabla}_Y\phi^2 X - \phi\overline{\nabla}_Y\phi X \\ &= \overline{\nabla}_Y\phi^2 X - \phi\overline{\nabla}_Y\phi X + \phi(\phi\overline{\nabla}_Y X) - \phi(\phi\overline{\nabla}_Y X) \\ &= -\overline{\nabla}_Y X + \eta(X)\overline{\nabla}_Y\xi - \phi(\overline{\nabla}_Y\phi X - \phi\overline{\nabla}_Y X) - \phi(\phi\overline{\nabla}_Y X) \\ &= -\overline{\nabla}_Y X + \eta(X)\overline{\nabla}_Y\xi - \phi(\overline{\nabla}_Y\phi)(X) - \overline{\nabla}_Y X - \eta(\overline{\nabla}_Y X)\xi.\end{aligned} \tag{5.3}$$

Using (5.3) in (5.2), we get

$$\begin{aligned}(\overline{\nabla}_{\phi X}\phi)(Y) = 2\alpha g(\phi X,Y)\xi - \eta(Y)\phi X + \beta\eta(Y)X - \beta\eta(X)\eta(Y)\xi \\ - \eta(X)\overline{\nabla}_Y\xi + \phi(\overline{\nabla}_Y\phi)(X) + \eta(\overline{\nabla}_Y X)\xi\end{aligned} \tag{5.4}$$

for any $X,Y \in T\overline{M}$, which completes the proof of the lemma. □

On a nearly trans-Sasakian manifold $\overline{M}$, Nijenhuis tensor is given by

$$N_\phi(X,Y) = (\overline{\nabla}_{\phi X}\phi)(Y) - (\overline{\nabla}_{\phi Y}\phi)(X) - \phi(\overline{\nabla}_X\phi)(Y) + \phi(\overline{\nabla}_Y\phi)(X) \tag{5.5}$$

for any $X,Y \in T\overline{M}$.

From (5.1) and (5.5), we get

$$\begin{aligned}N_\phi(X,Y) = 4\alpha g(\phi X,Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y + \beta\eta(Y)X \\ - \beta\eta(X)Y - \eta(X)\overline{\nabla}_Y\xi + \eta(Y)\overline{\nabla}_X\xi + \eta(\overline{\nabla}_Y X)\xi \\ - \eta(\overline{\nabla}_X Y)\xi + 2\phi(\overline{\nabla}_Y\phi)(X) - 2\phi(\overline{\nabla}_X\phi)(Y).\end{aligned} \tag{5.6}$$

Thus using (2.3) in the above equation and after some calculations, we obtain

$$\begin{aligned}N_\phi(X,Y) = 4\alpha g(\phi X,Y)\xi + (2\alpha-1)\eta(Y)\phi X + (2\alpha+1)\eta(X)\phi Y - \beta\eta(Y)X \\ - 3\beta\eta(X)Y + 4\beta\eta(X)\eta(Y)\xi - \eta(X)\overline{\nabla}_Y\xi + \eta(Y)\overline{\nabla}_X\xi \\ + \eta(\overline{\nabla}_Y X)\xi - \eta(\overline{\nabla}_X Y)\xi + 4\phi(\overline{\nabla}_Y\phi)(X)\end{aligned} \tag{5.7}$$

for any $X,Y \in T\overline{M}$.

Now we prove the following proposition.

Proposition 5.2. *Let M be a ξ-vertical CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then, the distribution D is integrable if the following conditions are satisfied:*

$$S(X,Z) \in D, \qquad h(X,\phi Z) = h(\phi X,Z) \tag{5.8}$$

for any $X,Z \in D$.

Proof. The torsion tensor $S(X,Y)$ of the almost contact structure (ϕ,ξ,η,g) is given by

$$S(X,Y) = N_\phi(X,Y) + 2d\eta(X,Y)\xi = N_\phi(X,Y) + 2g(\phi X,Y)\xi. \tag{5.9}$$

Thus, we have

$$S(X,Y) = [\phi X,\phi Y] - \phi[\phi X,Y] - \phi[X,\phi Y] + 2g(\phi X,Y)\xi \tag{5.10}$$

for any $X,Y \in TM$.

Suppose that the distribution D is integrable. So for $X,Y \in D$, $Q[X,Y] = 0$ and $\eta([X,Y]) = 0$ as $\xi \in D^\perp$.

If $S(X,Y) \in D$, then from (5.7) and (5.9) we have

$$\{2(2\alpha+1)g(\phi X,Y)\xi + \eta([X,Y])\xi + 4(\phi\nabla_Y\phi X + \phi h(Y,\phi X) + Q\nabla_Y X + h(X,Y))\} \in D \tag{5.11}$$

or

$$2(2\alpha+1)g(\phi X,Y)Q\xi+\eta([X,Y])Q\xi+4\big(\phi Q\nabla_Y\phi X+\phi h(Y,\phi X)+Q\nabla_Y X+h(X,Y)\big)=0 \tag{5.12}$$

for any $X,Y\in D$.

Replacing Y by ϕZ for $Z\in D$ in the above equation, we get

$$2(2\alpha+1)g(\phi X,\phi Z)Q\xi+4\big(\phi Q\nabla_{\phi Y}\phi X+\phi h(\phi Z,\phi X)+Q\nabla_{\phi Z}X+h(X,\phi Z)\big)=0. \tag{5.13}$$

Interchanging X and Z for $X,Z\in D$ in (5.13) and subtracting these relations, we obtain

$$\phi Q[\phi X,\phi Z]+Q[X,\phi Z]+h(X,\phi Z)-h(Z,\phi X)=0 \tag{5.14}$$

for any $X,Z\in D$ and the assertion follows. □

Now, we prove the following proposition.

Proposition 5.3. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then*

$$A_{\phi Y}Z-A_{\phi Z}Y=\alpha(\eta(Y)Z-\eta(Z)Y)+\frac{1}{3}\phi P[Y,Z] \tag{5.15}$$

for any $Y,Z\in D^{\perp}$.

Proof. For $Y,Z\in D^{\perp}$ and $X\in T(M)$, we get

$$\begin{aligned}
2g(A_{\phi Z}Y,X)&=2g(h(X,Y),\phi Z)\\
&=g(h(X,Y),\phi Z)+g(h(X,Y),\phi Z)\\
&=g(\overline{\nabla}_XY,\phi Z)+g(\overline{\nabla}_YX,\phi Z)\\
&=g(\overline{\nabla}_XY+\overline{\nabla}_YX,\phi Z)\\
&=-g(\phi(\overline{\nabla}_XY+\overline{\nabla}_YX),Z)\\
&=-g(\overline{\nabla}_Y\phi X+\overline{\nabla}_X\phi Y+\alpha\eta(X)Y+\alpha\eta(Y)X-2\alpha g(X,Y)\xi,Z)\\
&=-g(\overline{\nabla}_Y\phi X,Z)-g(\overline{\nabla}_X\phi Y,Z)-\alpha\eta(X)g(Y,Z)\\
&\quad-\alpha\eta(Y)g(X,Z)+2\alpha\eta(Z)g(X,Y)\\
&=g(\overline{\nabla}_YZ,\phi X)+g(A_{\phi Y}Z,X)-\alpha\eta(X)g(Y,Z)\\
&\quad-\alpha\eta(Y)g(X,Z)+2\alpha\eta(Z)g(X,Y).
\end{aligned} \tag{5.16}$$

The above equation is true for all $X\in T(M)$, therefore, transvecting the vector field X both sides, we obtain

$$2A_{\phi Z}Y=A_{\phi Y}Z-\phi\overline{\nabla}_YZ-\alpha g(Y,Z)\xi-\alpha\eta(Y)Z+2\alpha\eta(Z)Y \tag{5.17}$$

for any $Y,Z\in D^{\perp}$. Interchanging the vector fields Y and Z, we get

$$2A_{\phi Y}Z=A_{\phi Z}Y-\phi\overline{\nabla}_ZY-\alpha g(Y,Z)\xi-\alpha\eta(Z)Y+2\alpha\eta(Y)Z. \tag{5.18}$$

Subtracting (5.17) and (5.18), we get

$$A_{\phi Y}Z-A_{\phi Z}Y=\alpha(\eta(Y)Z-\eta(Z)Y)+\frac{1}{3}\phi P[Y,Z] \tag{5.19}$$

for any $Y,Z\in D^{\perp}$, which completes the proof. □

Theorem 5.4. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then, the distribution $D^\perp$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = \alpha(\eta(Y)Z - \eta(Z)Y), \quad \text{for any } Y, Z \in D^\perp. \tag{5.20}$$

Proof. First suppose that the distribution $D^\perp$ is integrable. Then $[Y,Z] \in D^\perp$ for any $Y, Z \in D^\perp$. Since P is a projection operator on D, so $P[Y,Z] = 0$. Thus from (5.15) we get (5.20). Conversely, we suppose that (5.20) holds. Then using (5.15), we have $\phi P[Y,Z] = 0$ for any $Y, Z \in D^\perp$. Since rank $\phi = 2n$. Therefore, either $P[Y,Z] = 0$ or $P[Y,Z] = k\xi$. But $P[Y,Z] = k\xi$ is not possible as P is a projection operator on D. Thus, $P[Y,Z] = 0$, which is equivalent to $[Y,Z] \in D^\perp$ for any $Y, Z \in D^\perp$ and hence $D^\perp$ is integrable. □

Corollary 5.5. *Let M be a ξ-horizontal CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then, the distribution $D^\perp$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = 0 \tag{5.21}$$

for any $Y, Z \in D^\perp$.

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