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THE NONEXISTENCE OF RANK 4 IP TENSORS IN SIGNATURE $(1,3)$

KELLY JEANNE PEARSON and TAN ZHANG

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Let V be a real vector space of dimension 4 with a nondegenerate symmetric bilinear form of signature $(1,3)$. We show that there exists no algebraic curvature tensor R on V so that its associated skew-symmetric operator $R(\cdot)$ has rank 4 and constant eigenvalues on the Grassmannian of nondegenerate 2-planes in V.

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1. Introduction. Let ∇ be the Levi-Civita connection of a smooth connected pseudo-Riemannian manifold (M,g) of signature (p,q). Let ${}^gR(x,y) := \nabla_x\nabla_y - \nabla_y\nabla_x - \nabla_{[x,y]}$ be the *Riemann curvature operator*. The *associated curvature tensor* ${}^gR(x,y,z,w) := g(R(x,y)z,w)$ has the symmetries

$$\begin{gathered} {}^gR(x,y,z,w) = -{}^gR(y,x,z,w) = -{}^gR(x,y,w,z), \\ {}^gR(x,y,z,w) = {}^gR(z,w,x,y), \\ {}^gR(x,y,z,w) + {}^gR(y,z,x,w) + {}^gR(z,x,y,w) = 0. \end{gathered} \tag{1.1}$$

We now extend the definition of curvature tensors to a more general algebraic framework. Let $\langle\cdot,\cdot\rangle$ be a nondegenerate symmetric bilinear form of signature (p,q) on a finite-dimensional real vector space V. A four tensor $R \in \otimes^4(V^*)$ is called an *algebraic curvature tensor* if R satisfies (1.1). The *associated algebraic curvature tensor* $R(x,y)$ is then defined by $\langle R(x,y)z,w\rangle := R(x,y,z,w)$.

The curvature tensor ${}^gR$ of a pseudo-Riemannian manifold (M,g) is an algebraic curvature tensor on the tangent space at every point P of M. Conversely, (M,g) is said to be a *geometric realization* of an algebraic curvature tensor R at a point P of M if there exists an isometry $\Theta : T_PM \to V$ such that ${}^gR(x,y,z,w) = R(\Theta x,\Theta y,\Theta z,\Theta w)$, for all $x,y,z,w \in T_PM$. An important fact from differential geometry shows that every algebraic curvature tensor has such a geometric realization.

A fundamental problem in differential geometry is to relate the algebraic properties of the curvature tensor R to the underlying geometry and topology of the manifold (M,g) itself. However, since the full curvature tensor R itself is quite complicated in general, we often use the curvature tensor R to define natural endomorphisms of the tangent bundle. The Jacobi operator, the Stanilov operator, the Szabó operator, and the skew-symmetric curvature operator are such examples. Many interesting geometric consequences, such as local 2-point homogeneous, locally symmetric, constant sectional curvature, and so forth, can be drawn once such a natural operator is assumed to have constant eigenvalues or constant rank on their corresponding domains. We refer to [2] for detailed discussions on this subject. In the remainder of our paper, we will discuss the skew-symmetric curvature operator.

If $\{v_1, v_2\}$ is an oriented basis for a nondegenerate 2-plane π in V, the *skew-symmetric curvature operator* is defined by

$$R(\pi) := \left| g(v_1,v_1)g(v_2,v_2) - g(v_1,v_2)^2 \right|^{-1/2} R(v_1,v_2). \tag{1.2}$$

It can be shown that this definition is independent of the basis chosen. A fundamental numerical invariant of a linear transformation is its rank. We say R has *rank* r if $\operatorname{rank}(R(\pi)) = r$ for every oriented nondegenerate 2-plane π.

If $p \le 2$, the constant-rank algebraic curvature tensors have rank at most 2 for most values of q. We refer to Gilkey [1], Gilkey et al. [3], and Zhang [8] for the following result.

Theorem 1.1. *Let R be a nontrivial rank r algebraic curvature tensor. The rank is determined in the following cases:*
(1) *let $p = 0$. If $q \ge 5$ and $q \ne 7$, then $r = 2$;*
(2) *let $p = 1$. If $q = 5$ or $q \ge 9$, then $r = 2$;*
(3) *let $p = 2$. If $q \ge 10$, then $r \le 4$. Furthermore, if neither q nor $q+2$ are powers of 2, then $r = 2$.*

Theorem 1.1 played an important role in the classification of rank 2 algebraic curvature tensors for $q \ge 5$. We refer to [5] for the proof of the following result.

Theorem 1.2. *Let R be an algebraic curvature tensor. Then R has constant rank 2 if and only if there exists a selfadjoint map ϕ whose kernel contains no nontrivial spacelike vectors so that $R = R_\phi$, where*

$$R_\phi(x,y)z = \pm\{\langle\phi y, z\rangle\phi x - \langle\phi x, z\rangle\phi y\}. \tag{1.3}$$

It is also shown in [5] that every rank 2 algebraic curvature tensor can be geometrically realized by the germ of pseudo-Riemannian hypersurface in flat space.

We say that an algebraic curvature tensor is (*Jordan*) *IP* if the Jordan normal form of the complexification of $R(\cdot)$ is constant on the Grassmannian of nondegenerate oriented 2-planes; such a tensor necessarily has constant rank. We say that a pseudo-Riemannian manifold (M,g) is *rank* r (*Jordan*) *IP* if ${}^gR$ is rank r Jordan IP at every point of M; the Jordan normal form is allowed to vary with the point but the rank is assumed to be constant.

Remark 1.3. The notion of *IP* follows from the pioneering classification result in 4-dimensional Riemannian geometry, that is, $p = 0$ and $q = 4$, due to Ivanov and Petrova [6]. In the Riemannian setting, that is, $p = 0$, R is Jordan IP if and only if $R(\pi)$ has constant eigenvalues on all oriented 2-planes π.

Theorem 1.1 also played an important role in the classification of rank 2 Jordan IP algebraic curvature tensors for $q \ge 5$. The following result is found in [5].

Theorem 1.4. *Let R be an algebraic curvature tensor. Then R is a rank 2 Jordan IP tensor if and only if $R = CR_\phi$ where $C > 0$ and ϕ is a selfadjoint map which satisfies one of the following three conditions:*

(1) *the map ϕ is an isometry, that is, $\langle \phi x, \phi y\rangle = \langle x, y\rangle$ for all x, y. This is equivalent to the condition $\phi^2 = \mathrm{id}$;*

(2) *the map ϕ is a para-isometry, that is, $\langle \phi x, \phi y\rangle = -\langle x, y\rangle$ for all x, y. This is equivalent to the condition $\phi^2 = -\mathrm{id}$;*

(3) *the map ϕ satisfies $\phi^2 = 0$ and $\ker\phi = \mathrm{range}\,\phi$.*

The classification of rank 2 Jordan IP pseudo-Riemannian manifolds for $q \geq 5$ is found in [4].

Theorem 1.5. *Let (M, g) be a connected spacelike rank 2 Jordan IP pseudo-Riemannian manifold of signature (p, q), where $q \geq 5$. Assume that ${}^gR$ is not nilpotent for at least one point P of M. We have the following trichotomy:*

(1) *for each point $P \in M$, we have ${}^gR_P = C(P)R_{\phi(P)}$, where $\phi(P)$ is selfadjoint map of T_PM so that $\phi(P)^2 = \mathrm{id}$; ${}^gR_P$ is never nilpotent;*

(2) *if $\phi = \pm\mathrm{id}$, then (M, g) has constant sectional curvature;*

(3) *if $\phi \neq \pm\mathrm{id}$, then (M, g) is locally isometric to one of the warped product manifolds of the form*

$$\begin{gathered} M := I \times S^\delta(r, s; \varrho), \qquad f(t) := \varepsilon\kappa t^2 + At + B, \\ ds_M^2 := \varepsilon\, dt^2 + f(t)\, d^2_{S^\delta(r,s;\varrho)}, \qquad C(t) := f^{-2}\Big\{f\kappa - \frac{1}{4}\varepsilon f_t^2\Big\}, \\ \phi := -\mathrm{id} \quad \text{on } TS^\delta(r, s; \varrho), \quad \phi(\partial_t) := \partial_t, \end{gathered} \tag{1.4}$$

where $\varrho > 0$, $\varepsilon = \pm 1$, $\delta = \pm 1$, $S^\delta(r, s; \varrho)$ is the pseudo-sphere of spacelike or timelike vectors of length $\pm\varrho^{-2}$ in a vector space of signature (r, s). Choose $\{\kappa, A, B\}$ so that $f\kappa - (1/4)\varepsilon f_t^2 \neq 0$ or equivalently so that $A^2 - 4\varepsilon\kappa B \neq 0$. Choose interval I so that $f(t) \neq 0$ on I.

However, the classification of such tensors in dimension 4 is exceptional. Kowalski et al. [7] showed that, in the Riemannian setting, an algebraic curvature tensor of rank 4 in dimension 4 must have both positive and negative sectional curvatures.

The study of rank 4 Jordan IP algebraic curvature tensors and manifolds adds some more interesting aspects to the story. We refer to [2, 6, 9] for the following theorem.

Theorem 1.6. *Let R be a Jordan IP algebraic curvature tensor on $\mathbb{R}^{p,q}$. We distinguish the following cases:*

(1) *if $(p, q) = (0, 4)$ and if $\mathrm{rank}\,R = 2$, then R is given by Theorem 1.4(1);*

(2) *if $(p, q) = (0, 4)$ and if $\mathrm{rank}\,R = 4$, then R is equivalent to a nonzero multiple of the "exotic" rank 4 tensor whose nonvanishing components are*

$$\begin{gathered} R_{1212} = 2, \quad R_{1313} = 2, \quad R_{1414} = -1, \quad R_{2424} = 2, \quad R_{2323} = -1, \\ R_{3434} = 2, \quad R_{1234} = -1, \quad R_{1324} = 1, \quad R_{1423} = 2; \end{gathered} \tag{1.5}$$

(3) *if $(p, q) = (2, 2)$ and if $\mathrm{rank}\,R = 2$, then R is given by Theorem 1.4(1), (2);*

(4) *if* $(p,q) = (2,2)$ *and if* $\operatorname{rank} R = 4$, *then* R *is equivalent to a nonzero multiple of the "exotic"* rank 4 *tensor whose nonvanishing components are*

$$\begin{gathered} R_{1212} = 2, \quad R_{1313} = -2, \quad R_{1414} = 1, \quad R_{2424} = -2, \quad R_{2323} = 1, \\ R_{3434} = 2, \quad R_{1234} = 1, \quad R_{1324} = -1, \quad R_{1423} = -2. \end{gathered} \tag{1.6}$$

However, in [6], Ivanov and Petrova further proved that such "exotic" rank 4 IP tensors cannot be geometrically realized by IP manifolds.

In summary, every rank 2 (Jordan) IP algebraic curvature tensor can be geometrically realized by the germ of a rank 2 (Jordan) IP pseudo-Riemannian manifold; but not every rank 4 (Jordan) IP algebraic curvature tensor can be geometrically realized by the germ of a rank 4 (Jordan) IP pseudo-Riemannian manifold. So in a sense, when $(p,q) = (0,4)$, the algebraic IP assumption gives the geometric obstruction. It is not known if such obstruction exists for $(p,q) = (2,2)$. The main result of this paper is Theorem 3.1 which deals with the remaining cases when $p+q = 4$.

Here is a brief outline of this paper. In Section 2, we present some notational conventions and employ techniques from linear algebra to establish some preliminary results needed for the proof of Theorem 3.1. In Section 3, we state and prove Theorem 3.1.

2. Preliminaries from linear algebra.

Let $O(p,q)$ be the group of all linear maps from V to V which preserves the nondegenerate symmetric bilinear form $\langle \cdot,\cdot \rangle$ and let $\mathrm{so}(p,q)$ be the associated Lie algebra. We have

$$\begin{aligned} O(p,q) &= \{A \in \operatorname{End}(V) : \langle Au, Av\rangle = \langle u,v\rangle \ \forall u,v \in V\}, \\ \mathrm{so}(p,q) &= \{A \in \operatorname{End}(V) : \langle Au, v\rangle + \langle u, Av\rangle = 0 \ \forall u,v \in V\}. \end{aligned} \tag{2.1}$$

A nonzero vector $v \in V$ is said to be *spacelike* (or *timelike*) if $\langle v,v\rangle > 0$ (or $\langle v,v\rangle < 0$). A nondegenerate 2-plane π in V is said to be *spacelike* (or *mixed*) if the restriction $\langle \cdot,\cdot\rangle|_\pi$ on π has signature $(0,2)$ (or $(1,1)$).

Let $\mathrm{Gr}^+_{(0,2)}(\mathbb{R}^{1,3})$ (or $\mathrm{Gr}^+_{(1,1)}(\mathbb{R}^{1,3})$) be the Grassmannian of nondegenerate oriented spacelike (or mixed) 2-planes in $\mathbb{R}^{1,3}$.

Throughout the remainder of our discussions, we will always fix an orthonormal basis $\{e_1,e_2,e_3,e_4\}$ for $\mathbb{R}^{1,3}$ so that $\langle e_i,e_j\rangle = 0$ for $i \neq j$, $\langle e_i,e_i\rangle = 1$ for $1 \le i \le 3$, and $\langle e_4,e_4\rangle = -1$.

If $\pi \in \mathrm{Gr}^+_{(0,2)}(\mathbb{R}^{1,3})$ (or $\pi \in \mathrm{Gr}^+_{(1,1)}(\mathbb{R}^{1,3})$), then $R(\pi) \in \mathrm{so}(1,3)$. Thus, relative to the orthonormal basis $\{e_1,e_2,e_3,e_4\}$, there are real constants $a(\pi)$, $b(\pi)$, $c(\pi)$, $d(\pi)$, $e(\pi)$, and $f(\pi)$ such that $R(\pi)$ has the form

$$R(\pi) = \begin{pmatrix} 0 & a(\pi) & b(\pi) & c(\pi) \\ -a(\pi) & 0 & d(\pi) & e(\pi) \\ -b(\pi) & -d(\pi) & 0 & f(\pi) \\ c(\pi) & e(\pi) & f(\pi) & 0 \end{pmatrix}. \tag{2.2}$$

We define

$$\begin{aligned} \sigma_{R(\pi)} &:= a(\pi)^2 + b(\pi)^2 + c(\pi)^2 - d(\pi)^2 - e(\pi)^2 - f(\pi)^2, \\ \delta_{R(\pi)} &:= a(\pi)f(\pi) - b(\pi)e(\pi) + c(\pi)d(\pi). \end{aligned} \tag{2.3}$$

Let $\chi_{R(\pi)}(\lambda) := \det(\lambda - R(\pi))$ be the characteristic polynomial of $R(\pi)$. We compute

$$\det R(\pi) = -\delta^2_{R(\pi)}, \qquad \chi_{R(\pi)}(\lambda) = \lambda^4 + \sigma_{R(\pi)}\lambda^2 - \delta^2_{R(\pi)}. \tag{2.4}$$

If R is a Jordan IP algebraic curvature tensor on $\mathbb{R}^{1,3}$, then $\chi_{R(\pi)}(\lambda)$ must be invariant on $\mathrm{Gr}^+_{(0,2)}(\mathbb{R}^{1,3})$ (or $\mathrm{Gr}^+_{(1,1)}(\mathbb{R}^{1,3})$). Thus the functions $\sigma_{R(\pi)}$ and $\delta^2_{R(\pi)}$ are invariant on $\mathrm{Gr}^+_{(0,2)}(\mathbb{R}^{1,3})$ (or $\mathrm{Gr}^+_{(1,1)}(\mathbb{R}^{1,3})$). Furthermore, $R(\pi)$ has rank 4 if and only if $\delta_{R(\pi)} \neq 0$ for all nondegenerate 2-planes π.

Lemma 2.1. *Let $\{x, y, z\}$ be an orthonormal set of spacelike vectors in $\mathbb{R}^{1,3}$ with $\pi_1 = \mathrm{Span}\{x, y\}$ and $\pi_2 = \mathrm{Span}\{x, z\}$. If R is a rank 4 Jordan IP algebraic curvature tensor, then $\delta_{R(\pi_1)} = \delta_{R(\pi_2)}$.*

Proof. Let $\theta \in [0, 2\pi]$. Let $\pi(\theta) := \mathrm{Span}\{x, \cos(\theta)y + \sin(\theta)z\}$ be a 1-parameter family of spacelike 2-planes in $\mathbb{R}^{1,3}$. We use (2.2) to see $R(\pi(\theta)) = \cos\theta R(\pi_1) + \sin\theta R(\pi_2)$ has the form

$$\begin{aligned} R(\pi(\theta)) = \cos\theta &\begin{pmatrix} 0 & a(\pi_1) & b(\pi_1) & c(\pi_1) \\ -a(\pi_1) & 0 & d(\pi_1) & e(\pi_1) \\ -b(\pi_1) & d(\pi_1) & 0 & f(\pi_1) \\ c(\pi_1) & e(\pi_1) & f(\pi_1) & 0 \end{pmatrix} \\ + \sin\theta &\begin{pmatrix} 0 & a(\pi_2) & b(\pi_2) & c(\pi_2) \\ -a(\pi_2) & 0 & d(\pi_2) & e(\pi_2) \\ -b(\pi_2) & d(\pi_2) & 0 & f(\pi_2) \\ c(\pi_2) & e(\pi_2) & f(\pi_2) & 0 \end{pmatrix}. \end{aligned} \tag{2.5}$$

Thus

$$\begin{aligned} \delta_{R(\pi(\theta))} &= (\cos\theta a(\pi_1) + \sin\theta a(\pi_2))(\cos\theta f(\pi_1) + \sin\theta f(\pi_2)) \\ &\quad - (\cos\theta b(\pi_1) + \sin\theta b(\pi_2))(\cos\theta e(\pi_1) + \sin\theta e(\pi_2)) \\ &\quad + (\cos\theta c(\pi_1) + \sin\theta c(\pi_2))(\cos\theta d(\pi_1) + \sin\theta d(\pi_2)) \\ &= \cos^2\theta \delta_{R(\pi_1)} + \sin^2\theta \delta_{R(\pi_2)} \\ &\quad + \sin\theta\cos\theta\{a(\pi_1)f(\pi_2) + a(\pi_2)f(\pi_1) - b(\pi_1)e(\pi_2) \\ &\qquad - b(\pi_2)e(\pi_1) + c(\pi_1)d(\pi_2) + c(\pi_2)d(\pi_1)\}. \end{aligned} \tag{2.6}$$

Since R is Jordan IP, we have $\delta^2_{R(\pi_1)} = \delta^2_{R(\pi_2)}$ and $\delta_{R(\pi(\theta))}$ is independent of θ.

Suppose $\delta_{R(\pi_1)} = -\delta_{R(\pi_2)}$, this implies that

$$\begin{aligned} \delta_{R(\pi(\theta))} &= \cos(2\theta)\delta_{R(\pi_1)} \\ &\quad + \sin\theta\cos\theta\{a(\pi_1)f(\pi_2) + a(\pi_2)f(\pi_1) - b(\pi_1)e(\pi_2) \\ &\qquad - b(\pi_2)e(\pi_1) + c(\pi_1)d(\pi_2) + c(\pi_2)d(\pi_1)\}. \end{aligned} \tag{2.7}$$

Since $\delta_{R(\pi(\theta))}$ is independent of θ, for any $\theta \in [0,2\pi]$ we have

$$\begin{aligned}0 = \frac{\partial}{\partial\theta}(\delta_{R(\pi(\theta))}) &= -2\sin(2\theta)\delta_{R(\pi_1)}\\ &\quad+\cos(2\theta)\{a(\pi_1)f(\pi_2)+a(\pi_2)f(\pi_1)-b(\pi_1)e(\pi_2)\\ &\qquad-b(\pi_2)e(\pi_1)+c(\pi_1)d(\pi_2)+c(\pi_2)d(\pi_1)\}.\end{aligned}\tag{2.8}$$

By choosing $\theta = \pi/4$ in (2.8), we have $\delta_{R(\pi_1)} = 0$, which is false. Our assertion now follows. □

We omit the proof of the following corollary since the same argument applies.

Corollary 2.2. *Let $\{x,y,z\}$ be an orthonormal set in $\mathbb{R}^{1,3}$ so that x is a unit timelike vector. Let $\pi_1 = \operatorname{Span}\{x,y\}$ and $\pi_2 = \operatorname{Span}\{x,z\}$. If R is a rank 4 Jordan IP algebraic curvature tensor, then $\delta_{R(\pi_1)} = \delta_{R(\pi_2)}$.*

3. The main result.

Let $\{e_1,e_2,e_3,e_4\}$ be the standard orthonormal basis for $\mathbb{R}^{1,3}$ discussed in Section 2. Let $R_{ijkl} := \langle R(e_i,e_j)e_k,e_l\rangle$ be the curvature components. We need the following list of variables in the proof of our main result:

$$\begin{array}{lll} x_1 := R_{1212}, & x_2 := R_{1213}, & x_3 := R_{1214},\\ x_4 := R_{1223}, & x_5 := R_{1224}, & x_6 := R_{1234},\\ x_7 := R_{1313}, & x_8 := R_{1314}, & x_9 := R_{1323},\\ x_{10} := R_{1324}, & x_{11} := R_{1334}, & x_{12} := R_{2314},\\ x_{13} := R_{2323}, & x_{14} := R_{2324}, & x_{15} := R_{2334},\\ y_1 := R_{1414}, & y_2 := R_{1424}, & y_3 := R_{1434},\\ y_4 := R_{2424}, & y_5 := R_{2434}, & y_6 := R_{3434}.\end{array}\tag{3.1}$$

Relative to the orthonormal basis $\{e_1,e_2,e_3,e_4\}$, we have

$$R(e_1,e_2) = \begin{pmatrix} 0 & x_1 & x_2 & x_3\\ -x_1 & 0 & x_4 & x_5\\ -x_2 & -x_4 & 0 & x_6\\ x_3 & x_5 & x_6 & 0\end{pmatrix},$$

$$R(e_1,e_3) = \begin{pmatrix} 0 & x_2 & x_7 & x_8\\ -x_2 & 0 & x_9 & x_{10}\\ -x_7 & -x_9 & 0 & x_{11}\\ x_8 & x_{10} & x_{11} & 0\end{pmatrix},$$

$$R(e_2,e_3) = \begin{pmatrix} 0 & x_4 & x_9 & x_{12}\\ -x_4 & 0 & x_{13} & x_{14}\\ -x_9 & -x_{13} & 0 & x_{15}\\ x_{12} & x_{14} & x_{15} & 0\end{pmatrix},$$

$$R(e_1,e_4) = \begin{pmatrix} 0 & x_3 & x_8 & y_1 \\ -x_3 & 0 & x_{12} & y_2 \\ -x_8 & -x_{12} & 0 & y_3 \\ y_1 & y_2 & y_3 & 0 \end{pmatrix},$$

$$R(e_2,e_4) = \begin{pmatrix} 0 & x_5 & x_{10} & y_2 \\ -x_5 & 0 & x_{14} & y_4 \\ -x_{10} & -x_{14} & 0 & y_5 \\ y_2 & y_4 & y_5 & 0 \end{pmatrix},$$

$$R(e_3,e_4) = \begin{pmatrix} 0 & x_6 & x_{11} & y_3 \\ -x_6 & 0 & x_{15} & y_5 \\ -x_{11} & -x_{15} & 0 & y_6 \\ y_3 & y_5 & y_6 & 0 \end{pmatrix}. \tag{3.2}$$

Theorem 3.1. *There exists no rank 4 (Jordan) IP algebraic curvature tensor if $(p,q) = (1,3)$ or $(3,1)$. Consequently, there exists no rank 4 (Jordan) IP Lorentzian manifolds.*

Proof. Since we can replace the metric g by $-g$ and reverse the roles of p and q, we may only consider the case $(p,q) = (1,3)$. Suppose there exists a rank 4 Jordan IP tensor R on $\mathbb{R}^{1,3}$; we argue for a contradiction. Since R is an algebraic curvature tensor, by the curvature symmetries (1.1), we have $x_6 + x_{12} - x_{10} = R_{1234} + R_{2314} + R_{3124} = 0$. Since R is Jordan IP, the characteristic polynomial of $R(\cdot)$ is invariant for the spacelike 2-planes $\operatorname{Span}\{e_1,e_2\}$, $\operatorname{Span}\{e_1,e_3\}$, and $\operatorname{Span}\{e_2,e_3\}$. We apply Lemma 2.1 and (2.3) to the 1-parameter families $R(e_1,\cos\theta e_2+\sin\theta e_3)$, $R(\cos\theta e_1+\sin\theta e_3,e_2)$, and $R(\cos\theta e_1+\sin\theta e_2,e_3)$ to see for any $\theta\in[0,2\pi]$,

$$\begin{aligned} \sigma_{R(e_1,e_2)} &= \sigma_{R(e_1,e_3)} = \sigma_{R(e_2,e_3)} = \sigma_{R(e_1,\cos\theta e_2+\sin\theta e_3)} \\ &= \sigma_{R(\cos\theta e_1+\sin\theta e_3,e_2)} = \sigma_{R(\cos\theta e_1+\sin\theta e_2,e_3)}, \\ \delta_{R(e_1,e_2)} &= \delta_{R(e_1,e_3)} = \delta_{R(e_2,e_3)} = \delta_{R(e_1,\cos\theta e_2+\sin\theta e_3)} \\ &= \delta_{R(\cos\theta e_1+\sin\theta e_3,e_2)} = \delta_{R(\cos\theta e_1+\sin\theta e_2,e_3)}. \end{aligned} \tag{3.3}$$

Thus we have

$$0 = x_1^2-x_3^2+x_4^2-x_5^2-x_6^2-x_7^2+x_8^2-x_9^2+x_{10}^2+x_{11}^2,$$

$$0 = x_1^2+x_2^2-x_3^2-x_5^2-x_6^2-x_9^2+x_{12}^2-x_{13}^2+x_{14}^2+x_{15}^2,$$

$$0 = x_1x_6-x_2x_5+x_3x_4-x_2x_{11}-x_8x_9+x_7x_{10},$$

$$0 = x_1x_6-x_2x_5+x_3x_4-x_4x_{15}-x_{12}x_{13}+x_9x_{14},$$

$$0 = x_1x_2+x_2x_7-x_3x_8+x_4x_9-x_5x_{10}-x_6x_{11},$$

$$0 = x_1x_4+x_2x_9-x_3x_{12}+x_4x_{13}-x_5x_{14}-x_6x_{15},$$

$$
\begin{aligned}
0 &= x_2x_4 + x_7x_9 - x_8x_{12} + x_9x_{13} - x_{10}x_{14} - x_{11}x_{15},\\
0 &= x_1x_{11} + x_2x_6 + x_3x_9 + x_4x_8 - x_2x_{10} - x_5x_7,\\
0 &= x_1x_{15} + x_4x_6 + x_3x_{13} + x_4x_{12} - x_2x_{14} - x_5x_9,\\
0 &= x_2x_{15} + x_4x_{11} + x_8x_{13} + x_9x_{12} - x_7x_{14} - x_9x_{10},\\
0 &= x_6 - x_{10} + x_{12}.
\end{aligned} \tag{3.4}
$$

The rank 4 condition further implies that

$$
\begin{aligned}
\delta_{R(e_1,e_2)} &= x_1x_6 - x_2x_5 + x_3x_4 \neq 0,\\
\delta_{R(e_1,e_3)} &= x_2x_{11} - x_7x_{10} + x_8x_9 \neq 0,\\
\delta_{R(e_2,e_3)} &= x_4x_{15} - x_9x_{14} + x_{12}x_{13} \neq 0.
\end{aligned} \tag{3.5}
$$

We use the computer algebra system Maple to solve the system of equations (3.4) subject to the rank conditions (3.5) to see that the only possible real solutions are parameterized by two free variables x_{10} and x_{13} as follows:

$$
\begin{gathered}
x_1 = x_7 = -2x_{13}, \qquad x_6 = -x_{10}, \qquad x_{12} = 2x_{10}, \qquad x_5^2 = x_{11}^2 = 3x_{10}^2 + 3x_{13}^2,\\
x_2 = x_3 = x_4 = x_8 = x_9 = x_{14} = x_{15} = 0.
\end{gathered} \tag{3.6}
$$

Thus we have

$$
R(e_1,e_2) = \begin{pmatrix} 0 & -2x_{13} & 0 & 0\\ 2x_{13} & 0 & 0 & x_5\\ 0 & 0 & 0 & -x_{10}\\ 0 & x_5 & -x_{10} & 0 \end{pmatrix},
$$

$$
R(e_1,e_3) = \begin{pmatrix} 0 & 0 & -2x_{13} & 0\\ 0 & 0 & 0 & x_{10}\\ 2x_{13} & 0 & 0 & x_{11}\\ 0 & x_{10} & x_{11} & 0 \end{pmatrix},
$$

$$
R(e_2,e_3) = \begin{pmatrix} 0 & 0 & 0 & 2x_{10}\\ 0 & 0 & x_{13} & 0\\ 0 & -x_{13} & 0 & 0\\ 2x_{10} & 0 & 0 & 0 \end{pmatrix},
$$

$$
R(e_1,e_4) = \begin{pmatrix} 0 & 0 & 0 & y_1\\ 0 & 0 & 2x_{10} & y_2\\ 0 & -2x_{10} & 0 & y_3\\ y_1 & y_2 & y_3 & 0 \end{pmatrix},
$$

$$
R(e_2,e_4) = \begin{pmatrix} 0 & x_5 & x_{10} & y_2\\ -x_5 & 0 & 0 & y_4\\ -x_{10} & 0 & 0 & y_5\\ y_2 & y_4 & y_5 & 0 \end{pmatrix},
$$

$$R(e_3,e_4) = \begin{pmatrix} 0 & -x_{10} & x_{11} & y_3 \\ x_{10} & 0 & 0 & y_5 \\ -x_{11} & 0 & 0 & y_6 \\ y_3 & y_5 & y_6 & 0 \end{pmatrix}. \tag{3.7}$$

Since R is Jordan IP, the characteristic polynomial of $R(\cdot)$ is also invariant for the mixed 2-planes $\mathrm{Span}\{e_1,e_4\}$, $\mathrm{Span}\{e_2,e_4\}$, and $\mathrm{Span}\{e_3,e_4\}$. We use (3.6), Corollary 2.2, and the above six matrices to produce the following system of equations:

$$\begin{aligned}
0 &= y_1^2+y_3^2-y_4^2-y_5^2+3x_{13}^2,\\
0 &= y_1^2+y_2^2-y_5^2-y_6^2+3x_{13}^2,\\
0 &= 2x_{10}y_1-x_5y_5+x_{10}y_4,\\
0 &= 2x_{10}y_1+x_{10}y_6+x_{11}y_5,\\
0 &= x_5^2-3x_{10}^2-3x_{13}^2,\\
0 &= x_{11}^2-3x_{10}^2-3x_{13}^2,\\
0 &= y_1y_2+y_2y_4+y_3y_5,\\
0 &= y_1y_3+y_2y_5+y_3y_6,\\
0 &= -x_5x_{10}+x_{10}x_{11}-y_2y_3-y_4y_5-y_5y_6,\\
0 &= -x_5x_{10}+x_{10}x_{11}-y_2y_3-y_4y_5-y_5y_6,\\
0 &= x_5y_3+x_{10}y_2,\\
0 &= x_{10}y_3-x_{11}y_2,\\
0 &= x_5y_6-2x_{10}y_5-x_{11}y_4.
\end{aligned} \tag{3.8}$$

We again use the computer algebra system MAPLE to solve the system of equations (3.8) subject to the rank conditions (3.5) and

$$2x_{10}y_1 \neq 0, \qquad x_5y_5-x_{10}y_4 \neq 0, \qquad x_{10}y_6+x_{11}y_5 \neq 0 \tag{3.9}$$

to see the only possible real solutions are given by

$$\begin{gathered}
y_4 = y_6 = -2y_1, \quad \text{where } y_1^2 = x_{13}^2,\ y_2 = y_3 = y_5 = 0,\\
x_5 = x_{11}, \qquad x_5^2 = 3x_{10}^2+3x_{13}^2.
\end{gathered} \tag{3.10}$$

For any $t \in \mathbb{R}$, $\sqrt{2}e_1 + (\sinh t)e_3 + (\cosh t)e_4$ and $(1/\sqrt{2})e_2 + (\cosh t/\sqrt{2})e_3 + (\sinh t/\sqrt{2})e_4$ are orthogonal unit spacelike vectors in $\mathbb{R}^{1,3}$. So for the spacelike 2-planes $\pi(t) := \mathrm{Span}\{\sqrt{2}e_1 + (\sinh t)e_3 + (\cosh t)e_4, (1/\sqrt{2})e_2 + (\cosh t/\sqrt{2})e_3 + (\sinh t/\sqrt{2})e_4\}$, we have

$$R(\pi(t)) = \begin{pmatrix} 0 & \alpha(t) & \beta(t) & \mu(t) \\ -\alpha(t) & 0 & \nu(t) & \phi(t) \\ -\beta(t) & -\nu(t) & 0 & \psi(t) \\ \nu(t) & \phi(t) & \psi(t) & 0 \end{pmatrix}, \tag{3.11}$$

where $\alpha(t) = (1/\sqrt{2})x_{10} - 2x_{13} - (\cosh t/\sqrt{2})x_5$, $\beta(t) = -(1/\sqrt{2})x_5 - (2\cosh t)x_{13} - (\cosh t/\sqrt{2})x_{10}$, $\mu(t) = (\sinh t)y_1 - (\sqrt{2}\sinh t)x_{10}$, $\nu(t) = (2\sinh t)x_{10} - (\sinh t/\sqrt{2})x_{13}$, $\phi(t) = x_5 + (\cosh t)x_{10} + (\sqrt{2}\cosh t)y_1$, and $\psi(t) = -x_{10} + \sqrt{2}y_1 + (\cosh t)x_5$. Consequently, from (2.3) we have

$$\begin{aligned}\delta_{R(\pi(t))} = {}& \frac{3}{\sqrt{2}}x_{10}^2 - x_{10}y_1 + x_{10}x_{13} - \frac{3}{\sqrt{2}}x_{13}y_1 + \frac{1}{\sqrt{2}}x_5^2 + (2\sqrt{2}\cosh t)x_5x_{10} \\ & - \frac{1}{\sqrt{2}}(\cosh^2 t)x_5^2 + (3\cosh^2 t)x_{10}x_{13} + \frac{3}{\sqrt{2}}(\cosh^2 t)x_{13}y_1 \\ & - \frac{3}{\sqrt{2}}(\cosh^2 t)x_{10}^2 + (3\cosh^2 t)x_{10}y_1,\end{aligned} \tag{3.12}$$

which is independent of t. Thus $(\partial^r/\partial t^r)\delta_{R(\pi(t))} \equiv 0$ for all $t \in \mathbb{R}$ and all $r \in \mathbb{N}$. However, by a direct calculation, we have

$$\frac{\partial^3}{\partial t^3}\delta_{R(\pi(t))} = (\sinh t)(6x_{10}y_1 + 6x_{10}x_{13} + 3\sqrt{2}x_{13}y_1 - 3\sqrt{2}x_{10}^2 - \sqrt{2}x_5^2). \tag{3.13}$$

Since $x_5^2 = 3x_{10}^2 + 3x_{13}^2$, the identity $0 \equiv (\partial^3/\partial t^3)\delta_{R(\pi(t))}$ holds if and only if $x_{10} = \sqrt{2}x_{13}$. Since $(\partial^2/\partial t^2)\delta_{R(\pi(t))} = 2\sqrt{2}x_5x_{10} + (\cosh t)(6x_{10}y_1 + 6x_{10}x_{13} + 3\sqrt{2}x_{13}y_1 - 3\sqrt{2}x_{10}^2 - \sqrt{2}x_5^2)$ and since $x_{10} = \sqrt{2}x_{13}$, the identity $0 \equiv (\partial^2/\partial t^2)\delta_{R(\pi(t))}$ holds if and only if $x_{10} = x_{13} = 0$. This contradicts the rank conditions (3.5), hence completes the proof of Theorem 3.1. □

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Kelly Jeanne Pearson: Department of Mathematics and Statistics, Murray State University, Murray, KY 42071-0009, USA
E-mail address: kelly.pearson@murraystate.edu

Tan Zhang: Department of Mathematics and Statistics, Murray State University, Murray, KY 42071-0009, USA
E-mail address: tan.zhang@murraystate.edu