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POWERSUM FORMULA FOR POLYNOMIALS WHOSE DISTINCT ROOTS ARE DIFFERENTIALLY INDEPENDENT OVER CONSTANTS

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We prove that the author's *powersum formula* yields a nonzero expression for a particular linear ordinary differential equation, called a *resolvent*, associated with a univariate polynomial whose coefficients lie in a differential field of characteristic zero provided the distinct roots of the polynomial are differentially independent over constants. By definition, the terms of a resolvent lie in the differential field generated by the coefficients of the polynomial, and each of the roots of the polynomial are solutions of the resolvent. One example shows how the powersum formula works. Another example shows how the proof that the formula is not zero works.

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1. Introduction. In 1993, the author began the study of polynomials of a single variable whose coefficients lie in a differential field of characteristic zero and an associated nonzero linear ordinary differential equation (LODE), with the roots of the polynomial as the dependent variable and one of the coefficients of the polynomial as the independent variable. If all the terms of the LODE lie in the differential field generated by the coefficients of the polynomial and are not all zero, then the LODE is called a *resolvent* of the polynomial. The author's original purpose for this line of research was to discover ways of solving nonlinear ODEs by a sequence of Picard-Vessiot extensions. The first linear differential resolvent of a polynomial had been discovered by Cockle in 1860 [4]. Reading the work of Cockle, Harley gave Cockle's newly discovered LODE a name in 1862: *differential resolvent* [7]. Cockle [5] and other authors in the 19th century had attempted to compute all the roots of a polynomial by solving one of its resolvents. Since various explicit formulae for all the roots of a polynomial in terms of the coefficients of the polynomial have since been discovered by Birkeland [2] and Umemura [14], the resolvent is not needed for this purpose. However, the author has continued to pursue explicit formulae for resolvents of any polynomial for the original purpose of solving nonlinear ODEs. For example, the author recently discovered [11] that a simple expression for a first-order inhomogeneous resolvent of a quadratic polynomial can be used to solve the nonlinear first-order Riccati ODE. Cormier et al. [6] have used the differential resolvent to compute the Galois group of a polynomial.

In the 19th century, Cayley [3], Cockle [4], Harley [7], and Lachtin [9] and in the early 20th century, Belardinelli [1] studied only trinomials (polynomials of the form $t^n + A\cdot t^p + B = 0$) with coefficients A and B in the field $\mathbb{Q}(x)$. Trinomials had been exclusively studied because polynomials of degree less than or equal to 5 can be reduced by algebraic manipulations to trinomials $z^n + z^p + C = 0$ involving just one root, z, and one other free parameter, C. These authors sought differential resolvents whose terms are polynomials in x. The author has since generalized the definition of Cockle's and Harley's resolvent to univariate polynomials over any differential field of characteristic zero.

The powersum formula is a remarkably simple application of linear algebra to the computation of a homogeneous LODE. It relies on the existence of an αth power resolvent for any polynomial with coefficients in a differential field of characteristic zero. It also relies on our ability to specialize the indeterminate α to an integer q and leave z^q as a solution of the resolvent. Unfortunately, it has not yet been proven that this formula does not simply yield zero, rather than a resolvent, which is by definition not zero, for every possible polynomial. Worse, it is not known for which polynomials, if any, the powersum formula yields zero. We must first overcome the obstacle of determining the number of derivatives and the number of powers of α in an αth power resolvent of the polynomial. This is necessary since the formula uses Cramer's rule by setting the unknown coefficients of α in the resolvent to the appropriate cofactor of the matrix consisting of integer multiples of the derivatives of the powersums (hence the formula's name) of the roots. A resolvent of lowest possible order and with no common power of α among its terms is called the *Cohnian* of the polynomial, after the author's dissertation advisor, Richard Cohn. No algorithm has yet been devised that is guaranteed to determine the number of powers of α in the Cohnian for all polynomials.

In some sense, all polynomials with coefficients in a differential field are differential specializations of polynomials whose coefficients are *differentially independent* over the integers, that is, there exist no algebraic relations over the integers of the coefficients of the polynomial or of any of their derivatives. It was therefore considered necessary first to prove that the powersum formula yields a nonzero resolvent for a polynomial whose coefficients are differentially independent over integers. For such polynomials, it is known [12, Theorem 40, page 71] that there exists an αth power resolvent of order n. Furthermore, the exact powers of α appearing in the resolvent, with no nontrivial factors, are known. Finally, it is known [12, Theorem 40, page 71] that there exist no αth power resolvents of lower order or with fewer powers of α. Therefore, it is possible to prove that the powersum formula yields a nonzero answer if we can prove that it yields a nonzero answer for, at least, one coefficient of α in, at least, one term of the resolvent. This paper will prove that the powersum formula yields a nonzero value for the coefficient $F_{1,0}$ of the first power of α in the zeroth derivative term of the resolvent.

The author would like to make one point about terminology. It feels more natural to say a single object, like a root of a polynomial, is differentially *transcendental* over some field rather than differentially *independent*. Indeed, without the preceding adverb *differentially,* it makes no sense to refer to a single object being *independent* over anything. However, it does make sense to say that a single object and all of its derivatives are algebraically independent over a field, which is the definition of the object being differentially transcendental over the field. Therefore, since the case of several objects being differentially independent covers the case of any one of them being differentially transcendental, the author has adopted the terminology *differentially independent* throughout this paper. However, in future papers, the author will refer to a polynomial, considered to be a single object, as being differentially transcendental (*dt polynomial*) if all its distinct roots are differentially independent over constants.

2. Example: polynomial with relations on the roots. It is worth mentioning that there exist polynomials whose coefficients are essentially the opposite of being differentially independent over constants for which the powersum formula yields a nonzero answer. The readers may be interested in verifying, for themselves, that the powersum formula works on the following polynomial which has many algebraic relations among its coefficients and roots. This is [12, Example 99, page 166]. The cubic polynomial $P(t) \equiv t^3 - x\cdot(1+x+x^2)\cdot t^2 + x^2\cdot(1+x+x^2)\cdot t - x^6$ has roots $z = \{x, x^2, x^3\}$ and coefficients $e_1 = x\cdot(1+x+x^2)$, $e_2 = x^3\cdot(1+x+x^2)$, and $e_3 = x^6$. We can verify that $x = e_1/e_2\cdot(e_2+e_3)/(1+e_1)$. So, x lies in the coefficient field $\mathbb{Q}(e_1,e_2,e_3)$ of P. This is a particular case of [12, Lemma 100, page 167]. It has an αth power resolvent of the form $c_{0,3}\cdot x^3\cdot D^3 y + (c_{0,2}+c_{1,2}\cdot\alpha)\cdot x^2\cdot D^2 y + (c_{0,1}+c_{1,1}\cdot\alpha + c_{2,1}\cdot\alpha^2)\cdot Dy + c_{3,0}\cdot\alpha^3\cdot y = 0$, where $Dx = 1$, $y = z^\alpha$, and all seven $c_{i,m} \neq 0$ and $c_{i,m} \in \mathbb{Z}[x]$. There clearly exists no αth power resolvent of lower order with fewer powers of α. This is a particular case of [12, Lemma 98, page 163] for which an αth power resolvent was computed for all polynomials of the form $P(t) = \prod_{i=1}^n (t - x^i)$ without using the powersum formula.

Although we could specialize α to any set of integers we like, it is natural to specialize α to the set of integers from one to one less than the number of nonzero coefficients, $c_{i,m}$. It is this choice of integers that defines the powersum formula. So, in the example above, if we specialize α to each of the integers $q \in \{1,2,3,4,5,6\}$, then y is specialized to z^q. If we sum the resulting equation over each of the three roots $z \in \{x, x^2, x^3\}$, we obtain the following homogeneous linear system of six equations in seven unknowns:

$$\begin{bmatrix} x^3D^3p_1 & x^2D^2p_1 & x^2\cdot 1\cdot D^2p_1 & x\cdot Dp_1 & x\cdot 1\cdot Dp_1 & x\cdot 1^2\cdot Dp_1 & 1^3\cdot p_1 \\ x^3D^3p_2 & x^2D^2p_2 & x^2\cdot 2\cdot D^2p_2 & x\cdot Dp_2 & x\cdot 2\cdot Dp_2 & x\cdot 2^2\cdot Dp_2 & 2^3\cdot p_2 \\ x^3D^3p_3 & x^2D^2p_3 & x^2\cdot 3\cdot D^2p_3 & x\cdot Dp_3 & x\cdot 3\cdot Dp_3 & x\cdot 3^2\cdot Dp_3 & 3^3\cdot p_3 \\ x^3D^3p_4 & x^2D^2p_4 & x^2\cdot 4\cdot D^2p_4 & x\cdot Dp_4 & x\cdot 4\cdot Dp_4 & x\cdot 4^2\cdot Dp_4 & 4^3\cdot p_4 \\ x^3D^3p_5 & x^2D^2p_5 & x^2\cdot 5\cdot D^2p_5 & x\cdot Dp_5 & x\cdot 5\cdot Dp_5 & x\cdot 5^2\cdot Dp_5 & 5^3\cdot p_5 \\ x^3D^3p_6 & x^2D^2p_6 & x^2\cdot 6\cdot D^2p_6 & x\cdot Dp_6 & x\cdot 6\cdot Dp_6 & x\cdot 6^2\cdot Dp_6 & 6^3\cdot p_6 \end{bmatrix} \cdot \begin{bmatrix} c_{0,3} \\ c_{0,2} \\ c_{1,2} \\ c_{0,1} \\ c_{1,1} \\ c_{2,1} \\ c_{3,0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.1}$$

Here,

$$\begin{aligned} p_1 &= x + x^2 + x^3, & p_2 &= x^2 + x^4 + x^6, & p_3 &= x^3 + x^6 + x^9, \\ p_4 &= x^4 + x^8 + x^{12}, & p_5 &= x^5 + x^{10} + x^{15}, & p_6 &= x^6 + x^{12} + x^{18} \end{aligned} \tag{2.2}$$

are the first six powersums of the roots of P. We now set each $c_{i,m}$ equal to its corresponding 6×6 cofactor and divide these seven cofactors by their common factor

$\vartheta = 34560 \cdot (x-1)^7 \cdot x^{18} \cdot (x+1) \cdot \wp(x)$, where

$$\begin{aligned}\wp(x) = &-1+6x+5x^2-40x^3-21x^4+158x^5+242x^6-282x^7-1192x^8\\ &-1710x^9-870x^{10}+1698x^{11}+2316x^{12}+846x^{13}+246x^{14}\\ &-594x^{15}-375x^{16}+324x^{17}+9x^{18}-54x^{19}+9x^{20}.\end{aligned} \tag{2.3}$$

The final result is $[c_{0,3}, c_{0,2}, c_{1,2}, c_{0,1}, c_{1,1}, c_{2,1}, c_{3,0}] = [1, 3, -6, 1, -6, 11, -6]$, which yields the correct minimal resolvent $x^3 \cdot y''' + (3-6\cdot\alpha)\cdot x^2 \cdot y'' + (1-6\cdot\alpha+11\cdot\alpha^2)\cdot x \cdot y' - 6\cdot\alpha^3\cdot y = 0$.

3. Notation. We will use the symbols $\exists$ for *there exists*, $\ni$ for *such that*, $\forall$ for *for all*, and $\equiv$ for *is defined as*. Let $\mathbb{Z}$ denote the ring of integers. Let $\mathbb{N}$ denote the set of positive integers. Let $\mathbb{N}_0$ denote the set of nonnegative integers. Let $\mathbb{Q}$ denote the field of rational numbers. Let $\mathbb{S}$ stand for either $\mathbb{Z}$ or $\mathbb{Q}$. Let $\mathbb{S}^\#$ denote $\mathbb{S}$ with zero removed. For any $m \in \mathbb{N}_0$, define $[m] \equiv \{k \in \mathbb{N} \ni 1 \le k \le m\}$ and $[m]_0 \equiv \{k \in \mathbb{N}_0 \ni 0 \le k \le m\}$. For any $m \in \mathbb{N}$ and any variable or number v, define $(v)_m \equiv \prod_{i=1}^m (v-i+1)$. Define $(v)_0 \equiv 1$.

Let $\mathbb{R}$ be a differential domain of characteristic 0 with derivation D. Let $\Bbbk$ be the subfield of constants of $\mathbb{R}$ with respect to the derivation D. It will cause almost no greater difficulty to consider a polynomial with multiple roots than one with simple roots, provided the distinct roots themselves are differentially independent over constants. Let P be a monic polynomial of a single variable t over $\mathbb{R}$, $P \in \mathbb{R}[t]$, of degree N with n distinct roots $z \equiv \{z_j\}_{j=1}^n$ with multiplicities $\{\pi_j\}_{j=1}^n$, respectively. So, $P = \prod_{j=1}^n (t-z_j)^{\pi_j}$, where $N = \sum_{j=1}^n \pi_j$. We will write P in the form $P(t) \equiv \sum_{i=0}^N (-1)^{N-i} e_{N-i} \cdot t^i$ with coefficients $e_{N-i} \in \mathbb{R}$. The notation e_j is used to denote the jth elementary symmetric function of the z. Let $e \equiv \{e_j\}_{j=1}^n$ denote the set of coefficients of P. For any $q \in \mathbb{Z}$, we denote and define the qth *powersum* of the roots of P by $p_q \equiv \sum_{l=1}^n \pi_l \cdot z_l^q$. We call q the *weight* of the powersum p_q. By a very minor generalization of [12, Theorem 1, page 23] to account for their multiplicities, the n distinct roots z are differentially independent over $\mathbb{Z}$ if and only if the first n powersums $\{p_q\}_{q=1}^n$ are differentially independent over $\mathbb{Z}$. Hence, we may refer to either of these conditions interchangeably. So, from now on, we will assume that the roots of P are differentially independent over $\Bbbk$. By some minor deductions made from the remarks of Kolchin immediately following [8, Corollary 1, page 87] differential independence over some field of constants $\Bbbk$ is the same as differential independence over any field of constants, such as $\mathbb{Q}$. So, from this point on, it is sufficient to assume that the roots of P are differentially independent over $\mathbb{Q}$.

It is important to keep in mind that only the n elementary symmetric functions $\{\bar{e}_j\}_{j=1}^n$ of the n distinct roots z, not the N elementary symmetric functions $e \equiv \{e_j\}_{j=1}^N$ of the N roots z including their multiplicities, are differentially independent over constants if and only if the n distinct roots z are differentially independent over constants. Independent of this fact, the powersum formula yields a resolvent whose terms lie in $\mathbb{Z}\{e\}$, the coefficient ring of the polynomial P. We will not consider $\{\bar{e}_j\}_{j=1}^n$ in this paper again.

We will use the Kolchin [8] notation for the adjunction of differential elements to rings and fields. For any set of elements $a \equiv \{a_1,\dots,a_n\}$, lying in an ordinary differential ring extension of $\mathbb{S}$, let $\mathbb{S}[a]$, $\mathbb{S}\{a\}$, $\mathbb{S}(a)$, and $\mathbb{S}\langle a\rangle$ denote, respectively, the nondifferential ring, the differential ring, the nondifferential field, and the differential field generated by $\mathbb{S}$ and a. For any $m \in \mathbb{N}_0$, let $\mathbb{S}\{a\}_m$ and $\mathbb{S}\langle a\rangle_m$ denote, respectively, the nondifferential ring and field generated by $\mathbb{S}$, a, and the derivatives of a up through mth order. Then, $\mathbb{S}\{a\}_0 = \mathbb{S}[a]$ and $\mathbb{S}\langle a\rangle_0 = \mathbb{S}(a)$. By an easy generalization of the material on [10, pages 19–25] to the differential case, we have $\mathbb{Q}\{p\}_m = \mathbb{Q}\{e\}_m$, $\mathbb{Z}\{p\}_m \subset \mathbb{Z}\{e\}_m$, and $\mathbb{Q}\langle p\rangle_m = \mathbb{Q}\langle e\rangle_m$ for any $m \in \mathbb{N}_0$ and $\mathbb{Q}\{p\} = \mathbb{Q}\{e\}$, $\mathbb{Z}\{p\} \subset \mathbb{Z}\{e\}$. Even though the powersum formula uses powersums p_q, whose weights q are much bigger than n, specifically up through weight $n(n^2-n+2)/2$, it is worth mentioning that $D^m p_q \in \mathbb{Z}\{e\}_m$, $\forall m \in \mathbb{N}_0$ and $\forall q \in \mathbb{N}_0$. That is, every entry in the matrix of the powersum formula lies in the differential ring $\mathbb{Z}\{e\}$, generated by the coefficients e of P over $\mathbb{Z}$. Therefore, the determinant of this matrix lies in $\mathbb{Z}\{e\}$.

Let α be transcendental over $\mathbb{Z}\{e\}$ with $D\alpha = 0$. For each root z_j of P, let y_j denote a nonzero solution of the first-order logarithmic differential equation $z_j \cdot Dy_j - \alpha \cdot y_j \cdot Dz_j = 0$. Formally, we may think of y as the αth power of z up to a constant multiple. By [12, Theorem 40, page 71], there exists a nonzero, nth order, linear ordinary differential equation with coefficients $\theta_{i,m} \in \mathbb{Z}\{e\}_n$ of the form $\sum_{m=0}^{n}\sum_{i=0}^{\Omega-m}\theta_{i,m}\cdot\alpha^i\cdot D^m y = 0$, where $\Omega \equiv n(n-1)/2+1$, $\theta_{0,0} = 0$, and all other $\theta_{i,m} \neq 0$. This ordinary differential equation is called an αth *power differential resolvent* of P. We call the $\theta_{i,m}$ the *coefficient functions* of the resolvent. Define $\Phi \equiv n\cdot\Omega+1$. Then, $\Phi = (n^3-n^2+2n+2)/2$. There is a total of Φ nonzero coefficient functions $\theta_{i,m}$ in this resolvent. Let $\underset{\frown}{\mathfrak{I}}$ denote the indices (i,m) of the nonzero coefficient functions $\theta_{i,m}$ in this resolvent. Then,

$$\underset{\frown}{\mathfrak{I}} = \{(i,m) \ni i \in [\Omega-m]_0,\ m \in [n]_0,\ (i,m) \neq (0,0)\}. \tag{3.1}$$

So, $|\underset{\frown}{\mathfrak{I}}| = \Phi$.

The choice of $\theta_{i,m}$ is not unique since we may multiply a resolvent of this form by an element of $\mathbb{Z}\{e\}$ to get another resolvent of this form. Ideally, we seek a set of $\theta_{i,m}$ that has no common factor over $\mathbb{Z}\{e\}$ except for the units ± 1. Define $\Psi \equiv n\cdot\Omega$. Then, $\Psi = \Phi - 1$. Let $F_{i,m}$ denote the particular choice of $\theta_{i,m}$ we get by applying the powersum formula with the choice of integers $q \in [\Psi]$. That is, $\sum_{(i,m)\in\underset{\frown}{\mathfrak{I}}} F_{i,m}\cdot\alpha^i D^m y = 0$, where

$$F_{i,m} \equiv (-1)^{\operatorname{sgn}(i,m)}\left|q^{i'}D^{m'}p_q\right|_{q\times(i,m)}. \tag{3.2}$$

We call (3.2) the *determinantal formula* for $F_{i,m}$. Here, $\operatorname{sgn}(i,m)$ denotes the order of the pair of indices (i,m) after ordering them in the set $\underset{\frown}{\mathfrak{I}}$. In this formula, the rows of the matrix $[q^{i'}D^{m'}p_q]$ are labelled by q as q spans the set $[\Psi]$, the columns are labelled by (i',m') as (i',m') spans the set $\underset{\frown}{\mathfrak{I}} - \{(i,m)\}$, and $|q^{i'}D^{m'}p_q|_{q\times(i,m)}$ denotes the determinant of $[q^{i'}D^{m'}p_q]$. We will assume these conditions and notation henceforth. We refer to $q^iD^mp_q$ as a *column of order* m in the determinantal formula for $F_{1,0}$.

From this point on, in the resolvent $\sum_{(i,m)\in\underset{\frown}{\mathfrak{I}}}\theta_{i,m}\cdot\alpha^i\cdot D^m y=0$, let $\theta_{i,m}$ denote the coefficient functions which have no common factor over $\mathbb{Z}\{e\}$ except ± 1. This resolvent, unique up to sign, is called the *Cohnian* of P. Currently, the Cohnian of polynomials whose distinct roots are differentially independent over constants is known only for the cases $n=2$ and $n=3$. It has been shown in [12, Lemma 66, page 121] that either $F_{i,m}=0\ \forall(i,m)\in\underset{\frown}{\mathfrak{I}}$ or there exists some very large common factor $\vartheta\in\mathbb{Z}\{e\}_n$, such that $F_{i,m}=\vartheta\cdot\theta_{i,m}\ \forall(i,m)\in\underset{\frown}{\mathfrak{I}}$. We will prove that $F_{i,m}\neq 0\ \forall(i,m)\in\underset{\frown}{\mathfrak{I}}$ when the distinct roots of P are differentially independent over $\mathbb{Q}$. We will not attempt to factor $F_{i,m}$ over the ring $\mathbb{Z}\{e\}$ in this paper. A general algorithm for completely factoring all the $F_{i,m}$ is unknown at this time, although a general algorithm for pulling out a large factor from some of the $F_{i,m}$ has been proven in [12, Theorem 62, page 114]. However, we will make use of a trivial factorization of the term $F_{1,0}$ in (5.2) to prove that $F_{1,0}\neq 0$, from which it follows that the powersum formula yields a (nonzero) resolvent.

4. Powersum nonvanishing theorem. The aim of this paper is to prove the following theorem.

Theorem 4.1 (powersum nonvanishing theorem). *Let the univariate polynomial* $P(t)\equiv\Pi_{j=1}^n(t-z_j)^{\pi_j}=\sum_{i=0}^N(-1)^{N-i}e_{N-i}\cdot t^i$ *have* n *distinct roots* $\{z_j\}_{j=1}^n$ *which are differentially independent over* $\mathbb{Q}$. *Let* $\underset{\frown}{\mathfrak{I}}$ *be defined by the set of pairs of integers given by (3.1). Define* $\Phi\equiv(n^3-n^2+2n+2)/2$ *and assume all other definitions in Section 3. Then, the powersum formula (3.2) yields a nonzero value for each of the* Φ *coefficient functions* $F_{i,m}$ *in the* α*th power differential resolvent* $\sum_{(i,m)\in\underset{\frown}{\mathfrak{I}}}F_{i,m}\cdot\alpha^i\cdot D^m y$ *of* P.

We will prove Theorem 4.1 in Section 9.

Define $\mathfrak{I}\equiv\underset{\frown}{\mathfrak{I}}-\{(1,0)\}$. Then, $\mathfrak{I}$ is the set of pairs of nonnegative integers (i,m) such that $i\in[\Omega-m]_0$, $m\in[n]_0$, and $(i,m)\notin\{(0,0),(1,0)\}$. The set $\mathfrak{I}$ represents all the terms $\alpha^i\cdot D^m y$, except $\alpha\cdot y$, that appear in the Cohnian of P. We will prove that the coefficient of $\alpha\cdot y$ in the differential resolvent $\sum_{m=0}^n\sum_{i=0}^{\Omega-m}F_{i,m}\cdot\alpha^i\cdot D^m y=0$, given by $F_{1,0}=(-1)^{\operatorname{sgn}(1,0)}\cdot|q^{i'}D^{m'}p_q|_{q\times(i',m')}$ where (i',m') spans $\mathfrak{I}$, is not identically zero. By the author's minimal form theorem [12, Theorem 40, page 71], P can have no αth power resolvent of order lower than n, and, among those resolvents of order n, none can have fewer than Φ nonzero coefficient functions of α. Therefore, if the powersum formula yields one nonzero coefficient, then the powersum formula for all the other coefficients must be nonzero. Therefore, to prove Theorem 4.1, it will be sufficient to prove $F_{1,0}\neq 0$.

We will now give in Sections 5 through 9 the prerequisite material and theorems for the proof of Theorem 4.1. From this point on, we assume that we have ordered the pairs (i,m) such that $(-1)^{\operatorname{sgn}(1,0)}=1$.

5. Factorization of the term $F_{1,0}$ in the resolvent. Consider the differential ring inclusion $\mathbb{Z}\{p_1,\dots,p_\Psi\}\subset\mathbb{Z}\{z_1,\dots,z_n\}$, where the smaller ring is generated by the first Ψ powersums of the roots $z_1,\dots,z_n$. The powersum formula shows that $F_{1,0}\in\mathbb{Z}\{p_1,\dots,p_\Psi\}_n\subset\mathbb{Z}\{p_1,\dots,p_\Psi\}$. Consider further the ordinary ring inclusion $\mathbb{Z}\{z_1,\dots,z_z\}\subset\mathbb{Z}\{z_1,\dots,z_n\}[z_1^{-1},\dots,z_n^{-1}]$. We will factor $F_{1,0}$ as the product of an element of the ordinary ring $\mathbb{Z}[z_1,\dots,z_n]$ and an element of the ring $\mathbb{Z}\{z_1,\dots,z_n\}[z_1^{-1},\dots,z_n^{-1}]$. The element from $\mathbb{Z}\{z_1,\dots,z_n\}[z_1^{-1},\dots,z_n^{-1}]$ will not depend upon the variable q.

We define a monomial in the derivatives of the roots to be formal products of the form $\prod_{i=1}^{n}\prod_{m\geq 0}(D^m z_i)^{\upsilon_{m,i}}$, with $\nu_{m,j}\in\mathbb{N}_0$, without any integer coefficients.

We factor $F_{1,0}$ in the following way. For each $m\in[n]$, express the mth derivative of p_q in the following way:

$$\begin{aligned} D^m p_q &= \sum_{l=1}^{n}\pi_l\cdot D^m z_l^q \\ &= \sum_{l=1}^{n}\sum_{j=0}^{m} B_{m,j}(Dz_l,D^2z_l,\dots)\cdot(q)_j z_l^{q-j} \\ &= \sum_{l=1}^{n}\pi_l\cdot\sum_{j=0}^{m}B_{m,j}\{z_l\}\cdot\sum_{k=0}^{j}s_k^j\cdot q^k\cdot z_l^{q-j} \\ &= \sum_{l=1}^{n}\pi_l\cdot z_l^q\cdot\sum_{k=0}^{m}A_{m,l,k}q^k, \end{aligned} \tag{5.1}$$

where $B_{m,j}\{z_l\}$ are the partial Bell polynomials in the derivatives of z_l as defined on [10, page 30], s_k^j are the Stirling numbers of the first kind as defined on [10, page 31] using the notation on [13, page 7], and $A_{m,l,k}\equiv\sum_{j=k}^{m}B_{m,j}\{z_l\}\cdot s_k^j\cdot z_l^{-j}$. Then, $A_{m,l,k}\in\mathbb{Z}\{z_1,\dots,z_n\}[z_1^{-1},\dots,z_n^{-1}]$ and does not depend upon q. Later we will state the definitions and properties of $B_{m,j}\{z_l\}$ and s_k^j that are necessary for the proofs.

Next, multiply $D^m p_q$ by q^i to get $q^iD^mp_q=\sum_{l=1}^{n}\sum_{k=0}^{m}A_{m,l,k}\cdot q^{i+k}\cdot z_l^q$. Define $t\equiv i+k$. So, $k=t-i$, and, hereafter, we need consider only $i\leq t\leq i+m$. So $q^iD^mp_q=\sum_{l=1}^{n}\pi_l\cdot\sum_{t=i}^{t=i+m}A_{m,l,t-i}\cdot q^t\cdot z_l^q$. Since $i+m\in[\Omega]$, $\forall(i,m)\in\mathfrak{I}$, we have $t\in[\Omega]$, $\forall(i,m)\in\mathfrak{I}$. Thus, we may factor $F_{1,0}$ as

$$\begin{aligned} F_{1,0} &= \left|q^iD^mp_q\right|_{q\times(i,m)} \\ &= \left|\sum_{l=1}^{n}\pi_l\cdot\sum_{t=i}^{t=i+m}A_{m,l,t-i}\cdot q^t\cdot z_l^q\right|_{q\times(i,m)} \\ &= \left|q^t\cdot\pi_l\cdot z_l^q\right|_{q\times(l,t)}\cdot\left|A_{m,l,t-i}\right|_{(l,t)\times(i,m)}. \end{aligned} \tag{5.2}$$

Thus, (l,t) labels the rows and (i,m) labels the columns in the first determinant on the right. The pair (l,t) spans the Cartesian product $[n]\times[\Omega]$ and the pair (i,m) spans the set $\mathfrak{I}$. Define the matrix

$$A\equiv\left[A_{m,l,t-i}\right]_{(l,t)\times(i,m)}. \tag{5.3}$$

Define $\aleph\equiv[n]\times[\Omega]$. Then, the rows of A are labelled by $(l,t)\in\aleph$ and the columns are labelled by $(i,m)\in\mathfrak{I}$.

6. First factor is nonzero

Theorem 6.1. *The determinant $|q^t\cdot\pi_l\cdot z_l^q|_{q\times(l,t)}$ in the factorization of $F_{1,0}$ is not zero as the row index q spans $[\Psi]$ and the column index (l,t) spans the Cartesian product $[n]\times[\Omega]$.*

Proof. The matrix $[q^t \cdot \pi_l \cdot z_l^q]_{q\times(l,t)}$ is $\Psi\times\Psi$. To see that $|q^t \cdot \pi_l \cdot z_l^q|_{q\times(l,t)} \neq 0$, pick out the highest powers of z_1 first. These will come from the $\Omega\times\Omega$ block $[q^t \cdot \pi_1 \cdot z_1^q]_{(\Psi-\Omega+1\le q\le\Psi)\times(t\in[\Omega])}$ with determinant

$$\begin{aligned} |q^t \cdot \pi_1 \cdot z_1^q|_{(\Psi-\Omega+1\le q\le\Psi)\times(t\in[\Omega])} &= \pi_1^\Omega \cdot \left(\prod_q z_1^q\right) \cdot |q^t|_{q\times t} \\ &= \pi_1^\Omega \cdot (z_1^{\beta_1}) \cdot \chi_1 \cdot \prod_{q''<q'} (q''-q') \neq 0, \end{aligned} \tag{6.1}$$

where $\beta_1 \equiv \sum_{q=\Psi-\Omega+1}^{q=\Psi} q$ and $\chi_1 \equiv \prod_{q=\Psi-\Omega+1}^{q=\Psi} q$. The highest power of z_2 in the remaining matrix comes from the $\Omega\times\Omega$ block $[q^t \cdot \pi_2 \cdot z_2^q]_{(\Psi-2\Omega+1\le q\le\Psi-\Omega)\times(t\in[\Omega])}$ with determinant $|q^t \cdot \pi_2 \cdot z_2^q|_{(\Psi-2\Omega+1\le q\le\Psi-\Omega)\times(t\in[\Omega])} = \pi_2^\Omega \cdot (\prod_q z_2^q) \cdot |q^t|_{q\times t} = \pi_2^\Omega \cdot (z_2^{\beta_2}) \cdot \chi_2 \cdot \prod_{q''<q'}(q''-q') \neq 0$, where $\beta_2 \equiv \sum_{q=\Psi-2\Omega+1}^{q=\Psi-\Omega} q$ and $\chi_2 \equiv \prod_{q=\Psi-2\Omega+1}^{q=\Psi-\Omega} q$. By similar procedures, we may continue and ultimately prove that $\det[q^t \cdot \pi_l \cdot z_l^q]_{(q\in[\Psi])\times(l\in[n],t\in[\Omega])} \neq 0$. This concludes the proof of Theorem 6.1. □

7. Properties of the second factor. As noted in [10, page 31], the Stirling number of the first kind s_k^j is $(-1)^{j-k}$ times the $(j-k)$th elementary symmetric function of the $j-1$ integers $[j-1]$ when $j>0$. Thus, $s_k^j \neq 0\ \forall j\ge k>0$, $s_0^j = 0\ \forall j>0$ and $s_k^0 = 0\ \forall j>0$. We define $s_0^0 \equiv 1$.

We continue with the definitions and some of the notation in [10, page 1]. We must use the letters i and m elsewhere in this paper; so, in place of these letters in Macdonald's definitions, we will use the letters u and θ. For our purposes, a *partition* λ is a finite decreasing sequence of positive integers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell$ called the *parts* of λ. The number of parts ℓ is the *length* of λ. To emphasize the particular partition, we will sometimes write $\ell(\lambda)$ for the length of λ. For our purposes in this paper, we need to consider only partitions all of whose parts are $\le n$. For each $u\in[n]$, define $\theta_u \in \mathbb{N}_0$ to be the number of parts of λ equal to u. We call θ_u the *multiplicity of u in λ*. Thus, $\theta_u = 0$ for all $u>n$ since all parts of λ are $\le n$. We will no longer deal directly with the individual parts λ_ν of a partition λ but rather with these multiplicities and write $\lambda = (1^{\theta_1}2^{\theta_2}\cdots n^{\theta_n})$. Hence, $\ell = \sum_{u=1}^n \theta_u$. We define $|\lambda|$ to be the sum of the parts of λ and call it the *weight* of λ. Hence, $|\lambda| = \sum_{u=1}^n u\cdot\theta_u$. We say λ is a *partition of the integer* $|\lambda|$. The weight of λ is not to be confused with the weight q of the qth powersum p_q, although they are related.

From these definitions, it follows that $1\le \ell(\lambda)\le m$ for all partitions λ of a positive integer m. We need to consider only the case $m\le n$. We note the two extreme cases on $\ell(\lambda)$. There exists exactly one partition λ such that $\ell(\lambda)=1$, namely, $\lambda=(m^1)$. Thus, $\theta_m = 1$ and $\theta_u = 0$ for all $u\neq m$. There exists exactly one partition λ such that $\ell(\lambda) = m$, namely, $\lambda = (1^m)$. Thus, $\theta_m = m$ and $\theta_u = 0$ for all $u\neq 1$.

Let us temporarily drop the subscript l on the root z_l. According to [10, page 31], $B_{m,j}\{z\} = \sum_\lambda c_\lambda \cdot \prod_{\nu=1}^n (D^\nu z)^{\theta_\nu}$, where the sum is over all partitions $\lambda = (1^{\theta_1}\cdots n^{\theta_n})$ of m of length j and $c_\lambda \equiv m!/\prod_{u=1}^n(\theta_u!\cdot(u!)^{\theta_u})$. The constant c_λ is always a positive integer. This formula for $B_{m,j}\{z\}$ implies the following three remarks. Firstly, for $r\in[n]$, the inequality $r\cdot\theta_r \le \sum_{u=1}^m u\cdot\theta_u = |\lambda| = m$ implies that there exist no partitions of m with $\theta_r>0$ if $r>m$. Secondly, if $r\le m$ and $j\in[m]$, there might exist no partitions of m of length j with $\theta_r > 0$. This occurs in the case $r < m$ and $j = 1$, for which the only partition, namely, $\lambda = (m^1)$, has $\theta_r = 0$ for $r < m$ and $\theta_m = 1$. Thirdly, $B_{m,j}\{z\} = 0$ if $j > m$. When any of these three conditions occurs, we say that $D^r z$ *does not appear* in $B_{m,j}\{z\}$ and define the degree of $D^r z$ in $B_{m,j}\{z\}$ to be 0.

Since all roots $\{z_l\}_{l=1}^n$ are differentially independent over constants, any root z is differentially transcendental over constants. Therefore, if $\lambda = (1^{\theta_1} \cdots n^{\theta_n})$ and $\lambda' = (1^{\theta'_1} \cdots n^{\theta'_n})$ are two distinct partitions, even if they have the same length and weight, the monomials $\prod_{\nu=1}^n (D^\nu z)^{\theta_\nu}$ and $\prod_{\nu=1}^n (D^\nu z)^{\theta'_\nu}$ cannot cancel. Therefore, the degree of $D^r z$ in $B_{m,j}\{z\}$ equals the maximum θ_r over all partitions λ of m of length j if they exist. By the inequalities $\theta_r \le \sum_{u=1}^m \theta_u = \ell(\lambda) = j$ and $r \cdot \theta_r \le \sum_{u=1}^m u \cdot \theta_u = |\lambda| = m$, the degree of $D^r z$ in $B_{m,j}\{z\}$ is $\le j$ and $\le m/r$. This includes the possibility that the degree of $D^r z$ in $B_{m,j}\{z\}$ is 0. Therefore, in the case $m = r$, the degree of $D^r z$ in $B_{r,j}\{z\}$ is $\le m/r = r/r = 1$, so it must equal 0 or 1.

If $r > 1$ and $m = r$, then the inequalities $\sum_{u=1}^r \theta_u = j$ and $\sum_{u=1}^r u \cdot \theta_u = r$ imply $(r-1) \cdot \theta_r \le \sum_{u=2}^r (u-1) \cdot \theta_u = (r-j)$. Thus, if $j > 1$, there exists no partition λ of r of length j such that $\theta_r > 0$. Therefore, $D^r z_l$ does not appear in $B_{r,j}\{z_l\}$ for $j > 1$. If $j = 1$, then there exists exactly one partition of r of weight j, namely, $\lambda = (r^1)$. Therefore, $D^r z_l$ appears in $B_{r,1}\{z_l\}$ with nonzero coefficient $c_\lambda = c_{(r^1)} = 1$. By the last statement of the previous paragraph, the degree of $D^r z_l$ in $B_{r,1}\{z_l\}$ equals 1.

For reasons that will become apparent in the induction step of the proof of Theorem 6.1, we need to determine the degree of $D^r z_l$ in $A_{m,l,k} \equiv \sum_{j=k}^{j=m} B_{m,j}\{z_l\} \cdot s_k^j \cdot z_l^{-j}$ only for the case $m = r$ and the degree of Dz_l in $A_{m,l,k}$ for any $m \in [n]$. If $k \in [r]$, since $s_k^j \ne 0 \ \forall j \ge k > 0$, the degree of $D^r z_l$ in $A_{r,l,k}$ equals the maximum over $k \le j \le r$ of the degrees of $D^r z_l$ in $B_{r,j}\{z_l\}$. This maximum is achieved when $j = r$ with the partition $\lambda = (r^1)$, $\theta_r = 1$. So, the degree of $D^r z_l$ in $A_{r,l,k}$ equals 1.

If $r > m$, $D^r z_l$ does not appear in $A_{m,l,k}$ because $D^r z_l$ does not appear in $B_{m,j}\{z_l\}$ for any j. If $r = m$, since $A_{r,l,k}$ involves only $B_{r,j}$ with $j \ge k$, it follows that $D^r z_l$ does not appear in $A_{r,l,k}$ for $k > 1$.

We define $B_{m,j}$ for $m = 0$ and $j = 0$ so that the defining property of the Bell polynomials, which in this paper is simply $D^m z^q = \sum_{j=0}^m B_{m,j} \cdot (q)_j \cdot z^{q-j}$, still holds. We can easily see that $B_{m,0} = 0$, for all $m > 0$, $B_{0,j} = 0$ for all $j > 0$, and $B_{0,0} = 1$. From the definition $A_{m,l,k} \equiv \sum_{j=k}^{j=m} B_{m,j}\{z_l\} \cdot s_k^j \cdot z_l^{-j}$, it follows that $A_{0,l,k} = 0$ for all $k > 0$. Because both $B_{0,j} = 0$ for all $j > 0$ and $s_0^j = 0$ for all $j > 0$, it follows that $A_{m,l,0} = \sum_{j=0}^{j=m} B_{m,j}\{z_l\} \cdot s_k^j \cdot z_l^{-j} = B_{m,0}\{z_l\} \cdot s_0^0 \cdot z_l^{-0} = B_{m,0}\{z_l\}$. Thus, $A_{m,l,0} = 0$ for all $m > 0$ and $A_{0,l,0} = 1$. Thus, it trivially follows that $D^r z$ does not appear in $A_{m,l,k}$ if $k \cdot m = 0$.

We must now summarize these results for the entries $A_{m,l,t-i}$ of the matrix A. We have already mentioned that it is necessary that $i \le t \le i + m$ in order for $A_{m,l,t-i} \ne 0$. And we just proved that it is necessary that $m \cdot (t-i) \ne 0$, excluding $A_{0,l,0} = 1$. For a given monomial of the form $D^r z_r$ for $r \in [n]$, we must determine necessary conditions on the indices (l,t) and (i,m) of the entry $A_{m,l,t-i}$ for the monomial $D^r z_r$ to appear in $A_{m,l,t-i}$ with nonzero coefficient. And, when these conditions are met, we must determine the degree of $D^r z_r$ in the entry $A_{m,l,t-i}$. Since $A_{m,l,t-i}$ involves only the lth root z_l, it is necessary that $l = r$. Hence, the following properties hold.

Property P. If $r \in [n]$ with $r > 1$, $l = r$, $t - i = 1$, the degree of $D^r z_r$ in $A_{r,l,t-i}$ equals 1. If $r \in [n]$ with $r > 1$, but $l \neq r$ or $t - i \neq 1$, then $D^r z_r$ does not appear in $A_{r,l,t-i}$.

Property Q. If $r \in [n]$, the degree of Dz_r in $A_{m,l,t-i}$ is m if $m \in [n]$, $r \le m$, $l = r$, and $i < t \le i + m$. If any of these conditions is not met, then Dz_r does not appear in $A_{m,l,t-i}$.

Property R. The entry $A_{m,l,t-i} = 0$ if $t < i$ or $t > i + m$ or $m \cdot (t - i) = 0$, excluding $A_{0,l,0} = 1$.

We include Property R for reference, even though we will not refer to it again. In the preceding discussion, Property R has been used implicitly to derive Properties P and Q.

8. Induction step. Define $x_r \equiv (r-2)(r-1)/2 + 1$. Observe that the formula in this definition is independent of n. Note also that $x_r = x_{r-1} + r - 2$ and $x_n = \Omega - n + 1$ so $x_n - 1 = \Omega - n$. Note that $x_1 = x_2 = 1$ and $x_r \in \mathbb{N}\ \forall r \in \mathbb{N}$. Our next goal is to prove the claim that the monomial

$$M \equiv \left(\prod_{r=1}^{n} (D^r z_r)^{x_r} \right) \cdot \left(\prod_{r=1}^{n} (Dz_r)^{(\Omega - x_r)(r-1)} \right), \tag{8.1}$$

made up of the smaller monomials $(D^r z_r)^{x_r}$ and $(Dz_r)^{(\Omega - x_r)(r-1)}$, appears with nonzero coefficient in $F_{1,0}$. Theorem 8.1 gives necessary conditions on the rows and columns of the matrix A of (5.3) which can contribute to the monomial M in determinant $|A| = |A_{m,l,t-i}|_{(l,t)\times(i,m)}$.

Theorem 8.1. *Let $n \in \mathbb{N}$ with $n \ge 3$.*

First half. For each $r \in [n]$, the monomial $(D^r z_r)^{x_r}$ in M can come only from the product of the x_r entries of A, with $l = r$, $t \in [x_r]$, $i = t - 1$, and $m = r$.

Second half. For each $r \in [n]$ with $r > 1$, the monomial $(Dz_r)^{(\Omega - x_r)(r-1)}$ in M can come only from the product of the $\Omega - x_r$ entries of A with $l = r$, $t = i + r - 1$, $x_{r-1} \le i \le \Omega - r + 1$, and $m = r - 1$.

Proof. We will prove Theorem 8.1 by downward induction on the index r in the product defining M. Any term in the expansion of the determinant of a $\Psi \times \Psi$ matrix such as A is the product of Ψ entries of A taken from exactly Ψ distinct rows, indexed by $(l,t) \in \aleph$, and exactly Ψ distinct columns, indexed by $(i,m) \in \Im$, of A. From now on, we will say "row" in place of "row of A" and "column" in place of "column of A." When we say that a particular monomial, $(D^r z_r)^{x_r}$ for instance, "comes from" a certain set of x_r (resp., $\Omega - x_r$) rows and x_r (resp., $\Omega - x_r$) columns, we mean that the monomial appears with nonzero coefficient in the determinant of the $x_r \times x_r$ (resp., $(\Omega - x_r) \times (\Omega - x_r)$) minor of A formed from these rows and columns. We say that we have "used up" these rows and columns, suggesting that the remaining monomials comprising M must come from the determinant of the minor formed from the remaining rows and columns of A not already considered in all the previous steps.

We define the following subsets of the rows and columns of A. We include the definitions of $\aleph$ and $\Im$ again for reference. We need to define $\aleph_r$, $\breve{\aleph}_r$, $\Im_r$, and $\breve{\Im}_r$ for each $r \in [n]$:

$\aleph \equiv [n] \times [\Omega]$.

$\aleph_r \equiv \{(l,t) \in \aleph \ni l \le r\}$. So, $\aleph_n = \aleph$.

$\breve{\aleph}_r \equiv (l,t) \in \aleph$ such that $l < r$ or $l = r$ and $x_r + 1 \le t \le \Omega$.

$\Im \equiv \{(i,m) \ni m \in [n]_0,\ i \in [\Omega - m]_0\}$ excluding $\{(0,0),(1,0)\}$.

$\Im_r \equiv (i,m) \in \Im$ such that $m < r$ or $m = r$ and $i \in [x_r - 1]_0$. Since $x_n - 1 = \Omega - n$, $\Im_n = \Im$.

$\breve{\Im}_r \equiv \{(i,m) \in \Im \ni m < r\}$.

We say a column of A indexed by (i,m) has *order* m.

Induction hypothesis. Let $r' \in [n]$.

First half. For each $r \ge r'$, $(D^r z_r)^{x_r}$ can come only from rows in $\aleph_r$ with $l = r$ and $t \in [x_r]$ and columns in $\Im_r$ with $i \in [x_r - 1]_0$ and $m = r$ paired up by $t = i + 1$.

Second half. Let $r' > 1$. For each $r \ge r'$, $(Dz_r)^{(\Omega - x_r)(r-1)}$ can come only from rows in $\breve{\aleph}_r$ with $l = r$ and $x_r + 1 \le t \le \Omega$, and columns in $\breve{\Im}_r$ with $x_{r-1} \le i \le \Omega - (r-1)$ and $m = r - 1$ paired up by $t = i + r - 1$.

Start of induction

First half. We begin the induction with $r' = n$. Then, $(D^r z_r)^{x_r} = (D^n z_n)^{x_n}$. Since the nth derivative is the highest derivative in the columns in $\Im_n = \Im$, it follows that the monomial $(D^n z_n)^{x_n}$ can come only from columns with $m = n$. By Property P, the monomial $(D^n z_n)^{x_n}$ can come only from rows and columns with $l = n$ and $t - i = 1$. By Property P, the degree of $D^n z_n$ in $A_{m,l,t-i}$ for $l = n$, $m = n$, and $t - i = 1$ equals 1. Therefore, the monomial $(D^n z_n)^{x_n}$ must come from x_n columns and, therefore, from x_n rows. But there are only $x_n = \Omega - n + 1$ columns $(i,m) \in \Im_n$ with $m = n$, namely, $(i,m) \in [x_n - 1]_0 \times \{n\} \subset \Im_n$. So, all x_n columns of order n in $\Im_n$ have been taken. For each column $i \in [x_n - 1]_0$, there exists a corresponding row t subject to $t - i = 1$, by Property P. Therefore, as i spans $i \in [x_n - 1]_0$, t spans $[x_n]$. Thus, we have used up x_n rows in $\aleph_n$ with $(l,t) \in \{n\} \times [x_n] \subset \aleph_n$.

Removing $\{n\} \times [x_n]$ from $\aleph_n$ leaves $\breve{\aleph}_n$. Removing $[x_n - 1]_0 \times \{n\}$ from $\Im_n$ leaves $\breve{\Im}_n$. Therefore, in the second half of this induction step, we may look for rows only in $\breve{\aleph}_n$ and columns only in $\breve{\Im}_n$.

Second half. The previous statement implies that the monomial $(Dz_n)^{(\Omega - x_n)(n-1)}$ must come from columns with $m < n$. By Property Q, the monomial $(Dz_n)^{(\Omega - x_n)(n-1)}$ can come only from rows with $l = n$, and, for each $m \in [n-1]$, the degree of Dz_n in $A_{m,n,t-i}$ is m. Suppose that $(Dz_n)^{(\Omega - x_n)(n-1)}$ came from a set of columns indexed by some subset $T \subset \breve{\Im}_n$ with $m \le n - 1\ \forall (i,m) \in T$. The degree of Dz_n, coming from the columns of T, is $\le \sum_{(i,m) \in T} m$ and must equal the degree of Dz_n in $(Dz_n)^{(\Omega - x_n)(n-1)}$, which is obviously $(\Omega - x_n)(n-1)$. Hence, $(\Omega - x_n)(n-1) \le \sum_{(i,m) \in T} m$.

If some column (i,m) in T had $m < n - 1$, then T would contain strictly more than $\Omega - x_n$ columns to make the inequality $(\Omega - x_n)(n-1) \le \sum_{(i,m) \in T} m$ hold. But this would imply that $(Dz_n)^{(\Omega-x_n)(n-1)}$ comes from strictly more than $\Omega-x_n$ columns in $\breve{\mathfrak{I}}_n$ and thus from strictly more than $\Omega-x_n$ rows in $\breve{\aleph}_n$ with $l=n$. This contradicts the range of l and t in the indexing set $\breve{\aleph}_n$. Therefore, we must have $m=n-1$ for all pairs $(i,m)\in T\subset\mathfrak{I}$. The only condition that $(i,m)\in\mathfrak{I}$ places upon i and m is that $i\in[\Omega-m]_0$. Therefore, i must span some subset $\Upsilon\subset[\Omega-(n-1)]_0$ with $|\Upsilon|=\Omega-x_n$. We will shortly prove that $\Upsilon=\{i\ni x_{n-1}\le i\le\Omega-(n-1)\}$.

Since we have used up x_n rows in $\aleph_n$ with $l=n$ and $t\in[x_n]$ to get $(D^nz_n)^{x_n}$, this implies that $(Dz_n)^{(\Omega-x_n)(n-1)}$ must come only from the $\Omega-x_n$ rows in $\breve{\aleph}_n$ with $l=n$ and t spanning the set $x_n+1\le t\le\Omega$.

So, we have now accounted for Ω rows with $l=n$ and Ω columns such that

(i) $(D^nz_n)^{x_n}$ can come only from the x_n rows in $\aleph_n$ with $l=n$ and t spanning $[x_n]$, and the x_n columns in $\mathfrak{I}_n$ with $m=n$ and i spanning $[x_n-1]_0$, with the rows and columns paired up by the relation $t=i+1$;

(ii) $(Dz_n)^{(\Omega-x_n)(n-1)}$ can come only from the $\Omega-x_n$ rows in $\breve{\aleph}_n$ with $l=n$ and t spanning $x_n+1\le t\le\Omega$ and $\Omega-x_n$ columns in $\breve{\mathfrak{I}}_n$ with $m=n-1$ and $i\in[\Omega-(n-1)]_0$, with the rows and columns subject to $i<t\le i+n-1$.

We will show that the three conditions on i and t in (ii) force i and t to be related by $t=i+n-1$. We have t spanning $x_n+1\le t\le\Omega$, $i\in[\Omega-(n-1)]_0$ and $t\le i+n-1$. When $t=\Omega$, the second two conditions force $i=\Omega-(n-1)$. This leaves t to span $x_n+1\le t\le\Omega-1$, $i\in[\Omega-(n-1)-1]_0$, and $t\le i+n-1$. When $t=\Omega-1$, the second two conditions force $i=\Omega-(n-1)-1$. Continuing in this manner, we see that i and t get paired up by $t=i+n-1$, forcing i to span the set $\Upsilon=\{i\ni x_n-(n-1)+1\le i\le\Omega-(n-1)\}$ of size $\Omega-x_n$. Since $x_{n-1}=x_n-n+2$, it follows that $\Upsilon=\{i\ni x_{n-1}\le i\le\Omega-(n-1)\}$.

Removing the $\Omega-x_n$ rows with $m=n-1$ and $x_{n-1}\le i\le\Omega-(n-1)$ from $\breve{\mathfrak{I}}_n$ leaves $\mathfrak{I}_{n-1}$. Removing the $\Omega-x_n$ columns with $l=n$ and $x_n+1\le t\le\Omega$ from $\breve{\aleph}_n$ leaves $\aleph_{n-1}$.

General step of induction. We assume that the induction hypothesis is true for $r'>r>1$. This means that we may choose only from rows in $\aleph_r$ and columns in $\mathfrak{I}_r$. We now wish to prove the induction hypothesis true for $r'=r>1$.

First half. Since the rth derivative is the highest derivative in the columns in $\mathfrak{I}_r$, it follows that the monomial $(D^rz_r)^{x_r}$ can come only from columns with $m=r$. By Property P, the monomial $(D^rz_r)^{x_r}$ can come only from rows in $\aleph_r$ and columns in $\mathfrak{I}_r$ with $l=r$ and subject to $t-i=1$. By Property P, the degree of D^rz_r in $A_{m,l,t-i}$ for $l=r$, $m=r$, and $t-i=1$ equals 1. Therefore, the monomial $(D^rz_r)^{x_r}$ must come from x_r columns and, therefore, from x_r rows. But there are only x_r columns $(i,m)\in\mathfrak{I}_r$ with $m=r$, namely, $(i,m)\in[x_r-1]_0\times\{r\}\subset\mathfrak{I}_r$. So, all x_r columns in $\mathfrak{I}_r$ of order r have been taken. By Property P, for each column $i\in[x_r-1]_0$, there exists a corresponding row t subject to $t-i=1$. Therefore, as i spans $i\in[x_r-1]_0$, t spans $[x_r]$. Thus, we have used up x_r rows in $\aleph_r$ with $(l,t)\in\{r\}\times[x_r]\subset\aleph_r$.

Removing the rows $\{r\}\times[x_r]$ from $\aleph_r$ leaves $\breve{\aleph}_r$. Removing the columns $[x_r-1]_0\times\{r\}$ from $\mathfrak{I}_r$ leaves $\breve{\mathfrak{I}}_r$. Therefore, in the second half of this induction step, we may look for rows only in $\breve{\aleph}_r$ and columns only in $\breve{\mathfrak{I}}_r$.

Second half. The previous statement implies that the monomial $(Dz_r)^{(\Omega-x_r)(r-1)}$ must come from columns with $m<r$. By Property Q, for each $m\in[r-1]$, the monomial $(Dz_r)^{(\Omega-x_r)(r-1)}$ can come only from rows in $\breve{\aleph}_r$ with $l=n$ and the degree of Dz_r in $A_{m,n,t-i}$ is m. Suppose that $(Dz_r)^{(\Omega-x_r)(r-1)}$ came from a set of columns indexed by some subset $T\subset\breve{\mathfrak{I}}_r$ with $m\le r-1\ \forall(i,m)\in T$. The degree of Dz_r coming from the columns of T is $\le\sum_{(i,m)\in T}m$ and must equal the degree of Dz_r in $(Dz_r)^{(\Omega-x_r)(r-1)}$, which is obviously $(\Omega-x_r)(r-1)$. Hence, $(\Omega-x_r)(r-1)\le\sum_{(i,m)\in T}m$.

If some column (i,m) in T had $m<r-1$, then T would contain strictly more than $\Omega-x_r$ columns to make the inequality $(\Omega-x_r)(r-1)\le\sum_{(i,m)\in T}m$ hold. But this would imply that $(Dz_r)^{(\Omega-x_r)(r-1)}$ comes from strictly more than $\Omega-x_r$ columns in $\breve{\mathfrak{I}}_r$ and thus from strictly more than $\Omega-x_r$ rows in $\breve{\aleph}_r$ with $l=r$. This contradicts the range of l and t in the indexing set $\breve{\aleph}_r$. Therefore, we must have $m=r-1$ for all pairs $(i,m)\in T\subset\breve{\mathfrak{I}}_r$. The only condition that $(i,m)\in\breve{\mathfrak{I}}_r$ places upon i and m is that $i\in[\Omega-m]_0$. Therefore, i must span some subset $\Upsilon\subset[\Omega-(r-1)]_0$ with $|\Upsilon|=\Omega-x_r$. We will shortly prove that $\Upsilon=\{i\ni x_{r-1}\le i\le x_r\}$.

Since we have used up x_r rows in $\aleph_r$ with $l=r$ and $t\in[x_r]$ to get $(D^rz_r)^{x_r}$, this implies $(Dz_r)^{(\Omega-x_r)(r-1)}$ must come only from the $\Omega-x_r$ rows in $\breve{\aleph}_r$ with $l=r$ and t spanning the set $x_r+1\le t\le\Omega$.

So, we have now accounted for Ω rows with $l=r$ and Ω columns such that

(i) $(D^rz_r)^{x_r}$ can come only from x_r rows in $\aleph_r$ with $l=r$ and t spanning $[x_r]$, and x_r columns in $\mathfrak{I}_r$ with $m=r$ and i spanning $[x_r-1]_0$, with the rows and columns paired up by the relation $t=i+1$;

(ii) $(Dz_r)^{(\Omega-x_r)(r-1)}$ can come only from $\Omega-x_r$ rows in $\breve{\aleph}_r$ with $l=r$ and t spanning $x_r+1\le t\le\Omega$, and $\Omega-x_r$ columns in $\breve{\mathfrak{I}}_r$ with $m=r-1$ and $i\in[\Omega-(r-1)]_0$, with the rows and columns subject to $i<t\le i+r-1$.

We will show that the three conditions on i and t in (ii) force i and t to be related by $t=i+r-1$. We have t spanning $x_r+1\le t\le\Omega$, $i\in[\Omega-(r-1)]_0$ and $t\le i+r-1$. When $t=\Omega$, the second two conditions force $i=\Omega-(r-1)$. This leaves t to span $x_r+1\le t\le\Omega-1$, $i\in[\Omega-(r-1)-1]_0$, and $t\le i+r-1$. When $t=\Omega-1$, the second two conditions force $i=\Omega-(r-1)-1$. Continuing in this manner, we see that i and t get paired up by $t=i+r-1$, forcing i to span the set $\Upsilon=\{i\ni x_r-(r-1)+1\le i\le\Omega-(r-1)\}$ of size $\Omega-x_r$. Since $x_{r-1}=x_r-r+2$, it follows that $\Upsilon=\{i\ni x_{r-1}\le i\le\Omega-(r-1)\}$.

Removing the $\Omega-x_r$ rows with $m=r-1$ and $x_{r-1}\le i\le\Omega-(r-1)$ from $\breve{\mathfrak{I}}_r$ leaves $\mathfrak{I}_{r-1}$. Removing the $\Omega-x_r$ columns with $l=r$ and $x_r+1\le t\le\Omega$ from $\breve{\aleph}_r$ leaves $\aleph_{r-1}$.

This proves Theorem 8.1. □

9. Termination of the induction

Theorem 9.1. *A sufficient condition for $|A|\neq 0$ is that $|A_{m,l,t-i}|_{(\aleph-\breve{\aleph}_1)\times(\mathfrak{I}-\breve{\mathfrak{I}}_1)}\neq 0$.*

Proof. The first half of Theorem 8.1 is true for $r'\ge 1$, and the second half is true for $r'\ge 2$. Therefore, after obtaining the monomial M, Theorem 8.1 leaves the rows $\breve{\aleph}_1=\{(l,t)\in\aleph\ni l=1,\ x_1+1\le t\le\Omega\}=\{(1,t)\ni 2\le t\le\Omega\}$, and the columns $\breve{\mathfrak{I}}_1=\{(i,m)\in\mathfrak{I}\ni m<1\}=\{(i,0)\ni 2\le i\le\Omega\}$. Therefore, $M=\prod_{r=1}^n(D^rz_r)^{x_r}\cdot\prod_{r=2}^n(Dz_r)^{(\Omega-x_r)(r-1)}$ must come from all the rows of A except $\breve{\aleph}_1$ and all the columns of A except $\breve{\mathfrak{I}}_1$. Let $|A_{m,l,t-i}|_{(\aleph-\breve{\aleph}_1)\times(\mathfrak{I}-\breve{\mathfrak{I}}_1)}$ denote the determinant of the $(\Psi-\Omega+1)\times(\Psi-\Omega+1)$ minor formed from all the rows of A except $\breve{\aleph}_1$ and all the columns of A except $\breve{\mathfrak{I}}_1$. Let $|A_{m,l,t-i}|_{\breve{\aleph}_1\times\breve{\mathfrak{I}}_1}$ denote the determinant of the $(\Omega-1)\times(\Omega-1)$ minor of A formed from the rows $\breve{\aleph}_1$ and the columns $\breve{\mathfrak{I}}_1$. Then, $|A| = |A_{m,l,t-i}|_{(\aleph-\breve{\aleph}_1)\times(\mathfrak{I}-\breve{\mathfrak{I}}_1)} \cdot |A_{m,l,t-i}|_{\breve{\aleph}_1\times\breve{\mathfrak{I}}_1} + X$ where X denotes terms which cannot cancel with M, and we have proven in Theorem 8.1 that, if M appears in $|A|$, then it must appear in $|A_{m,l,t-i}|_{(\aleph-\breve{\aleph}_1)\times(\mathfrak{I}-\breve{\mathfrak{I}}_1)}$. Since $|A_{m,l,t-i}|_{\breve{\aleph}_1\times\breve{\mathfrak{I}}_1} = |A_{0,1,t-i}|_{2\le t\le\Omega,\,2\le i\le\Omega}$, $A_{0,1,t-i} = 0$ if $t \ne i$ and $A_{0,1,t-i} = 1$ if $t = i$, we have $|A_{0,1,t-i}|_{2\le t\le\Omega,\,2\le i\le\Omega} = 1$. Therefore, $|A| = |A_{m,l,t-i}|_{(\aleph-\breve{\aleph}_1)\times(\mathfrak{I}-\breve{\mathfrak{I}}_1)} + X$. Therefore, if $|A_{m,l,t-i}|_{(\aleph-\breve{\aleph}_1)\times(\mathfrak{I}-\breve{\mathfrak{I}}_1)} \ne 0$, then $|A| \ne 0$. $\square$

We will now prove $|A_{m,l,t-i}|_{(\aleph-\breve{\aleph}_1)\times(\mathfrak{I}-\breve{\mathfrak{I}}_1)} \ne 0$.

10. Interpretation of Theorem 8.1. So far we have shown that if the monomial M is to appear in the determinant of A, it necessarily comes only from the following entries of $A = [A_{m,l,t-i}]_{(l,t)\times(i,m)}$: $l = r$, $t \in [x_r]$, $i \in [x_r-1]_0$, $m = r$ and $t = i+1$ for $r \in [n]$, and $l = r$, $x_r+1 \le t \le \Omega$, $x_{r-1} \le i \le \Omega-(r-1)$, $m = r-1$ and $t = i+r-1$ for $r \in [n]$ and $r > 1$.

In other words, in the expansion of the determinant of the minor $[A_{m,l,t-i}]_{(\aleph-\breve{\aleph}_1)\times(\mathfrak{I}-\breve{\mathfrak{I}}_1)}$, M can appear only in the product

$$P \equiv \left(\prod_{r=1}^{n}\prod_{t=1}^{x_r} A_{r,r,1}\right)\cdot\left(\prod_{r=2}^{n}\prod_{t=x_r+1}^{\Omega} A_{r-1,r,r-1}\right). \tag{10.1}$$

Thus, in the expansion of $|A_{m,l,t-i}|_{(\aleph-\breve{\aleph}_1)\times(\mathfrak{I}-\breve{\mathfrak{I}}_1)}$, M cannot cancel with terms not in P. Thus, Theorem 8.1 is equivalent to the statement that if M appears in P with nonzero coefficient, then $|A_{m,l,t-i}|_{(\aleph-\breve{\aleph}_1)\times(\mathfrak{I}-\breve{\mathfrak{I}}_1)} \ne 0$.

Now, we must prove that M appears in P with nonzero coefficient.

Theorem 10.1. *The monomial M appears in the expansion of the determinant of the minor $|A_{m,l,t-i}|_{(\aleph-\breve{\aleph}_1)\times(\mathfrak{I}-\breve{\mathfrak{I}}_1)}$ in the product (10.1) with nonzero coefficient.*

We wish to compute the product $\prod_{r=1}^{n}\prod_{t=1}^{x_r} A_{r,r,1}$ first.

The conditions $l = r$, $t \in [x_r]$, $i \in [x_r-1]_0$, $m = r$, and $t = i+1$ imply

$$\begin{aligned}
A_{m,l,t-i} = A_{r,r,1} &= \sum_{j=1}^{j=r} B_{r,j}\{z_r\}\cdot s_1^j \cdot z_r^{-j} \\
&= B_{r,1}\{z_r\}\cdot s_1^1\cdot z_r^{-1} + X = \frac{D^r z_r}{z_r} + X, \\
\prod_{r=1}^{n}\prod_{t=1}^{x_r} A_{r,r,1} &= \prod_{r=1}^{n}(A_{r,r,1})^{x_r} = \prod_{r=1}^{n}\left(\frac{D^r z_r}{z_r} + X\right)^{x_r} \\
&= \prod_{r=1}^{n}\left(\frac{D^r z_r}{z_r}\right)^{x_r} + X.
\end{aligned} \tag{10.2}$$

Next, we wish to compute the product $\prod_{r=2}^{n}\prod_{t=x_r+1}^{\Omega} A_{r-1,r,r-1}$.

The conditions $l = r$, $x_r + 1 \le t \le \Omega$, $x_{r-1} \le i \le \Omega - (r-1)$, $m = r-1$, and $t = i + r - 1$ imply

$$\begin{aligned} A_{m,l,t-i} = A_{r-1,r,r-1} &= \sum_{j=r-1}^{r-1} B_{r-1,j}\{z_r\} \cdot s_{r-1}^{j} \cdot z_r^{-j} \\ &= B_{r-1,r-1}\{z_r\} \cdot s_{r-1}^{r-1} \cdot z_r^{-(r-1)} = \left(\frac{Dz_r}{z_r}\right)^{r-1}, \end{aligned} \tag{10.3}$$

$$\prod_{r=2}^{n} \prod_{t=x_r+1}^{\Omega} A_{r-1,r,r-1} = \prod_{r=2}^{n} (A_{r-1,r,r-1})^{\Omega - x_r} = \prod_{r=2}^{n} \left(\frac{Dz_r}{z_r}\right)^{(\Omega - x_r)(r-1)}.$$

In other words, $P = M \cdot \prod_{r=1}^{n} z_r^{-x_r - (\Omega - x_r)\cdot(r-1)} + X$ where the X stands for terms that cannot cancel with M. Since $\prod_{r=1}^{n} z_r^{-x_r - (\Omega - x_r)\cdot(r-1)} \ne 0$, M appears in P with nonzero coefficient.

We may now prove Theorem 4.1.

Proof of Theorem 4.1. By Theorem 10.1, M appears in P with nonzero coefficient. Therefore, by Theorem 8.1, $|A_{m,l,t-i}|_{(\aleph - \breve{\aleph}_1)\times(\mathfrak{I} - \breve{\mathfrak{I}}_1)} \ne 0$. Therefore, by Theorem 9.1, $|A| \ne 0$. Therefore, $F_{1,0} = |q^t z_l^q|_{q \times (l \in [n],\, t \in [\Omega])} \cdot |A| \ne 0$ from the factorization (5.2). Therefore, the resolvent $\sum_{(i,m)\in \undertilde{\mathfrak{I}}} F_{i,m} \cdot \alpha^i \cdot D^m y = 0$, obtained by the powersum formula, is not identically zero. By remarks made in Section 4, all the terms $F_{i,m}$ of this resolvent are not zero. this completes the proof of Theorem 4.1. $\square$

11. Cubic example.

We would now like to demonstrate the idea behind the proof of Theorem 4.1 on the smallest possible nontrivial example. Even on this small example, the 12×12 matrix in the powersum formula will be too large to show. Therefore, we will instead reason as the author had originally formulated the proof of [12, Theorem 1]. Since the author has already provided one example using the powersum formula, we will not explain how it works in the following example.

Let $P(t) \equiv (t-u)(t-v)(t-w)$ be a monic cubic polynomial whose roots $z_1 = w$, $z_2 = v$, and $z_3 = u$ are differentially independent over $\mathbb{Z}$. Since $n = 3$, we have $\Omega = n\cdot(n-1)/2 + 1 = 4$, $\Psi = n \cdot \Omega = 12$. Therefore, the homogeneous α-power Cohnian of P has the form

$$\begin{aligned} &(\theta_{0,3} + \theta_{1,3}\cdot\alpha)\cdot D^3 y \\ &\quad + (\theta_{0,2} + \theta_{1,2}\cdot\alpha + \theta_{2,2}\cdot\alpha^2)\cdot D^2 y \\ &\quad + (\theta_{0,1} + \theta_{1,1}\cdot\alpha + \theta_{2,1}\cdot\alpha^2 + \theta_{3,1}\cdot\alpha^3)\cdot Dy \\ &\quad + (\theta_{1,0}\cdot\alpha + \theta_{2,0}\cdot\alpha^2 + \theta_{3,0}\cdot\alpha^3 + \theta_{4,0}\cdot\alpha^4)\cdot y = 0, \end{aligned} \tag{11.1}$$

where all $\theta_{i,m} \ne 0$ by [12, Theorem 40, page 71]. To obtain $\theta_{1,0}$, first compute $F_{1,0}$ by the powersum formula, which sets $F_{1,0}$ equal to the 12×12 cofactor of the matrix $[q^i \cdot D^m p_q]_{q \times (i,m)}$ where q spans $[12]$ and (i,m) spans $\mathfrak{I} = \{(0,3),(1,3),(0,2),(1,2),(2,2),(0,1),(1,1),(2,1),(3,1),(2,0),(3,0),(4,0)\}$. We show that the powersum formula yields a nonzero value for $F_{1,0}$. We expand out the rows of $[q^i \cdot D^m p_q]_{q\times(i,m)}$ for easier reference. To shorten the notation we may drop p_q and indicate the (i,m)

column simply by $q^i \cdot D^m$, so

$$[q^i \cdot D^m p_q]_{q\times(i,m)} = [D^3, q\cdot D^3, D^2, q\cdot D^2, q^2D^2, D, q\cdot D, q^2D, q^3D, q^2, q^3, q^4]. \quad (11.2)$$

Denote the terms in each powersum p_q and their derivatives in the following way:

$$\begin{aligned}
&p_q \sim z^q,\\
&Dp_q \sim (q)_1 \cdot z^{q-1} \cdot Dz,\\
&D^2p_q \sim (q)_2 \cdot z^{q-2}(Dz)^2 + (q)_1 \cdot z^{q-1} \cdot D^2 z,\\
&D^3p_q \sim (q)_3 \cdot z^{q-3}(Dz)^3 + 3\cdot (q)_2 \cdot z^{q-2}(Dz\cdot D^2z) + (q)_1 \cdot z^{q-1}D^3 z.
\end{aligned} \quad (11.3)$$

The given $D^m p_q$ equals the expression following the $\sim$ mark if we sum that expression over the three roots. For instance, the column $q\cdot D^2p_q$ of the matrix $[q^i \cdot D^m p_q]_{q\times(i,m)}$ can be expressed as the sum of the six columns $q\cdot(q)_2\cdot z^{q-2}(Dz)^2$ and $q\cdot(q)_1\cdot z^{q-1}\cdot D^2z$, one for each of the three roots z, with each column involving exactly one root and exactly one monomial of the form $\prod_{r>0}(D^r z)^{\nu_r}$. Thus, the determinant of $[q^i \cdot D^m p_q]_{q\times(i,m)}$ can be expressed as the sum of the determinants of the six matrices formed by replacing $q\cdot D^2 p_q$ with each of these six columns.

We have $x_1 = 1$, $x_2 = 1$, and $x_3 = (r-2)(r-1)/2+1|_{r=3} = 2$. So, $(\Omega - x_3)(3-1) = 2\cdot 2 = 4$ and $(\Omega - x_2)(2-1) = 3\cdot 1 = 3$. So,

$$M \equiv \prod_{r=1}^{n} (D^r z_r)^{x_r} \cdot \prod_{r=1}^{n} (Dz_r)^{(\Omega - x_r)(r-1)} = (D^3u)^2(D^2v)^1(Dw)^1(Du)^4(Dv)^3. \quad (11.4)$$

We determine the coefficient of $M = (D^3u)^2(Du)^4(D^2v)^1(Dv)^3(Dw)^1$ in the expansion of the determinant of $[q^i \cdot D^m p_q]_{q\times(i,m)}$.

The monomial $(D^3u)^2$ can come only from the D^3p_q and $q\cdot D^3p_q$ columns since they are the only columns of third order. Since $D^3u\cdot Du$ does not appear in either D^3p_q or $q\cdot D^3p_q$, the monomial $(Du)^4$ must come only from the columns of second and first order. Furthermore, since u appears in D^2p_q only in the form $(q)_2\cdot u^{q-2}(Du)^2 + (q)_1\cdot u^{q-1}\cdot D^2u$, it follows that the three columns of second order D^2p_q, $q\cdot D^2p_q$, and $q^2D^2p_q$ will contribute at least two powers of Du. Therefore, $(Du)^4$ must come from either two columns of second order, one column of second order and two columns of first order, or four columns of first order.

If $(Du)^4$ came from one column of second order and two columns of first order, or four columns of first order, then, at least, two columns of second order would remain. These two columns of second order would contribute $(D^2v)^2$ or $(D^2v)^1(Dw)^2$ or $(D^2v)^1(Dv)^2$. Since $(D^2v)^2$ and $(D^2v)^1(Dw)^2$ do not appear in M, it follows that the other two columns of second order contribute $(D^2v)^1(Dv)^2$ to M. Then, one more power of Dv would come from the columns of first order. Then, $(Dw)^2$ would come from the remaining two columns of first order. But $(Dw)^2$ does not appear in M.

Therefore, $(Du)^4$ must come from two columns of second order. We have D^2p_q contributing $(q)_2\cdot u^{q-2}\cdot(Du)^2$ (degree 2 in q), $q\cdot D^2p_q$ contributing $q\cdot(q)_2\cdot u^{q-2}\cdot(Du)^2$ (degree 3 in q), and $q^2\cdot D^2p_q$ contributing $q^2\cdot(q)_2\cdot u^{q-2}\cdot(Du)^2$ (degree 4 in q). Since $(D^3u)^2$ comes from D^3p_q, which contributes $(q)_1\cdot u^{q-1}D^3u$ (degree 1 in q), $q\cdot D^3p_q$, which contributes $q\cdot(q)_1\cdot u^{q-1}D^3u$ (degree 2 in q), and since the column

$(q)_2 \cdot u^{q-2} \cdot (Du)^2$ (from $D^2 p_q$) is a linear combination of the columns $(q)_1 \cdot u^{q-1} D^3 u$ (from $D^3 p_q$) and $q \cdot (q)_1 \cdot u^{q-1} D^3 u$ (from $q \cdot D^3 p_q$), it follows that the column $D^2 p_q$ would contribute nothing to $(D^3 u)^2 \cdot (Du)^4$ in the determinant of $[q^i \cdot D^m p_q]_{q\times(i,m)}$.

Therefore, $(Du)^4$ must come from the columns $q \cdot D^2 p_q$ and $q^2 D^2 p_q$. Then, $D^2 v$ must come from the $D^2 p_q$ column. Therefore, $(Dv)^3$ must come from three of the four columns of first order, Dp_q, $q \cdot Dp_q$, $q^2 Dp_q$, or $q^3 Dp_q$. Since $D^2 p_q$ contributes $(q)_1 \cdot v^{q-1} \cdot D^2 v$ (degree 1 in q), Dp_q contributes $(q)_1 \cdot v^{q-1} \cdot Dv$ (degree 1 in q), and the columns $(q)_1 \cdot v^{q-1} \cdot D^2 v$ and $(q)_1 \cdot v^{q-1} \cdot Dv$ are multiples of one another, it follows that the column $D^2 p_q$ would contribute nothing to $(D^2 v) \cdot (Dv)^3$ in the determinant of $[q^i \cdot D^m p_q]_{q\times(i,m)}$.

Therefore, $(Dv)^3$ must come only from the $q \cdot Dp_q$, $q^2 Dp_q$, and $q^3 Dp_q$ columns. Therefore, $(Dw)^1$ must come only from the Dp_q column. The remaining columns $q^2 \cdot w^q$, $q^3 \cdot w^q$, and $q^4 \cdot w^q$ must come from the columns $q^2 \cdot p_q$, $q^3 \cdot p_q$, and $q^4 \cdot p_q$, respectively.

Putting this all together, the coefficient of M in the determinant of $[q^i \cdot D^m p_q]_{q\times(i,m)}$ equals the determinant of

$$\begin{aligned}&[q \cdot u^{q-1}, q^2 \cdot u^{q-1}, q \cdot v^{q-1}, q^3 \cdot u^{q-1}, q^4 \cdot u^{q-1}, q \cdot w^{q-1},\\ &\quad q^2 \cdot v^{q-1}, q^3 \cdot v^{q-1}, q^4 \cdot v^{q-1}, q^2 \cdot w^{q-1}, q^3 \cdot w^{q-1}, q^4 \cdot w^{q-1}],\end{aligned} \tag{11.5}$$

where we have reordered the columns as $[D^3, q \cdot D^3, q \cdot D^2, q^2 D^2, D^2, q \cdot D, q^2 D, q^3 D, D, q^2, q^3, q^4]$ to demonstrate that each of the $n = 3$ roots occupies $\Omega = 4$ columns and has a coefficient of q^i for each $i \in [\Omega]$. By Theorem 6.1, $|q^l \cdot z_l^t|_{q\times(l,t)} \neq 0$, so M appears in $F_{1,0}$ with nonzero coefficient.

To obtain the Cohnian coefficient function $\theta_{1,0}$, we must divide $F_{1,0}$ by the greatest common divisor of all the $F_{i,m}$ in the ring $\mathbb{Z}\{e_1, e_2, e_3\}$.

12. Conclusions.

In [12], the author has factored some terms of a resolvent, given by the powersum formula, of a polynomial whose roots are differentially independent over constants using some *partial* differential resolvents of the polynomial. These partial differential resolvents are the A-hypergeometric relations of Gel'fand and Sturmfels. But much more algebraic factorization remains to be done to make the powersum formula implementable on a computer for polynomials of degree larger than 3.

Furthermore, much work remains to prove that the powersum formula works on polynomials with differential and algebraic relations among their roots.

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