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EXTREME POINTS AND ROTUNDITY OF ORLICZ-SOBOLEV SPACES

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It is well known that Sobolev spaces have played essential roles in solving nonlinear partial differential equations. Orlicz-Sobolev spaces are generalized from Sobolev spaces. In this paper, we present sufficient and necessary conditions of extreme points of Orlicz-Sobolev spaces. A sufficient and necessary condition of rotundity of Orlicz-Sobolev spaces is obtained.

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Definition 1. Let $A(u) = \int_0^{|u|} p(t)dt$, where $p(t)$ satisfies the following properties:

(1) $p(t)$ is right-continuous and nondecreasing;
(2) $p(t) > 0$ $(t > 0)$;
(3) $p(0) = 0$, $\lim_{t\to\infty} p(t) = \infty$.

Then $A(u)$ is called an N-function and $p(t)$ is called the right derivative of $A(u)$.

Definition 2. Let $A(u)$ be an N-function, $p(t)$ the right derivative of $A(u)$. Let

$$q(v) = \sup\{u \ge 0 : p(u) \le v\} = \inf\{u \ge 0 : p(u) \ge v\}. \tag{1}$$

Then $\bar{A}(v) = \int_0^{|v|} q(t)dt$ is called the complementary function of $A(u)$.

Definition 3. Let $A(u)$ be an N-function, $u \in \mathbb{R}$, if $v, w \in \mathbb{R}$, $v + w = 2u$, $u \ne v$, implies $A((v+w)/2) < (1/2)(A(v)+A(w))$. Then u is called a strictly convex point of A. The set of strictly convex points of A is denoted by S_A.

Definition 4. Let $A(u)$ be an N-function, $\Omega \subset \mathbb{R}^n$, Orlicz space is defined as follows:

$$L_A(\Omega) = \left\{u(t) : \exists\lambda > 0,\ \text{such that } \int_\Omega A(\lambda u(t))dt < \infty\right\}. \tag{2}$$

Definition 5. Let $A(u)$ be an N-function, and Ω be a bounded and connected field of $\mathbb{R}^n$. Orlicz-Sobolev space is defined as follows:

$$W^0_{m,A} = \{u \in L_A(\Omega) : \partial^\alpha u \in L_A(\Omega),\ |\alpha| \le m\}, \tag{3}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $\partial^\alpha u$ is a distribution of u.

For $u \in W^0_{m,A}$, its norm is defined as

$$\|u\|^0_{m,A} = \left\{ \sum_{0\le|\alpha|\le m} (||\partial^\alpha(u)||^0)^p \right\}^{1/p}, \quad 1 \le p < \infty. \tag{4}$$

Orlicz-Sobolev spaces with the norm defined above are Banach spaces, see [1].

Definition 6. For any $x \ne 0$, $x \in L_A(\Omega)$, let

$$\begin{aligned} K^*_x &= \inf\left\{K > 0 : \int_\Omega \bar{A}(p(kx(t)))dt \ge 1\right\}, \\ K^{**}_x &= \sup\left\{K > 0 : \int_\Omega \bar{A}(p(kx(t)))dt \le 1\right\}. \end{aligned} \tag{5}$$

Then $k^*_x \le k^{**}_x$. We set $K(x) = [k^*_x, k^{**}_x]$.

Definition 7. Let X be a Banach space, $B(X)$ the closed unit ball of X, and $S(X)$ its unit sphere. Let $x \in S(X)$. If $y, z \in B(X)$, $y + z = 2x$ implies $x = y = z$, then x is called an extreme point of $B(X)$. The set of extreme points of $B(X)$ is denoted by $\operatorname{ext} B(X)$. If $S(X) = \operatorname{ext} B(X)$, then X is called a rotund space.

Lemma 8. *For any* $x \in L^0_A$, $\|x\|^0_A = (1/k)\{1 + \int_\Omega A(kx(t))dt\}$ *if and only if* $k \in K(x)$.

Theorem 9. *Let* $x \in S(W^0_{m,A})$. *If* $\mu\{t \in \Omega : kx(t) \notin S_A\} = 0$, $k \in K(x)$, *then* $x \in \operatorname{ext} B(W^0_{m,A})$.

Proof. Let $y, z \in B(W^0_{m,A})$, and $y + z = 2x$. By the convexity of $f(u) = u^p$, $(1 \le p < \infty)$

$$\begin{aligned} 1 &= \frac{(\|y\|^0_{m,A})^p + (\|z\|^0_{m,A})^p}{2} = \sum_{0\le|\alpha|\le m} \frac{(||\partial^\alpha y||^0)^p + (||\partial^\alpha z||^0)^p}{2} \\ &\ge \sum_{0\le|\alpha|\le m} \left(\frac{||\partial^\alpha y||^0 + ||\partial^\alpha z||^0}{2}\right)^p \ge \sum_{0\le|\alpha|\le m} \left(\left\|\frac{\partial^\alpha y + \partial^\alpha z}{2}\right\|^0\right)^p \\ &= \sum_{0\le|\alpha|\le m} (||\partial^\alpha x||^0)^p = 1^p = 1. \end{aligned} \tag{6}$$

So the equality holds in the above inequalities. Since for any $0 \le |\alpha| \le m$, we have

$$\frac{(||\partial^\alpha y||^0)^p + (||\partial^\alpha z||^0)^p}{2} \ge \left(\frac{||\partial^\alpha y||^0 + ||\partial^\alpha z||^0}{2}\right)^p \ge \left(\left\|\frac{\partial^\alpha y + \partial^\alpha z}{2}\right\|^0\right)^p. \tag{7}$$

From (6) and (7), we know that the equality holds in (7). In particular, when $p > 1$,

$$||\partial^\alpha y||^0 + ||\partial^\alpha z||^0 = 2||\partial^\alpha x||^0. \tag{8}$$

Take $h \in K(y)$, $l \in K(z)$, and let $k = hl/(h+l)$. Then

$$\begin{aligned} 2\|x\|^0 &= \|y\|^0 + \|z\|^0 \\ &= \frac{1}{h}\left(1+\int_\Omega A(hy(t))\,dt\right) + \frac{1}{l}\left(1+\int_\Omega A(lz(t))\,dt\right) \\ &= \frac{h+l}{hl} + \frac{1}{h}\int_\Omega A(hy(t))\,dt + \frac{1}{l}\int_\Omega A(lz(t))\,dt \\ &= \frac{h+l}{hl}\left[1+\int_\Omega\left(\frac{l}{h+l}A(hy(t)) + \frac{h}{h+l}A(lz(t))\right)dt\right] \\ &\geq \frac{h+l}{hl}\left[1+\int_\Omega A\left(\frac{hl}{h+l}(y(t)+z(t))\right)dt\right] \\ &\geq 2\cdot\frac{1}{2k}\left[1+\int_\Omega A(2kx(t))\,dt\right] \\ &\geq 2\|x\|^0. \end{aligned} \tag{9}$$

So the equality holds in the above inequalities. Hence $2k \in K(x)$ and for a.e. $t \in \Omega$, $(l/(h+l))A(hy(t)) + (h/(h+l))A(lz(t)) = A(2kx(t))$. By the known conditions, for almost all $t \in \Omega$, $hy(t) = lz(t) = 2kx(t)$. Therefore,

$$l = l\|z\|^0_{m,A} = \|lz\|^0_{m,A} = \|hy\|^0_{m,A} = h\|y\|^0_{m,A} = h. \tag{10}$$

This implies $x = y = z$. So $x \in \operatorname{ext} B(W^0_{m,A})$. □

Theorem 10. *Let $x \in S(W^0_{m,A})$. If for any $i = 1,2,\ldots,n$, $\mu\{t \in \Omega : k_i\partial_i x(t) \notin S_A\} = 0$, where $K_i \in K(\partial_i x(t))$. Then $x \in \operatorname{ext} B(W^0_{m,A})$.*

Proof. Let $y, z \in B(W^0_{m,A})$, and $y + z = 2x$. By the proof of Theorem 9, for any $0 \leq |\alpha| \leq m$ we have

$$2\|\partial^\alpha x\|^0 = \|\partial^\alpha y\|^0 + \|\partial^\alpha z\|^0. \tag{11}$$

In particular, if $|\alpha| = 1$, then $2\|\partial_i x\|^0 = \|\partial_i y\|^0 + \|\partial_i z\|^0$. Take $h_i \in K(\partial_i y)$, $l_i \in K(\partial_i z)$, and let $k_i = h_i l_i/(h_i + l_i)$. By the proof of Theorem 9, we have

$$h_i\partial_i y(t) = l_i\partial_i z(t) = 2k_i\partial_i x(t), \quad i = 1,2,\ldots,n \tag{12}$$

and $l_i = h_i = 2k_i$. Hence $\partial_i y(t) = \partial_i x(t) = \partial_i z(t)$. Thus there exists a constant c such that $y(t) = x(t) + c$, $z(t) = x(t) - c$. Now, we show that $c = 0$. If not, $c \neq 0$. Without loss of generality, we may assume that $c > 0$. If $|x| < c$, then $y(t) > 0$, $z(t) < 0$. Since $0 \in S_A$, when $a > 0$, $b < 0$, for any $\lambda \in (0,1)$, we have $A(\lambda a + (1-\lambda)b) < \lambda A(a) + (1-\lambda)A(b)$. By (9), $|x(t)| < c$ does not hold. Then for a.e. $t \in \Omega$, $|x(t)| \geq c$.

Let $E_1 = \{t \in \Omega : x(t) \geq c\}$, $E_2 = \{t \in \Omega : x(t) \leq -c\}$. Then $\mu(E_1 \cup E_2) = \mu\Omega$. Since Ω is connected, for any $p \in E_1$, $q \in E_2$, p can continuously move to q in Ω by a transform of finite single-variable. If $\mu E_1 > 0$ and $\mu E_2 > 0$, there exists at least a $p \in E_1$, $q \in E_2$ such that the connecting line between p and q over $E_1 \cup E_2$ is condense. So there exists a line $l = \{(t_1, t_2, \ldots, t_{i-1}, \lambda t_{i+1}, \ldots, t_n)\ \lambda \in [a,b]\}$ on that connecting line, such that $l \cap E_1 \neq \emptyset$, $l \cap E_2 \neq \emptyset$. But $x(t) \geq c$ over E_1 and $x(t) \leq -c$ over E_2 whereas $E_1 \cup E_2$ is condense of l. This is a contradiction to the fact that $\partial_i x(t) \in L_A \subset L_1$ implies that $x(t)$ is absolutely continuous with respect to t_i. So, either $\mu E_1 = 0$ or $\mu E_2 = 0$. Without loss of generality, let $\mu E_2 = 0$. Then for almost all $t \in \Omega$, $x(t) \geq c$. So, $y(t) > x(t)$. Thus $\|y\|^0_{m,A} > \|x\|^0_{m,A} = 1$. This contradicts $y \in B(W^0_{m,A})$. From above, we know that $c = 0$. So $x(t) = y(t) = z(t)$. This means $x \in \operatorname{ext} B(W_{m,A})$. □

Theorem 11. *Let $x \in S(W^0_{m,A})$. For any $i = 1, 2, \ldots, n$,*

$$\mu\{t \in \Omega : kx(t) \notin S_A\} \cap \{t \in \Omega : k_i \partial_i x(t) \notin S_A\} = 0, \quad k_i \in K(\partial_i x),\ k \in K(x), \tag{13}$$

then $x \in \operatorname{ext} B(W^0_{m,A})$.

Proof. Let $y, z \in B(W^0_{m,A})$ and $y + z = 2x$. Let $B = \{t \in \Omega : kx(t) \notin S_A\}$, $B_i = \{t \in \Omega : k_i \partial_i x(t) \notin S_A\}$, and $y(t) = x(t) + \delta(t)$.

Case 1. For almost all $t \in \Omega \setminus B$, $\delta(t) = 0$ by Theorem 10. Therefore $x(t) = y(t) = z(t)$.

Case 2. For any $i = 1, 2, \ldots, n$, $\mu(B \cap B_i) = 0$, so for almost all $t \in B$, $t \notin B_i$. Hence $\partial_i x(t) \in S_A$. By the proof of Theorem 10, we know that $\partial_i \delta(t) = 0$, when $\delta(t) = c$. Similarly, $x(t) = y(t) = z(t)$ by Theorem 10. By Cases 1 and 2 we know $x \in \operatorname{ext} B(W^0_{m,A})$. □

Theorem 12. *Let $x \in S(W^0_{m,A})$. If there exists an affine interval (a_α, b_α) and $\epsilon > 0$ such that*

$$\operatorname{int} \bigcap_{0 \leq |\alpha| \leq m} \left\{t \in \Omega : \partial^\alpha k_\alpha x(t) \in (a_\alpha + \epsilon, b_\alpha - \epsilon)\right\} \neq \emptyset, \tag{14}$$

then $x \notin \operatorname{ext} B(W^0_{m,A})$.

Proof. Let $G = \bigcap_{0 \leq |\alpha| \leq m} \{t \in \Omega : k_\alpha \partial^\alpha x(t) \in (a_\alpha + \epsilon, b_\alpha - \epsilon)\}$ and $\operatorname{int} G \neq \emptyset$. Take $t', t'' \in \operatorname{int} G$, $r > 0$ such that $B(t', r) = B_1 \subset G$, $B(t'', r) = B_2 \subset G$, and $B_1 \cap B_2 = \emptyset$. For any $t^* \in \Omega$ satisfying $B(t^*, r) \subset \Omega$. Define

$$J_{t^*}(t) = \begin{cases} e^{-1/(r^2 - \sum_{i=1}^n (t_i - t_i^*)^2)}, & t \in B(t^*, r), \\ 0, & t \in \Omega \setminus B(t^*, r). \end{cases} \tag{15}$$

Then $J_{t^*}(t)$ is an infinitely differentiable function on Ω and for any $0 \leq |\alpha| \leq m$, $\partial^\alpha J_{t^*}(t) = 0$ on $\Omega \setminus B(t^*, r)$. Let

$$c = \epsilon \min_{0 \leq |\alpha| \leq m} \left\{ \frac{1}{\max_{t \in \Omega} |\partial^\alpha J_{t^*}(t)|} \right\}. \tag{16}$$

Then $c > 0$ and for all $t \in \Omega$, $c\partial^\alpha J_{t^*}(t) \le \epsilon$. Define

$$y(t) = x(t) + cJ_{t'}(t) - cJ_{t''}, \qquad z(t) = x(t) - cJ_{t'}(t) + cJ_{t''}. \tag{17}$$

Then $y, z \in W^0_{m,A}$, and $y + z = 2x$, $y \ne z$. Let $A(u) = h_\alpha u + b_\alpha$ on $(a_\alpha + \epsilon, b_\alpha - \epsilon)$. For any $k_\alpha \in K(\partial^\alpha x)$,

$$\begin{aligned}
||\partial^\alpha y||^0 &= \frac{1}{k_\alpha}\left[1 + \int_\Omega A(k_\alpha \partial^\alpha y(t))dt\right] \\
&= \frac{1}{k_\alpha}\left[1 + \int_{\Omega\setminus(B_1\cup B_2)} A(k_\alpha\partial^\alpha x(t))dt + \int_{B_1} A(k_\alpha\partial^\alpha x(t) + k_\alpha\partial^\alpha(cJ_{t'}(t)))dt\right. \\
&\qquad \left. + \int_{B_2} A(k_\alpha\partial^\alpha x(t) - k_\alpha\partial^\alpha(cJ_{t''}(t)))dt\right] \\
&= \frac{1}{k_\alpha}\left[1 + \int_{\Omega\setminus(B_1\cup B_2)} A(k_\alpha\partial^\alpha x(t))dt\right. \\
&\qquad + \int_{B_1}(h_\alpha k_\alpha\partial^\alpha x(t) + b_\alpha)dt + \int_{B_1} h_\alpha k_\alpha \partial^\alpha(cJ_{t'}(t))dt \\
&\qquad \left. + \int_{B_2}(h_\alpha k_\alpha\partial^\alpha x(t) + b_\alpha)dt - \int_{B_2} h_\alpha k_\alpha \partial^\alpha(cJ_{t''}(t))dt\right] \\
&= \frac{1}{k_\alpha}\left[1 + \int_\Omega A(k_\alpha\partial^\alpha x(t))dt\right] \\
&= ||\partial^\alpha x||^0.
\end{aligned} \tag{18}$$

Hence for any $0 \le |\alpha| \le m$, we have $\|\partial^\alpha y\|^0 = \|\partial^\alpha x\|^0$.

Likewise, for any $0 \le |\alpha| \le m$, we have $\|\partial^\alpha z\|^0 = \|\partial^\alpha x\|^0$. Then

$$\|y\|^0_{m,A} = \|z\|^0_{m,A} = \|x\|^0_{m,A} = 1. \tag{19}$$

Therefore $y, z \in S(W^0_{m,A})$. We know that $x \notin \operatorname{ext} B(W^0_{m,A})$ since $y \ne z$. □

Theorem 13. *We show that $W^0_{m,A}$ is rotund if and only if A is strictly convex.*

Proof

Sufficiency. It is immediately obtained from Theorem 9.

Necessity. Suppose A is not strictly convex. Then there exists $0 < a < b$ such that $A(u)$ is an affine function on (a, b). Since Ω is bounded, we can take $t' \in \bar{\Omega}$, $t'' \in \bar{\Omega}$ such that

$$\sum_{i=1}^n t_i' = \inf_{(t_1,t_2,\dots,t_n)\in\Omega} \sum_{i=1}^n t_i, \qquad \sum_{i=1}^n t_i'' = \sup_{(t_1,t_2,\dots,t_n)\in\Omega} \sum_{i=1}^n t_i. \tag{20}$$

(1) When $\int_\Omega \bar{A}(p((a+b)/2))dt < 1$, we set $g(c) = \int_\Omega \bar{A}(p(((a+b)/2)e^{c\sum_{i=1}^n (t_i - t_i')}))dt$. Then by the continuity of $\bar{A}$ and the right continuity of p, $g(c)$ is right continuous

with respect to c and $g(0) = \int_\Omega \bar{A}(p((a+b)/2))dt < 1$, $\lim_{c\to\infty} g(c) = \infty$. Take $c_0 = \inf\{c > 0 : g(c) \geq 1\}$, then the following two statements hold:

(a) $g(c_0) \geq 1$, so $c_0 > 0$;

(b) for any $l \in (0,1)$, $\int_\Omega \bar{A}(p(((a+b)/2)le^{c_0\sum_{i=1}^n (t_i - t_i')}))dt < 1$.

Indeed, take $c_n \searrow c_0$ such that $g(c_n) \geq 1$. Then $g(c_0) = \lim_{n\to\infty} g(c_n) \geq 1$ since $g(c)$ is right continuous. So (a) holds.

Let $\lambda = \sup_{(t_1,t_2,\dots,t_n)} \sum_{i=1}^n (t_i - t_i')$. Then for any $t \in \Omega$, $\lambda \geq \sum_{i=1}^n (t_i - t_i') > 0$. For any $0 < l < 1$, since $\ln l < 0$,

$$0 < l\frac{a+b}{2}e^{c_0\sum_{i=1}^n (t_i - t_i')} = \frac{a+b}{2}e^{\ln l + c_0\sum_{i=1}^n (t_i - t_i')} \leq \frac{a+b}{2}e^{(c_0 + \ln l/\lambda)\sum_{i=1}^n (t_i - t_i')}. \tag{21}$$

By the definition of c_0,

$$\int_\Omega \bar{A}\left(p\left(l\frac{a+b}{2}e^{c_0\sum_{i=1}^n (t_i - t_i')}\right)\right)dt \leq g\left(c_0 + \frac{\ln l}{\lambda}\right) < 1. \tag{22}$$

Let $x(t) = ((a+b)/2)e^{c_0\sum_{i=1}^n (t_i - t_i')}$. By the above discussion, $1 \in K(x)$. Then $\|x\|^0 = 1 + \int_\Omega A(x(t))dt$. Let $x_0(t) = x(t)/\|x\|_{m,A}^0$. Then $x_0(t) \in S(W_{m,A}^0)$ and

$$\begin{aligned} ||x_0||^0 &= \frac{\|x\|^0}{\|x\|_{m,A}^0} = \frac{1}{\|x\|_{m,A}^0}\left(1 + \int_\Omega A(x(t))dt\right) \\ &= \frac{1}{\|x\|_{m,A}^0}\left(1 + \int_\Omega A(\|x\|_{m,A}^0 x_0(t))dt\right). \end{aligned} \tag{23}$$

Therefore $\|x\|_{m,A}^0 \in K(x_0(t))$. Set $1/b_0 = \|x\|_{m,A}^0$. Since $(t_1,t_2,\dots,t_n) \in \Omega$, $x(t) \to (a+b)/2$ as $t_i \to t_i'$, we can choose a ball $B \subset \Omega$ such that $x(B) \subset (a,b)$. It means that

$$\{t \in \Omega : x(t) \notin S_A\} \supset B. \tag{24}$$

Therefore,

$$\left\{t \in \Omega : \frac{1}{b_0}x_0(t) \notin S_A\right\} \supset B. \tag{25}$$

On the other hand, as $1 \leq |\alpha| \leq m$,

$$\partial^\alpha x_0(t) = \frac{\partial^\alpha x(t)}{\|x\|_{m,A}^0} = \frac{c_0^{|\alpha|}}{\|x\|_{m,A}^0}x(t) = b_\alpha x(t), \tag{26}$$

where $b_\alpha = c_0^{|\alpha|}/\|x\|_{m,A}^0$. By Lemma 8, $1/b_\alpha \in K(\partial^\alpha x(t))$. So

$$\left\{t \in \Omega : \frac{1}{b_\alpha}\partial^\alpha x_0(t) \notin S_A\right\} \supset B. \tag{27}$$

Then,

$$\operatorname{int} \bigcap_{0\le|\alpha|\le m} \left\{t \in \Omega : \frac{1}{b_\alpha}\partial^\alpha x_0(t) \notin S_A\right\} \ne \varnothing. \tag{28}$$

By Theorem 12, we know $x_0 \notin \operatorname{ext} B(W^0_{m,A})$. This is a contradiction.

(2) When $\int_\Omega \bar{A}(p((a+b)/2))dt \ge 1$.

Set $g(c) = \int_\Omega \bar{A}(p(((a+b)/2)e^{c\sum_{i=1}^n (t_i - t_i'')}))dt$. Then $g(c)$ is left-continuous with respect to c. For any $(t_1, t_2, \dots, t_n) \in \Omega$, $\sum_{i=1}^n (t_i - t_i'') < 0$, and $g(0) = \int_\Omega \bar{A}(p((a+b)/2))dt \ge 1$, $\lim_{c\to\infty} g(c) = 0$. Take $c_0 = \sup\{c > 0 : g(c) \le 1\}$. As in (1), we can prove $g(c_0) \le 1$ and for any $l > 1$,

$$\int_\Omega \bar{A}\left(p\left(l\frac{a+b}{2}e^{c_0\sum_{i=1}^n (t_i - t_i'')}\right)\right)dt \ge 1. \tag{29}$$

Let $x(t) = ((a+b)/2)e^{c_0\sum_{i=1}^n (t_i - t_i'')}$, $x_0(t) = x(t)/\|x\|^0_{m,A}$. Then $x_0 \in S(W^0_{m,A})$. Likewise, we can show $x_0 \notin \operatorname{ext} B(W^0_{m,A})$. This is also a contradiction.

By (1) and (2) we know that A is strictly convex. □

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