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CHERN CLASSES OF INTEGRAL SUBMANIFOLDS OF SOME CONTACT MANIFOLDS

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A complex subbundle of the normal bundle to an integral submanifold of the contact distribution in a Sasakian manifold is given. The geometry of this bundle is investigated and some results concerning its Chern classes are obtained.

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1. Introduction. Let $\tilde{M}$ be a $(2m+1)$-dimensional manifold endowed with the almost contact metric structure F, ξ, η, g. These tensor fields satisfy the conditions

$$F^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad g(FX, FY) = g(X, Y) - \eta(X)\eta(Y), \tag{1.1}$$

for all vector fields X, Y tangent to $\tilde{M}$.

Let $\mathcal{D}$ be the contact distribution of $\tilde{M}$, defined by the equation $\eta = 0$. The study of the integral submanifolds of $\mathcal{D}$ is very difficult for, at least, three reasons: (a) their abundance (see, e.g., [1, 5, 8]), (b) the nonexistence of a natural structure induced on the submanifold M, resulting from the equalities $\eta = 0$, $d\eta = 0$, true along M, and (c) for any vector field X tangent to M, the vector field FX is normal to M and therefore, freely speaking, the geometry of an integral submanifold of $\mathcal{D}$ is normal to the submanifold. However, for maximal integral submanifolds (i.e., $\dim M = m$), we know many properties (see, e.g., [1, Chapter V]); while for nonmaximal integral submanifolds, we have so few results.

In this paper, we associate to each nonmaximal integral submanifold M of $\tilde{M}$ a nontrivial vector bundle $\tau(M)$. The geometry and the topology of this vector bundle are also studied. In Section 2, we give, in an "appropriate" form, the structure equations of an integral submanifold in a Sasakian manifold. In Section 3, we study the geometry of $\tau(M)$, namely, we prove that it has a natural structure of complex symplectic vector bundle.

It is well known that integral submanifolds of an almost contact manifold are anti-invariant, [8]. Thus, such a submanifold is analogous to the isotropic (or totally real) submanifolds of a Kähler manifold, investigated by Chen and Morvan in [2, 4], and we can use some of their technics in order to study Chern classes of the vector bundle $\tau(M)$. In Section 4, by combining these ideas with some Vaisman's results [7] concerning the characteristic classes of quaternionic bundles, we obtain stronger results than for isotropic submanifolds. Namely, we prove that if $m-n$ is even, then all odd Chern classes of $\tau(M)$ are zero. In absence of this supposition on the dimensions, we prove that the first Chern class of $\tau(M)$ is zero when $\tilde{M}$ is a Sasakian space form.

2. Structure equations of an integral submanifold. Let $\tilde{M}$ be an almost contact metric manifold. Furthermore, we assume that $\tilde{M}$ is Sasakian and let $\mathscr{X}(\tilde{M})$ denote the set of all vector fields tangent to $\tilde{M}$. We have [1, page 73]

$$(\tilde{\nabla}F)Y = g(X,Y)\xi - \eta(Y)X, \quad X,Y \in \mathscr{X}(\tilde{M}), \tag{2.1}$$

where $\tilde{\nabla}$ is the Levi-Civita connection associated to the metric g on $\tilde{M}$. Moreover, we have the well-known equalities

$$F\xi = 0, \qquad \eta\circ F = 0, \qquad \eta(X) = g(X,\xi), \qquad \tilde{\nabla}_X\xi = -FX. \tag{2.2}$$

Now, let M be an n-dimensional submanifold of the Sasakian manifold $\tilde{M}$ and denote by h, $\tilde{\nabla}^{\perp}$, and A its second fundamental form, normal connection, and Weingarten operator, respectively. It is well known that $n \le m$ (see [8] or [1, page 36]), and we can consider in $\tilde{M}$ local fields of orthonormal frames $\mathscr{B} = \{e_1,\dots,e_n,e_{n+1},\dots,e_m,e_{1^*} = Fe_1,\dots,e_{n^*} = Fe_n,e_{(n+1)^*} = Fe_{n+1},\dots,e_{m^*} = Fe_m,e_{(m+1)^*} = \xi\}$, with the property that the restrictions of $e_1,\dots,e_n$ to the submanifold M are tangent to M, so that $\mathscr{B}$ are local frames such that $TM\oplus \operatorname{span}\{e_{n+1},\dots,e_m\}$ is a Legendrian subbundle of $T\tilde{M}$.

Afterwards, we will use the following convention on the indices: $j \in \{1,\dots,m\}$; $j^* = j+m$; $a,b,c \in \{1,\dots,n\}$; $a^* = a+m$, $b^* = b+m$, $c^* = c+m$; $\lambda,\mu,\nu \in \{n+1,\dots,m\}$; $\lambda^* = \lambda+m$; $\alpha,\beta,\gamma,\delta \in \{1,\dots,2m+1\}$.

If $\mathscr{B}^* = \{\omega^1,\dots,\omega^n,\omega^{n+1},\dots,\omega^m,\omega^{1^*},\dots,\omega^{n^*},\omega^{(n+1)^*},\dots,\omega^{m^*},\omega^{(m+1)^*} = \eta\}$ is the local field of coframes of $\mathscr{B}$, then, at the points of M, we have (locally)

$$\omega^{\lambda} = \omega^{j^*} = \omega^{(m+1)^*} = 0. \tag{2.3}$$

On the other hand, by computations we prove that if $(\omega_{\alpha}^{\beta})$ is the connection form of $\tilde{\nabla}$, expressed with respect to $\mathscr{B}$, then, on the submanifold M, we have

$$\omega^a_{(m+1)^*} = \omega^{\lambda}_{(m+1)^*} = \omega^{\lambda^*}_{(m+1)^*} = 0, \qquad \omega^{(m+1)^*}_{a^*}(X) = g(X,e_a), \tag{2.4}$$

$$\omega^{j^*}_a = \omega^{a^*}_j, \qquad \omega^{j^*}_{a^*} = \omega^j_a, \qquad \omega^{j^*}_{\lambda} = \omega^{\lambda^*}_j, \qquad \omega^{j^*}_{\lambda^*} = \omega^j_{\lambda}. \tag{2.5}$$

The curvature forms of $\tilde{M}$ and M are, respectively,

$$\tilde{\Omega}^{\alpha}_{\beta} = \frac{1}{2}\sum_{\alpha,\beta=1}^{2m+1} \tilde{R}^{\alpha}_{\beta\gamma\delta}\omega^{\gamma}\wedge\omega^{\delta}, \qquad \Omega^a_b = \frac{1}{2}\sum_{c,d=1}^{n} R^a_{bcd}\omega^c\wedge\omega^d, \tag{2.6}$$

where $\tilde{R}^{\alpha}_{\beta\gamma\delta}$ and R^a_{bcd} are the components (with respect to $\mathscr{B}$) of the curvature tensors of $\tilde{M}$ and M, respectively. Then, at the points of M, we have

$$\Omega^a_b = \tilde{\Omega}^a_b - \sum_{\lambda=n+1}^{m}\omega^a_{\lambda}\wedge\omega^{\lambda}_b - \sum_{j=1}^{m}\omega^a_{j^*}\wedge\omega^{j^*}_b, \tag{2.7}$$

$$\Omega^{\lambda}_{\mu} = \tilde{\Omega}^{\lambda}_{\mu} - \sum_{a=1}^{n}\omega^{\lambda}_a\wedge\omega^a_{\mu} = \frac{1}{2}\sum_{a,b=1}^{n} R^{\lambda}_{\mu ab}\omega^a\wedge\omega^b, \tag{2.8}$$

where $R^{\lambda}_{\mu ab}$ are the components of the curvature tensor of $\nabla^{\perp}$. Finally, from (2.3), (2.4), and (2.5) and from the general form of the structure equations (see, e.g., [3, page 121]), we deduce the structure equations of an integral submanifold of a Sasakian manifold under the form

$$
\begin{gathered}
d\omega^a = -\sum_{b=1}^{n} \omega_b^a \wedge \omega^b, \qquad d\omega_b^a = -\sum_{c=1}^{n} \omega_c^a \wedge \omega_b^c + \Omega_b^a, \\
d\omega_\mu^\lambda = -\sum_{\nu=n+1}^{m} \omega_\nu^\lambda \wedge \omega_\mu^\nu - \sum_{j=1}^{m} \omega_{j^*}^\lambda \wedge \omega_\mu^{j^*} + \Omega_\mu^\lambda.
\end{gathered}
\tag{2.9}
$$

3. Geometry of the maximal invariant normal bundle. The normal space $T_x^\perp M$ at each point $x \in M$ has the following orthogonal decomposition

$$
T_x^\perp M = F(T_x M) \oplus \tau_x(M) \oplus \operatorname{span}\{\xi_x\}, \tag{3.1}
$$

where $\tau_x(M)$ is the $2(m-n)$-dimensional subspace of $T_x M$, orthogonal to $F(T_x M) \oplus \operatorname{span}\{\xi_x\}$. Then, $\tau(M) = \bigcup_{x\in M} \tau_x(M)$ is the total space of a subbundle $\tau(M)$ of $T^\perp M$ and $\mathcal{B}_\tau = \{e_\lambda, e_{\lambda^*}\} = \{e_{n+1}, \dots, e_m, e_{(n+1)^*}, \dots, e_{m^*}\}$ is a local basis in the module $\Gamma(\tau)$ of its sections. We also denote this bundle by $\tau(M)$ and call it the *maximal invariant normal bundle* of the integral submanifold M.

Theorem 3.1. *Let M be an integral submanifold of the Sasakian manifold $\tilde{M}$. Its maximal invariant normal bundle $\tau(M)$ has the following properties:*

(a) *$\tau(M)$ is invariant by F, that is, $F(\tau_x(M)) = \tau_x(M)$ for each $x \in M$;*

(b) *$\tau(M)$ has a natural structure of complex vector bundle;*

(c) *if $m - n = (\dim \tilde{M} - \dim M)/2$ is even, then $\tau(M)$ has a quaternionic structure.*

Proof. (a) follows easily from (3.1).

(b) Denote by $(n^\lambda, n^{\lambda^*})$ the components of the vector $\vec{n}_x \in \tau_x(M)$, relative to the basis $\mathcal{B}_\tau$, and let $\rho : \tau(M) \to M$ be the natural projection. Then, using the classical notations, the vector charts

$$
\Phi : \rho^{-1}(U) \longrightarrow U \times \mathbb{C}^{m-n}, \quad \Phi(\vec{n}_x) = (x, (n^\lambda + i n^{\lambda^*})), \quad x \in U, \tag{3.2}
$$

define on $\tau(M)$ a complex vector bundle structure.

(c) From (a), we deduce that the space $\Gamma(\tau)$ can be considered as a complex space with the following multiplication by complex numbers:

$$
(\alpha + i\beta)\vec{n} = \alpha\vec{n} + \beta F\vec{n}, \quad \alpha, \beta \in \mathbb{R},\ \vec{n} \in \Gamma(\tau). \tag{3.3}
$$

Endowed with this complex structure, $\Gamma(\tau)$ is an $(m-n)$-dimensional space, denoted by $\Gamma^c(\tau)$. Moreover, we can define the map $F^\tau : \Gamma^c(\tau) \to \Gamma^c(\tau)$ by $F^\tau(\mathbf{n}) = F\vec{n} - iF\vec{n}^*$ for all $\mathbf{n} = \vec{n} + i\vec{n}^*$, $\vec{n}, \vec{n}^* \in \Gamma(\tau)$, and it has the following properties:

$$
F^\tau(\mathbf{n}_1 + \mathbf{n}_2) = F^\tau\mathbf{n}_1 + F^\tau\mathbf{n}_2, \qquad F^\tau(\lambda\mathbf{n}) = \bar{\lambda}F^\tau\mathbf{n}, \qquad (F^\tau)^2\mathbf{n} = -\mathbf{n}, \tag{3.4}
$$

for $\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2 \in \Gamma^c(\tau)$ and $\lambda \in \mathbb{C}$. Hence (see [7, Section 1]), F^τ defines on $\tau(M)$ a quaternionic structure. □

A natural connection can be defined on $\tau(M)$. Firstly, we remark that $g(\nabla^{\perp}_X\vec{n},\xi)=0$ for all $X\in\mathscr{X}(M)$ and $\vec{n}\in\Gamma(\tau)$, hence the normal vector field $\nabla^{\perp}_X\vec{n}$ has the following decomposition:

$$\nabla^{\perp}_X\vec{n}=B_{\vec{n}}X+\nabla^{\tau}_X\vec{n}, \tag{3.5}$$

where $B_{\vec{n}}X\in\Gamma(FTM)$ and $\nabla^{\tau}_X\vec{n}\in\Gamma(\tau)$. Moreover, the maps $B:\Gamma(\tau)\times\mathscr{X}(M)\to\Gamma(FTM)$ and $\nabla^{\tau}:\mathscr{X}(M)\times\Gamma(\tau)\to\Gamma(\tau)$ have the following properties.

Proposition 3.2. (a) ∇^{τ} *is an almost complex connection on the maximal invariant normal bundle of the integral submanifold M, that is,* $(\nabla^{\tau}_X F)\vec{n}=0$.
(b) $B_{\vec{n}}X=FA_{F\vec{n}}X$ *for all* $X\in\mathscr{X}(M)$ *and* $\vec{n}\in\Gamma(\tau)$.

The proof follows from (3.5) by computation, taking into account (2.1) and (2.2) and using the Weingarten formula for the submanifold M.

Now, if we extend the scalar product g over $\Gamma^c(\tau)$ by

$$g^{\tau}(\mathbf{n}_1,\lambda\mathbf{n}_2)=\overline{\lambda}g^{\tau}(\mathbf{n}_1,\mathbf{n}_2),\qquad g^{\tau}(\mathbf{n}_2,\mathbf{n}_1)=\overline{g^{\tau}(\mathbf{n}_1,\mathbf{n}_2)}, \tag{3.6}$$

for $\lambda\in\mathbb{C}$ and $\mathbf{n}_1,\mathbf{n}_2\in\Gamma^c(\tau)$, then we have

$$g^{\tau}(F^{\tau}\mathbf{n}_1,F^{\tau}\mathbf{n}_2)=\overline{g^{\tau}(\mathbf{n}_1,\mathbf{n}_2)}, \tag{3.7}$$

hence g^{τ} is a Hermitian scalar product on the complex vector bundle $\tau(M)$. Moreover, $\mathscr{B}^c_{\tau}=\{f_{\lambda}=(1/\sqrt{2})(e_{\lambda}+ie_{\lambda^*}),\ f_{\lambda^*}=(1/\sqrt{2})(e_{\lambda}-ie_{\lambda^*})\}$ is an orthonormal local basis of $\Gamma^c(\tau)$ with respect to g^{τ} and $f_{\lambda^*}=-if_{\lambda}$, $F^{\tau}f_{\lambda}=i\overline{f_{\lambda}}=f_{\lambda^*}$.

For any $\mathbf{n}_1,\mathbf{n}_2\in\Gamma^c(\tau)$, we put

$$\Omega^{\tau}(\mathbf{n}_1,\mathbf{n}_2)=-\overline{g^{\tau}(F^{\tau}\mathbf{n}_1,\mathbf{n}_2)}, \tag{3.8}$$

and a simple computation shows that Ω^{τ} is $\mathbb{C}$-linear with respect to the first argument and

$$\Omega^{\tau}(\mathbf{n}_1,\mathbf{n}_2)=-\overline{\Omega^{\tau}(\mathbf{n}_2,\mathbf{n}_1)},\qquad \Omega^{\tau}(F^{\tau}\mathbf{n}_1,F^{\tau}\mathbf{n}_2)=\overline{\Omega^{\tau}(\mathbf{n}_1,\mathbf{n}_2)}. \tag{3.9}$$

From these relations and because $\mathscr{B}^c_{\tau}$ is an orthonormal local basis, we deduce that Ω^{τ} is a nondegenerate skew-symmetric 2-form on the complex vector bundle $\tau(M)$. Hence, we have the following proposition.

Proposition 3.3. *For $m-n$ even, the maximal invariant normal bundle $\tau(M)$ of the integral submanifold M of a Sasakian manifold has a structure of complex symplectic vector bundle with the symplectic form Ω^{τ}.*

4. Normal Chern classes of an integral submanifold. As a complex vector bundle, the basic characteristic classes of the maximal invariant normal bundle $\tau(M)$ are the Chern classes $[\gamma_k(\tau)]$, represented by the Chern forms

$$\gamma_k=\frac{i^k}{(2\pi)^k k!}\delta^{\mu_1\cdots\mu_k}_{\lambda_1\cdots\lambda_k}\overset{\tau}{\Omega}{}^{\lambda_1}_{\mu_1}\wedge\cdots\wedge\overset{\tau}{\Omega}{}^{\lambda_k}_{\mu_k}, \tag{4.1}$$

where $\overset{\tau}{\Omega}{}^{\lambda}_{\mu}$ are the curvature forms of ∇^{τ} and $\delta^{\cdots}_{\cdots}$ is the multiindex Kronecker symbol. We say that $\gamma_k(\tau)$ is the *kth normal Chern form* of the submanifold M and the purpose of this section is to obtain some results concerning the computation of $\gamma_k(\tau)$ and the *kth normal Chern class* $[\gamma_k(\tau)]$ of M.

Theorem 4.1. *Let M be an n-dimensional integral submanifold of a Sasakian manifold of dimension $2m+1$. If $m-n$ is even, then*

$$[\gamma_{2k+1}(\tau)] = 0 \quad \text{for } k = 0,1,\ldots,\left[\frac{m-n-1}{2}\right]. \tag{4.2}$$

Proof. By Theorem 3.1(c), the maximal invariant normal bundle $\tau(M)$ has a quaternionic structure, and then we can apply [7, Proposition 2.1]. □

Now, we will analyse the first normal Chern form and its associated class in absence of the supposition that $m-n$ is even.

Theorem 4.2. *The first normal Chern form of the n-dimensional integral submanifold M in a Sasakian manifold of dimension $2m+1$, $m>n$, is given by*

$$\gamma_1(\tau) = \frac{1}{2\pi}\sum_{\lambda=n+1}^{m}\Omega^{\lambda^*}_{\lambda}. \tag{4.3}$$

Proof. Using (3.5), (2.1), and the Weingarten formula, we obtain the components of the curvature tensor $\overset{\tau}{R}$ of ∇^{τ} under the form

$$\overset{\tau}{R}{}^{\lambda^*}_{\lambda ab} = R^{\lambda^*}_{\lambda ab} + g\left(B_{e_\lambda}e_b, B_{e_{\lambda^*}}e_a\right) - g\left(B_{e_\lambda}e_a, B_{e_{\lambda^*}}e_b\right), \tag{4.4}$$

and then its curvature form is $\overset{\tau}{\Omega}{}^{\lambda^*}_{\lambda} = \Omega^{\lambda^*}_{\lambda}$. On the other hand, from (2.9), it follows the complex form of the second structure equation of $\tau(M)$, namely,

$$\begin{gathered} d\phi^{\lambda}_{\mu} = -\sum_{\nu=n+1}^{m}\phi^{\lambda}_{\nu}\wedge\phi^{\nu}_{\mu} + \Phi^{\lambda}_{\mu} \quad \text{with } \phi^{\lambda} = \omega^{\lambda} + i\omega^{\lambda^*}, \\ \phi^{\lambda}_{\mu} = \omega^{\lambda}_{\mu} + i\omega^{\lambda}_{\mu^*}, \qquad \Phi^{\lambda}_{\mu} = \overset{\tau}{\Omega}{}^{\lambda}_{\mu} + i\,\overset{\tau}{\Omega}{}^{\lambda}_{\mu^*}. \end{gathered} \tag{4.5}$$

But $\Phi^{\lambda}_{\lambda} = i\,\overset{\tau}{\Omega}{}^{\lambda}_{\lambda^*}$, and then we have

$$\gamma_1(\tau) = \frac{i}{2\pi}\sum_{\lambda=n+1}^{m}\Phi^{\lambda}_{\lambda} = -\frac{1}{2\pi}\sum_{\lambda=n+1}^{m}\overset{\tau}{\Omega}{}^{\lambda}_{\lambda^*} = -\frac{1}{2\pi}\sum_{\lambda=n+1}^{m}\Omega^{\lambda}_{\lambda^*} \tag{4.6}$$

and the proof is complete. □

Let $\vec{n}$ be a vector field normal to the integral submanifold M of the Sasakian manifold $\tilde{M}$. For $X \in \mathscr{X}(M)$, the equality $\alpha_{\vec{n}}(X) = g(F\vec{n}, X)$ defines a 1-form $\alpha_{\vec{n}}$ on M. In [6], this form is used for the study of some remarkable vector fields on M (Legendrian, Hamiltonian, and harmonic variations). Another 1-form on M is defined by $\theta = \sum_{a=1}^{n}\omega^{a^*}_{a}$, and we can state the following proposition.

Proposition 4.3. *The forms $\alpha_{\vec{n}}$ and θ have the following properties:*
(a) $\alpha_\xi = 0$ *and* $\theta = -n\alpha_H$, *where* H *is the mean curvature vector of* M*;*
(b) $\alpha_{\vec{n}}$ *is closed if and only if*

$$g(\nabla^\perp_X \vec{n}, FY) = g(\nabla^\perp_Y \vec{n}, FX) \tag{4.7}$$

for all $X, Y \in \mathcal{X}(M)$*;*
(c) *the exterior derivative of* θ *is given by*

$$d\theta = \sum_{b,c=1}^{n} \left(\tilde{S}_{bc^*} - \sum_{\lambda} R^{\lambda^*}_{\lambda bc} - \frac{1}{2} \sum_{a=1}^{n} \tilde{R}^{a^*}_{abc} \right) \omega^b \wedge \omega^c, \tag{4.8}$$

where $\tilde{S}$ *is the Ricci tensor of* $\tilde{M}$.

Proof. (a) We have the well-known equality

$$\tilde{\nabla}_X e_\alpha = \sum_{\beta=1}^{m} \omega^\beta_\alpha(X) e_\beta \tag{4.9}$$

for any $X \in \mathcal{X}(\tilde{M})$, and, by using (2.4) and (2.5), we obtain

$$\theta(e_b) = \sum_{a=1}^{n} g(\tilde{\nabla}_{e_b} e_a, e_{a^*}), \quad b \in \{1, 2, \ldots, n\}. \tag{4.10}$$

Taking into account (2.1) and the Gauss formula, we deduce

$$\begin{aligned} \theta(e_b) &= \sum_{a=1}^{n} g(h(e_a, e_b), e_{a^*}) = \sum_{a=1}^{n} g(\tilde{\nabla}_{e_a} e_b, F e_a) \\ &= -\sum_{a=1}^{n} g(F(\tilde{\nabla}_{e_a} e_b), e_a) = -\sum_{a=1}^{n} g(\tilde{\nabla}_{e_a} e_{b^*}, e_a) = \sum_{a=1}^{n} g(e_{b^*}, \tilde{\nabla}_{e_a} e_a) \\ &= \sum_{a=1}^{n} g(e_{b^*}, h(e_a, e_a)) = n g(e_{b^*}, H) = -n g(e_b, FH) = -n\alpha_H(e_b). \end{aligned} \tag{4.11}$$

(b) From the definition of the 1-form $\alpha_{\vec{n}}$, we obtain

$$d\alpha_{\vec{n}}(X, Y) = g(\tilde{\nabla}_X (F\vec{n}), Y) - g(\tilde{\nabla}_Y (F\vec{n}), X). \tag{4.12}$$

On the other hand, using (2.1), we have

$$\tilde{\nabla}_X (F\vec{n}) = F(\tilde{\nabla}_X \vec{n}) - \eta(\vec{n}) X, \tag{4.13}$$

and then, applying the Weingarten formula in (4.12), it follows that

$$d\alpha_{\vec{n}}(X, Y) = g(\tilde{\nabla}^\perp_Y \vec{n}, FX) - g(\tilde{\nabla}^\perp_X \vec{n}, FY). \tag{4.14}$$

(c) From (2.4) and (2.5) and taking into account the second structure equation of $\tilde{M}$

$$d\omega^\alpha_\beta = -\sum_{\gamma=1}^{2m} \omega^\alpha_\gamma \wedge \omega^\gamma_\beta + \tilde{\Omega}^\alpha_\beta, \tag{4.15}$$

we obtain

$$d\theta = -\sum_{\lambda=n+1}^{m}\sum_{a=1}^{n}\omega_{\lambda}^{a^*}\wedge\omega_a^{\lambda} - \sum_{b=1}^{n}\sum_{a=1}^{n}\omega_b^{a}\wedge\omega_a^{b^*} - \sum_{\lambda=n+1}^{m}\sum_{a=1}^{n}\omega_{\lambda}^{a}\wedge\omega_a^{\lambda^*} + \sum_{a=1}^{n}\tilde{\Omega}_a^{a^*}. \tag{4.16}$$

Using (2.5) again, the above equality becomes

$$d\theta = 2\sum_{a=1}^{n}\sum_{\lambda=n+1}^{m}\omega_a^{\lambda}\wedge\omega_a^{\lambda^*} + \sum_{a=1}^{n}\tilde{\Omega}_a^{a^*}. \tag{4.17}$$

Now, applying the Gauss formula for the submanifold M in (4.9), we have

$$\sum_{b=1}^{n}\left[\omega_a^{b}(X)e_b + \omega_a^{b^*}(X)e_{b^*}\right] + \sum_{\lambda=n+1}^{m}\left[\omega_a^{\lambda}(X)e_\lambda + \omega_a^{\lambda^*}(X)e_{\lambda^*}\right] = \nabla_X e_a + h(X,e_a) \tag{4.18}$$

for any $X \in \mathscr{X}(M)$. It follows that

$$\begin{aligned}\omega_a^{\mu}(X) &= g(h(X,e_a),e_\mu)\\ &= \sum_{b=1}^{n}X^b g(h(e_a,e_b),e_\mu) = \sum_{b=1}^{n}X^b h_{ba}^{\mu} = \sum_{b=1}^{n}h_{ba}^{\mu}\omega^b(X),\end{aligned} \tag{4.19}$$

where h_{ac}^{α} are the components of $h(e_a,e_c)$ with respect to the basis $\mathscr{B}_T$. Therefore, we have

$$\omega_a^{\alpha} = \sum_{a=1}^{n}h_{ac}^{\alpha}\omega^c \tag{4.20}$$

for any $\alpha = \lambda$ or $\alpha = \lambda^*$. Finally, from (4.17) and (4.20) we deduce

$$d\theta = \sum_{a,b,c=1}^{n}\left(h_{ab}^{\lambda}h_{ac}^{\lambda^*} - h_{ac}^{\lambda}h_{ab}^{\lambda^*}\right)\omega^b\wedge\omega^c + \sum_{a=1}^{n}\tilde{\Omega}_a^{a^*}. \tag{4.21}$$

Because $\tilde{M}$ is Sasakian, its curvature tensor $\tilde{R}$ satisfies the following equality [1, page 75]:

$$\tilde{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y, \quad X,Y \in \mathscr{X}(\tilde{M}), \tag{4.22}$$

hence the Ricci tensor $\tilde{S}$ of $\tilde{M}$ is given by

$$\tilde{S}(X,Y) = \sum_{\alpha=1}^{2m}\tilde{\mathscr{R}}(e_\alpha,X,e_\alpha,Y) - g(X,Y), \tag{4.23}$$

for all $X,Y \in \mathscr{X}(\tilde{M})$ orthogonal to ξ, where $\tilde{\mathscr{R}}$ is the Riemann-Christoffel curvature tensor field of $\tilde{M}$.

Using (2.3), from the first equality in (2.6), we deduce

$$\sum_{a=1}^{n} \tilde{\Omega}_a^{a*} = \frac{1}{2} \sum_{a,b,c=1}^{n} \tilde{R}_{abc}^{a*} \omega^b \wedge \omega^c \tag{4.24}$$

at any point of the submanifold M. Moreover, using the first Bianchi identity relative to $\tilde{M}$, we have

$$\tilde{R}_{abc}^{a*} = \tilde{\mathfrak{R}}(e_{a*}, e_a, e_b, e_c) = \tilde{\mathfrak{R}}(e_c, e_{a*}, e_a, e_b) + \tilde{\mathfrak{R}}(e_c, e_a, e_b, e_{c*}). \tag{4.25}$$

On the other hand, on a Sasakian manifold, the following equalities are true [1, page 93]:

$$\tilde{\mathfrak{R}}(FX, FY, FZ, FU) = \tilde{\mathfrak{R}}(X, Y, Z, U), \tag{4.26}$$

$$\begin{aligned} \tilde{\mathfrak{R}}(FX, Y, Z, U) + \tilde{\mathfrak{R}}(X, FY, Z, U) &= d\eta(Y, Z) g(U, X) + d\eta(Z, X) g(Y, U) \\ &\quad + d\eta(U, Y) g(X, Z) + d\eta(X, U) g(Y, Z), \end{aligned} \tag{4.27}$$

for all $X, Y, Z, U \in \mathscr{X}(\tilde{M})$ orthogonal to ξ. But $d\eta(e_a, e_b) = 0$, hence, from (4.27), we deduce

$$\tilde{\mathfrak{R}}(e_{a*}, e_c, e_a, e_b) + \tilde{\mathfrak{R}}(e_a, e_{c*}, e_a, e_b) = 0 \tag{4.28}$$

and therefore, using (4.23) and (4.26), from (4.25) we obtain

$$\begin{aligned} \sum_{a=1}^{n} \tilde{R}_{abc}^{a*} &= \sum_{a=1}^{n} [\tilde{\mathfrak{R}}(e_a, e_b, e_a, e_{c*}) + \tilde{\mathfrak{R}}(e_{a*}, e_b, e_{a*}, e_{c*})] \\ &= \tilde{S}(e_b, e_{c*}) + \sum_{\lambda=n+1}^{m} [\tilde{\mathfrak{R}}(e_{c*}, e_\lambda, e_\lambda, e_b) + \tilde{\mathfrak{R}}(e_\lambda, e_{b*}, e_\lambda, e_c)]. \end{aligned} \tag{4.29}$$

Now, from (4.27), we give

$$\tilde{\mathfrak{R}}(e_{a*}, e_\lambda, e_\lambda, e_b) + \tilde{\mathfrak{R}}(e_a, e_{\lambda*}, e_\lambda, e_b) = 0 \tag{4.30}$$

and then

$$\sum_{a=1}^{n} \tilde{R}_{abc}^{a*} = \tilde{S}(e_b, e_{c*}) + \sum_{\lambda=n+1}^{m} [\tilde{\mathfrak{R}}(e_{\lambda*}, e_c, e_\lambda, e_b) + \tilde{\mathfrak{R}}(e_{\lambda*}, e_b, e_c, e_\lambda)]. \tag{4.31}$$

Applying the second Bianchi identity in the above equality, we obtain

$$\sum_{a=1}^{n} \tilde{R}_{abc}^{a*} = \tilde{S}(e_b, e_{c*}) - \sum_{\lambda=n+1}^{m} \tilde{\mathfrak{R}}(e_{\lambda*}, e_\lambda, e_b, e_c); \tag{4.32}$$

and taking into account the Ricci equation

$$\tilde{\mathfrak{R}}(e_{\lambda*}, e_\lambda, e_b, e_c) = \mathfrak{R}^\perp(e_{\lambda*}, e_\lambda, e_b, e_c) - g([A_{e_{\lambda*}}, A_{e_\lambda}] e_c, e_b), \tag{4.33}$$

we deduce

$$\sum_{a=1}^{n} \tilde{R}^{a*}_{abc} = \tilde{S}_{bc*} - \sum_{\lambda=n+1}^{m} R^{\lambda*}_{\lambda bc} + \sum_{\lambda=n+1}^{m} \sum_{d=1}^{n} (A^d_{\lambda c} A^b_{\lambda* d} - A^d_{\lambda* c} A^b_{\lambda d}), \tag{4.34}$$

where, by $A^b_{\lambda a}$, we denote the components of the Weingarten operator of M, relative to $\mathcal{B}$. Now, (4.8) follows from (4.17), (4.34), and (2.6). □

Theorem 4.4. *Let M be an integral submanifold of the Sasakian space form $\tilde{M}(c)$.*

(a) *The first normal Chern class $[\gamma_1(\tau)]$ of M is zero.*

(b) *If the mean curvature vector of M is parallel, then its first normal Chern form $\gamma_1(\tau)$ is zero.*

Proof. (a) Recall that in a Sasakian space form $\tilde{M}(c)$, the curvature tensor $\tilde{R}$ and the Ricci tensor $\tilde{S}$ have the following expressions (see, e.g., [1, pages, 97–98]):

$$\begin{aligned}
\tilde{R}(X,Y)Z &= \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] \\
&\quad + \frac{c-1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\
&\qquad + g(Z,FY)FX - g(Z,FX)FY - 2g(X,FY)FZ], \\
\tilde{S}(X,Y) &= \frac{m(c+3)+c-1}{2} g(X,Y) - \frac{(m+1)(c-1)}{2}\eta(X)\eta(Y),
\end{aligned} \tag{4.35}$$

for all $X,Y,Z \in \mathcal{X}(\tilde{M})$. From these equalities, we easily deduce $\tilde{R}^{a*}_{abc} = 0$, $\tilde{S}_{bc*} = 0$, and taking into account (2.8) from Proposition 4.3(c), we obtain

$$d\theta = -2 \sum_{\lambda=n+1}^{m} \Omega^{\lambda*}_{\lambda}. \tag{4.36}$$

From Theorem 4.2 and from Proposition 4.3(a) and (c), it follows that

$$d\alpha_H = -\frac{1}{n} d\theta = \frac{4\pi}{n}\gamma_1(\tau), \tag{4.37}$$

and then the assertion (a) is proved.

(b) From (4.36) and using Proposition 4.3(b), we obtain $\gamma_1(\tau) = 0$. □

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