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A NOTE ON THE SUBCLASS ALGEBRA

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ABSTRACT. Each irreducible character of the subclass algebra is paired up with its irreducible module.

KEY WORDS AND PHRASES. *Finite group, irreducible character, subclass algebra.*

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INTRODUCTION.

Let G be a finite group and let H be a subgroup of G. If $g \varepsilon G$, the subclass of G containing g is the set $E_g = \{hgh^{-1} | h \varepsilon H\}$ and the subclass sum containing g is $B_g = \sum_{x \varepsilon E_g} x$. The algebra over the complex numbers, K, generated by these subclass sums is called the subclass algebra (denoted by S) associated with G and H.

Let $\{M_1, \ldots, M_s\}$ be the irreducible KG-modules with M_j affording the irreducible character, χ_j, of G and let $\{N_1 \ldots N_t\}$ be the irreducible KH-modules with N_i

affording the irreducible character Φ_i, of H. Suppose $\{e_i\}_{i=1}^{t}$ is a set of primitive orthogonal idempotents of KH and $\{f_i\}_{i=1}^{t}$ is the set of primitive central orthogonal idempotents of KH where the sets are indexed so that $N_i \cong KHe_i$ and $f_i = (\dim N_i)e_i = \frac{\dim \Phi_i}{|H|} \sum_{h\varepsilon H} \Phi_i(h^{-1})h$. We define the non-negative integers $\{c_{ij}\}$ by $\chi_j\Big|_H = \sum_{i=1}^{t} c_{ij}\,\Phi_i$.

In [2], it was demonstrated that the irreducible S-modules are $\{e_iM_j\}$. D. Travis [3] has shown that the irreducible characters of S are parameterized by pairs χ_j, Φ_i ($c_{ij} \neq 0$) and are given by

$$\psi_{ij}(B_g) = \frac{|E_g|}{|H|} \sum_{h\varepsilon H} \chi_j(gh)\Phi_i(h^{-1}). \tag{1}$$

Independent of Travis's work we show that the irreducible character afforded by e_iM_j is ψ_{ij}.

LEMMA: $\chi_i(sB_g) = |E_g|\chi(sg)\ \forall\, s\varepsilon S,\ \forall\, g\varepsilon G$

PROOF: Since $B_g = \frac{|E_g|}{|H|} \sum_{h\varepsilon H} hgh^{-1}$, we have $\chi_i(sB_g) = \frac{|E_g|}{|H|} \sum_{h\varepsilon H} \chi_i(shgh^{-1})$

$$= \frac{|E_g|}{|H|} \sum_{h\varepsilon H} \chi_i(hsgh^{-1}) \text{ since } hs = sh,\ \forall\, h\varepsilon H$$

$$= |E_g|\ \chi_i(sg)\ .$$

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THEOREM: Let ψ_{ij} be the character afforded by the irreducible S-module $e_iM_j(c_{ij} \neq 0)$. Then ψ_{ij} is as defined by equation (1).

PROOF: By proposition 2.3 of [2], we have $M_j\Big|_S = \sum_{k=1}^{t} (\dim N_k)e_kM_j$

$$= \sum_{k=1}^{t} f_kM_j\ .$$

Therefore, for $s\varepsilon S$, $\chi_j(sf_i) = \sum_{k=1}^{t} (\dim \Phi_k)\psi_{kj}(sf_i)$

$$= (\dim \Phi_i)\psi_{ij}(sf_i)$$

$$= (\dim \Phi_i)\psi_{ij}(s)$$

since the trace of the action of sf_i on f_iM_j is the same as the trace of the action of s on f_iM_j and the trace of the action of sf_i on f_kM_j $(i \neq k)$ is 0.

Thus $\psi_{ij}(B_g) = \frac{1}{\dim\Phi_i} \chi_j(B_g f_i)$

$$= \frac{|E_g|}{\dim\Phi_i} \chi_j(gf_i) \text{ by the Lemma}$$

$$= \frac{|E_g|}{|H|} \sum_{h\in H} \chi_j(gh)\Phi_i(h^{-1}) \ .$$

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