THE HEEGAARD GENUS OF MANIFOLDS OBTAINED BY SURGERY ON LINKS AND KNOTS

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ABSTRACT. Let $L \subset S^3$ be a fixed link. It is shown that there exists an upper bound on the Heegaard genus of any manifold obtained by surgery on L. The tunnel number of L, $T(L)$, is defined and used as an upper bound. If K' is a double of the knot K, it is shown that $T(K') \leq T(K) + 1$. If M is a manifold obtained by surgery on a cable link about K which has n components, it is shown that the Heegaard genus of M is at most $T(K) + n + 1$.

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1. INTRODUCTION.

Lickorish in [5] demonstrated that all compact connected orientable 3-manifolds without boundary could be represented as a result of surgery on a link in S^3. Since then, considerable work has been done on the so-called closed 3-manifolds by studying various knots and links in S^3.

Both in [3] and [8] a relationship between a knot and the Heegaard splitting of manifolds obtained by surgery on that knot was hinted at. In [2], Dibner showed that the Mazur homology sphere (see [6]) has a genus-2 Heegaard splitting. A similar process was used in [1] to show that $n + 1$ is an upper bound to the genus of a manifold obtained by surgery on a torus link with n components. The purpose of this paper is to extend these techniques to other knots and links and thereby obtain a bound on the Heegaard genus of manifolds obtained by surgery on those links and knots.

If X is a point set, we shall use $int(X)$ for the interior of X, $cl(X)$ for the closure of X, and ∂X for the boundary of X. The genus of a manifold is defined to be the minimal genus of a Heegaard splitting of the manifold. Let X be a polyhedron contained in the P.L. n-manifold M. $N(X) \subset M$ is called a regular neighborhood of X in M if $X \subset N(X)$ and $N(X)$ is an n-manifold which can be simplicially collapsed to X.

This paper deals with piecewise linear topology. As such, all manifolds are considered to be simplicial and maps to be piecewise linear. This will be assumed as additional hypotheses throughout the paper.

2. THE TUNNEL NUMBER.

Let L be a link in $\mathbb{R}^3$. Let P be any plane and $p : \mathbb{R}^3 \to P$ the orthogonal projection. P is regular for L if for every $x \in P$, $p^{-1}(x) \cap L$ consists of at most two points; and if $p^{-1}(x) \cap L$ does contain two points, neither is a vertex of L. The crossing number for L, $c(L)$, is the minimum number of double-points in any regular projection of any link L' of the same link type as L.

A link $L = L_1 \cup L_2$ is splittable if there exist disjoint 3-cells B_1 and B_2 with $L_1 \subset int\ B_1$ and $L_2 \subset int\ B_2$. Thus if L is any link we can write $L = L_1 \cup \ldots \cup L_n$ where $L_i \subset int\ B_i$, $B_i \cap B_j = \emptyset$ for $i \neq j$ and L_i is unsplittable for $1 \leq i \leq n$.

Let L be a link in S^3 and $N(L)$ a regular neighborhood of L in S^3.
We associate with L the manifold C , $cl(S^3 - N(L))$.

DEFINITION. $t \subset C$ is a _tunnel_ if t is a 3-cell and $t \cap \partial C$ is a pair of
disjoint 2-cells.

THEOREM 2.1. If M is a 3-manifold obtained by surgery on
$L = L_1 \cup \dots \cup L_n$, then the genus of M is at most $\sum_{i=1}^{n} c(L_i) + n$.

PROOF: M can be written as $M_1 \# \dots \# M_n$ where M_i is the manifold
obtained by surgery on the link L_i. By [4] we know that the genus of a
connected sum is the sum of the genera of the components. Thus it will suffice
to show that the genus of M_i is at most $c(L_i) + 1$.

If $c(L_i) = m$, we can find a regular projection for L_i which has m
double points. We can find $\ell_1, \dots, \ell_m$ where ℓ_j is the straight line segment
between the points of L_i which form the j-th double point of the projection.
Let $N(L_i)$ be a regular neighborhood of L_i in S^3, and C its associated
manifold. For each ℓ_i we can find a tunnel t_i such that t_i is a regular
neighborhood of $\ell_i \cap C$ and such that $t_i \cap t_j = \emptyset$ for $i \neq j$.
$H = \bigcup_{j=1}^{m} t_j \cup N(L_i)$ is a handlebody. Clearly $\bigcup_{j=1}^{m} \ell_j \cup L_i$ is a core for
this handlebody and this core can be deformed into the plane of projection.
Therefore H' , $cl(S^3 - H)$ is also a handlebody. Since L_i is unsplittable,
the core of H has one component. After deforming this core into the plane,
a simple Euler characteristic argument shows that the genus of H is m + 1.

We note that if $N'(L_i)$ is a regular neighborhood of L_i in M_i, then
$H'' = \bigcup_{j=1}^{m} t_j \cup N'(L_i)$ and H' form a Heegaard splitting of M_i.

The above theorem gives a rather crude estimate of the maximum genus of a
manifold obtained by surgery on a link. As we shall see, in specific situations
we will be able to give a better estimate. We also see that if L is an
unsplittable link, we can always find a set of pairwise disjoint tunnels which
can convert the associated manifold to a handlebody.

DEFINITION. If L is an unsplittable link, the <u>tunnel number</u> of L, T(L),
is the minimum number of disjoint tunnels needed to convert the associated
manifold into a handlebody.

3. <u>SURGERY ON KNOTS</u>.

In [7] H.F. Trotter described a class of knots known as pretzel knots. It is
clear from this description that any pretzel knot K can be embedded in the
boundary of a cube-with-two-handles H in such a way that there are cutting
disks D_1 and D_2 for H so that $D_1 \cap K$ and $D_2 \cap K$ each consist of two
points. Also $cl(S^3 - H)$ is itself a cube-with-two-handles.

PROPOSITION 3.1. If M is obtained by surgery on a pretzel knot, then M
has Heegaard genus at most 3.

PROOF. We assume that the pretzel knot $K \subset \partial H$ is as described above. We
can find an arc $\alpha_i \subset D_i$ with $\partial \alpha_i = K \cap D_i$ for i = 1 or 2. Without loss
of generality we may assume that a regular neighborhood of the cutting disk
D_i in H is of the form $D_i \times I$ and that $K \cap (D_i \times I) = \partial \alpha_i \times I$.

If N(K) is a regular neighborhood of K in H, then obviously
$H' = N(K) \cup (D_1 \times I) \cup (D_2 \times I)$ is a cube-with-three-handles. Let
$B = cl(H - [(D_1 \times I) \cup (D_2 \times I)])$. We note that
$(K \cap B) \cup (\alpha_1 \times \{0,1\}) \cup (\alpha_2 \times \{0,1\})$ is a simple closed curve in $\partial B = S^2$.
Let D be the disk in B bounded by this simple closed curve. Then D along
with the two cutting disks of $cl(S^3 - H)$ will form a system of cutting disks
for $cl(S^3 - H')$. Using the same reasoning as in Theorem 2.1, we have found a
genus 3 Heegaard splitting for any manifold obtained by surgery on a pretzel knot.

PROPOSITION 3.2. If M is a manifold obtained by surgery on the composite
knot $K_1 \# K_2$, then the genus of M is at most $T(K_1) + T(K_2) + 2$.

PROOF. We can find a 2-sphere $S \subset S^3$ with the following properties.
S separates S^3 into two 3-cells B_1 and B_2. $(K_1 \# K_2) \cap S$ consists of
two points. If $\alpha \subset S$ is an arc connecting these two points,

$[B_1 \cap (K_1 \# K_2)] \cup \alpha$ is a knot of type K_1 and $[B_2 \cap (K_1 \# K_2)] \cup \alpha$ is a knot of type K_2.   Without loss of generality we may assume that $S \times I$ is a regular neighborhood of S in S^3 and that $(S \times I) \cap (K_1 \# K_2) = \partial \alpha \times I$.

Since $K_1' = [(K_1 \# K_2) \cap cl(B_1 - S \times I)] \cup [\alpha \times \{0\}]$ is a knot of type K_1, it has tunnel number $T(K_1)$.   We can find a Heegaard splitting of B_1 of genus $T(K_1) + 1$.   Likewise since $K_2' = [(K_1 \# K_2) \cap cl(B_2 - S \times I)] \cup [\alpha \times \{1\}]$ is a knot of type K_2 it has tunnel number $T(K_2)$ and we can find a Heegaard splitting of B_2 of genus $T(K_2) + 1$.   We can connect these splittings by adding the 3-cell which is a regular neighborhood of $\alpha \times I$ in $S \times I$.   This will yield a Heegaard splitting of M of genus $T(K_1) + T(K_2) + 2$.

PROPOSITION 3.3.   If K' is a double of K and M is a manifold obtained by surgery on K',   then M has genus at most $T(K) + 2$.

PROOF.   Since K' is a double of K we can find a singular disk D with the following properties.   A copy of K is contained in D. K' is spanned by D and the only singularity in D is a single double arc α with $\partial \alpha \subset K'$.

Let $N(K' \cup \alpha)$ be a regular neighborhood of $K' \cup \alpha$ in S^3.   We note that $D' = cl(D - N(K' \cup \alpha))$ is a real disk and that $N(D)$ is a solid torus with a knot of type K for a core.   This means that if we add $T(K)$ tunnels to $N(D)$ we can obtain a Heegaard splitting of genus $T(K) + 1$.   When we remove $N(D')$ from $N(D)$ we will obtain a Heegaard splitting of M of genus $T(K) + 2$.

In addition to these specific knots, we can find a better estimate than Theorem 2.1 gives for the Heegaard genus of manifolds obtained by surgery on specific links.

4. SURGERY ON CABLE LINKS.

THEOREM 4.1.   If M is a 3-manifold obtained by surgery along a cable link about K with n components, then M has genus at most $T(K) + n + 1$.

PROOF.   Let T be a solid torus in S^3 which has K for a core.   We split ∂T into $2n$ annuli $\{A_1, B_1, A_2, B_2, \ldots, A_n, B_n\}$ each following an (x,y) curve

on ∂T. Let $\partial T \times I$ be a collar attached to T with $\partial T = \partial T \times \{0\}$. Let $C = cl(S^3 - (T \cup (\partial T \times I)))$. Without loss of generality, we may assume the existence of $T(K)$ tunnels $t_1, \ldots, t_{T(K)}$ contained in C such that $cl(C - \bigcup_{i=1}^{T(K)} t_i)$ is a cube-with-handles. We may also assume that $t_i \cap \partial C \subset int(B_1 \times \{1\}) \subset \partial T \times I$.

We can think of M as being obtained by surgery on $A_i \times [1/2, 1]$ where $A_i \times I \subset \partial T \times I$. Let D_i be a disk in A_i with the property that $cl(A_i - D_i)$ is a disk. Let α_i and β_i be arcs in ∂A_i with the property that $\partial A_i \cap (cl(A_i - D_i) = \alpha_i \cup \beta_i$. Finally, we let $N_i = h(A_i \times [1/2, 1]) \subset M$ be the image of $A \times [1/2, 1]$ in M after surgery is performed on the cable link.

Since $D_i \times [0, 1/2] \subset \partial T \times I$ is a 3-cell and $(D_i \times [0, 1/2]) \cap T$ and $(D_i \times [0, 1/2]) \cap N_i$ are disks, we have that $H_1 = T \cup^n_{i=1} [(D_i \times [0, 1/2]) \cup N_i$ is a cube-with-n+1-handles. We know that $[\bigcup_{i=1}^{T(K)} t_i] \cap \partial C$ is a set of $2T(K)$ pairwise disjoint disks contained in $int(B_1 \times \{1\})$. If $t_i \cap \partial C = d_{i,1} \cup d_{i,2}$, we can extend t_i to the 3-cell $t'_i = t_i \cup (d_{i,1} \times I) \cup (d_{i,2} \times I)$. Then $H'_1 = H_1 \cup^{T(K)}_{i=1} t'_i$ is a cube-with-n+T(K)+1-handles.

We note that since C is homeomorphic to $C \cup^n_{i=1} (B_i \times I)$, we must have that $H_2 = cl[(C \cup^n_{i=1} (B_i \times I)) - \bigcup^{T(K)}_{j=1} t'_j]$ is a cube-with-T(K)+1-handles. Since $\alpha_i \times I$ and $\beta_i \times I$ are disks on ∂H_2 where $\alpha_i \times I$ and $\beta_i \times I$ are contained in $\partial T \times I$, we have that $cl(A_i - D_i) \times I \subset \partial T \times I$ is a handle added to H_2. Thus $M - int H'_1 = H'_2$ is also a cube-with-n+T(K)+1-handles.

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