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PROBABILISTIC DERIVATION OF A BILINEAR SUMMATION FORMULA FOR THE MEIXNER-POLLACZEK POLYNOMIALS

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ABSTRACT. Using the technique of canonical expansion in probability theory, a bilinear summation formula is derived for the special case of the Meixner-Pollaczek polynomials $\{\lambda_n^{(k)}(x)\}$ which are defined by the generating function

$$\sum_{n=0}^{\infty} \lambda_n^{(k)}(x) z^n/n! = (1 + z)^{\frac{1}{2}(x-k)}/(1 - z)^{\frac{1}{2}(x+k)}, \quad |z| < 1.$$

These polynomials satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} p_k(x)\lambda_m^{(k)}(ix)\lambda_n^{(k)}(ix)dx = (-1)^n n!(k)_n \delta_{m,n}, \quad i = \sqrt{-1}$$

with respect to the weight function

$$p_1(x) = \text{sech } \pi x$$

$$p_k(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \text{sech } \pi x_1 \text{ sech } \pi x_2 \cdots$$

$$\text{sech } \pi(x - x_1 - \cdots - x_{k-1})dx_1 dx_2 \cdots dx_{k-1}, \quad k = 2, 3, \ldots$$

KEY WORDS AND PHRASES. *Meixner-Pollaczek polynomials, orthogonal polynomials, bilinear summation formula, bivariate distribution, canonical expansion, Runge identity, G-functions.*

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1. INTRODUCTION

Let U be a Cauchy random variable with the probability density function (p.d.f.)

$$f(u) = \frac{1}{\pi}\frac{1}{1+u^2}, \quad -\infty < u < \infty.$$

Consider the transformation $U = \sinh \pi V$. The p.d.f. of V is

$$p(v) = \operatorname{sech} \pi v, \quad -\infty < v < \infty. \tag{1}$$

This is the hyperbolic secant distribution considered by Baten [2], and is a special case of the generalized hyperbolic secant distribution treated by Harkness and Harkness [10].

Let X_1 and X_2 be two random variables having additive random elements in common [6], i.e.

$$X_1 = V_1 + V_2$$

$$X_2 = V_2 + V_3$$

where V_i $(i = 1, 2, 3)$ are mutually independent random variables each having the p.d.f. given in (1). The joint p.d.f. $p(x_1, x_2)$ of X_1 and X_2 is easily shown to be

$$p(x_1, x_2) = \int_{-\infty}^{\infty} \operatorname{sech} \pi z \operatorname{sech} \pi(x_1 - z) \operatorname{sech} \pi(x_2 - x_1 + z)dz$$

$$= \tfrac{1}{2} \operatorname{sech} \frac{\pi x_1}{2} \operatorname{sech} \frac{\pi x_2}{2} \operatorname{sech} \frac{\pi(x_2 - x_1)}{2}, \quad -\infty < x_1 < \infty, \quad -\infty < x_2 < \infty. \tag{2}$$

The marginal p.d.f.'s for X_1 and X_2 are respectively

$$g(x_1) = \int_{-\infty}^{\infty} p(x_1, x_2)dx_2 = 2x_1 \text{ cosech } \pi x_1,$$
$$h(x_2) = \int_{-\infty}^{\infty} p(x_1, x_2)dx_1 = 2x_2 \text{ cosech } \pi x_2. \quad (3)$$

The orthogonal polynomials with the above marginals as weight function are related to the Euler numbers and have been discussed by Carlitz in [4]. Specifically, for the weight function

$$p_k(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \text{sech } \pi x_1 \text{ sech } \pi x_2 \cdots \text{sech } \pi(x - x_1 - x_2 - \cdots - x_{k-1})dx_1 dx_2 \cdots dx_{k-1}, \quad k = 2, 3, \ldots . \quad (4)$$

$$p_1(x) = \text{sech } \pi x,$$

the polynomials $\left\{\lambda_n^{(k)}(x)\right\}$ with generating function

$$\sum_{n=0}^{\infty} \lambda_n^{(k)}(x)z^n/n! = (1 + z)^{\frac{1}{2}(x-k)}/(1 - z)^{\frac{1}{2}(x+k)}, \quad |z| < 1 \quad (5)$$

are the orthogonal polynomials in the interval $(-\infty, \infty)$ and satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} p_k(x)\lambda_m^{(k)}(ix)\lambda_n^{(k)}(ix)dx = (-1)^n n!(k)_n \delta_{m,n} \quad (6)$$

where $i = \sqrt{-1}$ and $\delta_{m,n}$ denotes the Kronecker delta.

The explicit form of the orthogonal polynomial is given by Carlitz [4] as

$$\lambda_n^{(k)}(x) = \sum_{r=0}^{n} 2^r \binom{\frac{1}{2}(x - k)}{r}\binom{n + k - 1}{n - r}$$

$$= (-1)^n(k)_n \, {}_2F_1[-n, \tfrac{1}{2}(x + k); k; 2]. \quad (7)$$

The last result follows easily from the following well-known generating function for the Gaussian hypergeometric function ${}_2F_1$ [7, p. 82]

$$(1 + z)^{b-c}[1 + (1 - x)z]^{-b} = \sum_{n=0}^{\infty} \binom{-c}{n} {}_2F_1[-n, b; c; x]z^n$$

$$|z| < 1, \quad |z - zx| < 1.$$

A related system of polynomials has been discussed by Bateman [1] who referred to them as the Mittag-Leffler polynomials. It happens that both the polynomials discussed by Bateman and Carlitz are but special cases of the system of orthogonal polynomials first discussed by Meixner [11] and later independently by Pollaczek [12]. Following the notation of [8, p. 219] (See also [5, p. 184]), the Meixner-Pollaczek polynomials are given explicitly by

$$P_n^{(\alpha)}(x;\ \phi) = \frac{(2\alpha)n}{n!} e^{in\phi} {}_2F_1[-n, \alpha + ix; 2\alpha; 1 - e^{-2i\phi}] \tag{8}$$

where $\alpha > 0$, $0 < \phi < \pi$ and $-\infty < x < \infty$.

These polynomials are orthogonal with respect to the weight function

$$\omega^{(\alpha)}(x;\ \phi) = \frac{(2 \sin \phi)^{2\alpha-1}}{\pi} e^{-(\pi-2\phi)x}|\Gamma(\alpha + ix)|^2.$$

The orthogonality relation is given by

$$\int_{-\infty}^{\infty} \omega^{(\alpha)}(x;\ \phi)P_m^{(\alpha)}(x;\ \phi)P_n^{(\alpha)}(x;\ \phi)dx = \frac{\Gamma(2\alpha + n)}{n!} \operatorname{cosec} \phi\ \delta_{m,n}.$$

A generating function for $P_n^{(\alpha)}(x;\ \phi)$ is

$$\sum_{n=0}^{\infty} t^n P_n^{(\alpha)}(x;\ \phi) = (1 - te^{i\phi})^{-\alpha+ix}(1 - te^{-i\phi})^{-\alpha-ix}, \quad |t| < 1. \tag{9}$$

It is clear when comparing (7) with (8) or (5) with (9) that

$$(-i)^n\lambda_n^{(k)}(ix) = n!\ P_n^{(k/2)}(x/2;\ \pi/2), \quad k = 1, 2, \ldots; n = 0, 1, 2, \ldots$$

and thus $\lambda_n^{(k)}(x)$ may be regarded as a special case of the Meixner-Pollaczek polynomials.

2. A BILINEAR SUMMATION FORMULA

From the generating function in (5) it is immediately clear that $\lambda_n^{(k)}(x)$ satisfies the following so-called Runge-type identity

$$\lambda_n^{(k_1+k_2)}(x_1 + x_2) = \sum_{r=0}^{n} \binom{n}{r} \lambda_r^{(k_1)}(x_1)\lambda_{n-r}^{(k_2)}(x_2), \quad k_1, k_2 = 1, 2, 3, \ldots \text{ and all } n. \tag{10}$$

It has been shown that the result in (10) is both necessary and sufficient for the joint p.d.f. in (2) to possess a bilinear expansion (also called a canonical expansion in statistical literature) of the form [6]

$$p(x_1, x_2) = g(x_1)h(x_2) \sum_{r=0}^{\infty} \rho_n \theta_n(x_1)\phi_n(x_2)$$

where the canonical variables $\{\theta_n(x)\}$ $(\{\phi_n(x)\})$ are a complete set of orthonormal polynomials with weight function $g(x)$ $(h(x))$. The canonical correlation is

$$\rho_n = E[\theta_n(X_1)\phi_n(X_2)]$$

where E denotes the expectation operation.

For the joint p.d.f. in (2) with the equal marginal p.d.f.'s given in (3), we note that the canonical variable in this case is

$$\theta_n(x) = \phi_n(x) = \frac{i^{-n}}{\sqrt{n!(2)_n}} \lambda_n^{(2)}(ix).$$

The canonical correlation is

$$\begin{aligned}
\rho_n &= E[\theta_n(X_1)\phi_n(X_2)] \\
&= \frac{(-1)^{-n}}{n!(2)_n} E\{\lambda_n^{(2)}[i(V_1 + V_2)]\lambda_n^{(2)}[i(V_2 + V_3)]\} \\
&= \frac{(-1)^{-n}}{n!(2)_n} \sum_{s=0}^{n} \sum_{r=0}^{n} \binom{n}{s}\binom{n}{r} E[\lambda_r^{(1)}(iV_1)] \ .
\end{aligned}$$

$$\cdot\ E[\lambda_{n-r}^{(1)}(iV_2)\lambda_s^{(1)}(iV_2)]E[\lambda_{n-s}^{(1)}(iV_3)]$$

$$= \frac{(-1)^{-n}}{n!(2)_n} E[\lambda_n^{(1)}(iV_2)]^2$$

$$= \frac{1}{n+1}\ . \tag{11}$$

We therefore have the following interesting bilinear summation formula for the Meixner-Pollaczek polynomials

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{[(n+1)!]^2} \lambda_n^{(2)}(ix_1)\lambda_n^{(2)}(ix_2) = \sinh\frac{\pi x_1}{2}\sinh\frac{\pi x_2}{2}\operatorname{sech}\frac{\pi(x_2 - x_1)}{2}\Big/(2x_1x_2)$$

$$-\infty < x_1 < \infty \quad \text{and} \quad -\infty < x_2 < \infty. \tag{12}$$

3. A GENERALIZATION

Consider the following more general scheme of additive random variables as in [9].

Let $\{\xi_i\}$ for $i = 1, 2, \ldots, n - m$, $\{\eta_i\}$ for $i = 1, 2, \ldots, m$ and $\{\zeta_i\}$ for $i = 1, 2, \ldots, n_2 - m$ where $1 \leq m < \min(n_1, n_2)$ be $(n_1 + n_2 - m)$ mutually independent random variables each having the p.d.f. given in (1).

Define

$$U = \sum_{i=1}^{n_1-m} \xi_i, \quad V = \sum_{i=1}^{m} \eta_i, \quad W = \sum_{i=1}^{n_2-m} \zeta_i$$

$$X_1 = U + V$$

$$X_2 = V + W.$$

It is clear that the joint characteristic function $\phi(\omega_1, \omega_2)$ of X_1 and X_2 is

$$\phi(\omega_1, \omega_2) = E[\exp(i\omega_1X_1 + i\omega_2X_2)]$$

$$= E\{\exp[i\omega_1 U + i\omega_2 W + i(\omega_1 + \omega_2)V]\}$$

$$= \operatorname{sech}^{n_1-m}\left(\frac{\omega_1}{2}\right) \operatorname{sech}^{n_2-m}\left(\frac{\omega_2}{2}\right) \operatorname{sech}^{m}\left(\frac{\omega_1 + \omega_2}{2}\right)$$

since

$$E\left[e^{i\omega\xi_i}\right] = E\left[e^{i\omega\eta_j}\right] = E\left[e^{i\omega\xi_k}\right]$$

$$= \int_{-\infty}^{\infty} e^{i\omega v} \operatorname{sech} \pi v \, dv$$

$$= \operatorname{sech}\left(\frac{\omega}{2}\right) \quad \text{for} \quad 1 \leqslant i \leqslant n_1 - m,$$

$$1 \leqslant j \leqslant m,$$

$$1 \leqslant k \leqslant n_2 - m.$$

The joint p.d.f. in question is therefore

$$p(x_1, x_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega_1 x_1 - i\omega_2 x_2} \operatorname{sech}^{n_1-m}\left(\frac{\omega_1}{2}\right) \operatorname{sech}^{n_2-m}\left(\frac{\omega_2}{2}\right) \cdot \operatorname{sech}^{m}\left(\frac{\omega_1 + \omega_2}{2}\right) d\omega_1 d\omega_2 \tag{13}$$

and the marginal p.d.f.'s for X_1 and X_2 are respectively

$$g(x_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega_1 x_1} \operatorname{sech}^{n_1}\left(\frac{\omega_1}{2}\right) d\omega_1$$

$$= \frac{1}{\pi} \frac{2^{n_1-1}}{(n_1 - 1)!} \left|\Gamma\left(\frac{n_1}{2} + ix_1\right)\right|^2$$

$$h(x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega_2 x_2} \operatorname{sech}^{n_2}\left(\frac{\omega_2}{2}\right) d\omega_2$$

$$= \frac{1}{\pi} \frac{2^{n_2-1}}{(n_2 - 1)!} \left|\Gamma\left(\frac{n_2}{2} + ix_2\right)\right|^2$$

on using the fact that [3, p. 31]

$$\int_{-\infty}^{\infty} e^{-ivx}[\operatorname{sech}(\beta x + \gamma)]^{\mu+1}dx = \frac{2^{\mu}}{\beta} \frac{\left|\Gamma\left(\frac{1 + \mu + iv/\beta}{2}\right)\right|^2}{\Gamma(\mu + 1)} e^{iv\gamma/\beta}. \tag{14}$$

The respective canonical variables are

$$\theta_n(x_1) = \frac{i^{-n}}{\sqrt{n!(n_1)_n}} \lambda_n^{(n_1)}(ix_1)$$

$$\phi_n(x_2) = \frac{i^{-n}}{\sqrt{n!(n_2)_n}} \lambda_n^{(n_2)}(ix_2).$$

By a repeated application of the Runge-type identity in (10) analogous to the derivation leading to the result in (11), it may be shown that the canonical correlation in this case is

$$\rho_n = \frac{(m)_n}{\sqrt{(n_1)_n(n_2)_n}}, \quad 1 \leq m < \min(n_1, n_2).$$

On the other hand, note that from (14)

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega_1 x_1 - i\omega_2 x_2} \operatorname{sech}^m\left(\frac{\omega_1 + \omega_2}{2}\right)d\omega_1 d\omega_2$$

$$= \frac{2^{m-1}}{\pi(m-1)!} \left|\Gamma\left(\frac{m}{2} + ix_1\right)\right|^2 \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[-i\omega_2(x_2 - x_1)]d\omega_2$$

$$= \frac{2^{m-1}}{\pi(m-1)!} \left|\Gamma\left(\frac{m}{2} + ix_1\right)\right|^2 \delta(x_2 - x_1)$$

where $\delta(x)$ denotes the Dirac delta function.

A double convolution operation applied to (13) then yields the following expression for $p(x_1, x_2)$

$$p(x_1, x_2) = \frac{2^{n_1+n_2+m-3}}{\pi^3(m-1)!(n_1 - m - 1)!(n_2 - m - 1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{n_1 - m}{2} + iu\right)\right|^2 \cdot$$

$$\left|\Gamma\left(\frac{n_2 - m}{2} + iv\right)\right|^2 \left|\Gamma\left(\frac{m}{2} + i(x_1 - u)\right)\right| \delta(x_2 - v - x_1 + u)dudv$$

$$= \frac{2^{n_1+n_2+m-3}}{\pi^3(m-1)!(n_1-m-1)!(n_2-m-1)!}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{n_1-m}{2}+iu\right)\right|^2 \cdot$$

$$\left|\Gamma\left(\frac{m}{2}+i(x_1-u)\right)\right|^2\left|\Gamma\left(\frac{n_2-m}{2}+i(x_2-x_1+u)\right)\right|^2 du. \tag{15}$$

Finally, the result in (15) may be rewritten into the following Barnes type contour integral

$$p(x_1, x_2) = \frac{2^{n_1+n_2+m-2}}{\pi^2(m-1)!(n_1-m-1)!(n_2-m-1)!} \cdot \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma\left(\frac{n_1-m}{2}+s\right)\Gamma\left(\frac{m}{2}-ix_1+s\right) \cdot$$

$$\Gamma\left(\frac{n_2-m}{2}+i(x_2-x_1)+s\right)\Gamma\left(\frac{n_1-m}{2}-s\right)\Gamma\left(\frac{m}{2}+ix_1-s\right)\Gamma\left(\frac{n_2-m}{2}-i(x_2-x_1)-s\right)ds$$

which may be evaluated in terms of a sum of ${}_3F_2$ functions [13, p. 133] or, perhaps more conveniently, in terms of Meijer's G-function as follows [7, Sec. 5.3]

$$p(x_1, x_2) = \frac{2^{n_1+n_2-m-2}}{\pi^2(m-1)!(n_1-m-1)!(n_2-m-1)!} \cdot$$

$$G^{3,3}_{3,3}\left(1\left|\begin{matrix}1-\frac{n_1-m}{2}, & 1-\frac{m}{2}+ix_1, & 1-\frac{n_2-m}{2}-i(x_2-x_1)\\ \frac{n_1-m}{2}, & \frac{m}{2}+ix_1, & \frac{n_2-m}{2}-i(x_2-x_1)\end{matrix}\right.\right).$$

The existence of a diagonal expansion then implies the following summation formula

$$\sum_{n=0}^{\infty}\frac{(-1)^n(m)_n}{(n_1)_n(n_2)_n n!}\lambda_n^{(n_1)}(x_1)\lambda_n^{(n_2)}(x_2)$$

$$= \frac{(n_1-1)!(n_2-1)!}{2^m(m-1)!(n_1-m-1)!(n_2-m-1)!} \cdot \frac{1}{\Gamma\left(\frac{n_1}{2}+x_1\right)\Gamma\left(\frac{n_1}{2}-x_1\right)\Gamma\left(\frac{n_2}{2}+x_2\right)\Gamma\left(\frac{n_2}{2}-x_2\right)} \cdot$$

$$\cdot\, G^{3,3}_{3,3}\left\{1\left|\begin{matrix} 1-\frac{n_1-m}{2}, & 1-\frac{m}{2}+x_1, & 1-\frac{n_2-m}{2}-(x_2-x_1) \\ \frac{n_1-m}{2}, & \frac{m}{2}+x_1, & \frac{n_2-m}{2}-(x_2-x_1) \end{matrix}\right.\right\} \qquad (16)$$

for $1 \leqslant m < \min(n_1, n_2)$, $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$.

It is perhaps interesting to note in passing that a comparison of the two results in (12) and (16) allows us to deduce the following special case of the G-function, viz.

$$G^{3,3}_{3,3}\left\{1\left|\begin{matrix} \frac{1}{2}, & \frac{1}{2}+x_1, & \frac{1}{2}+x_1-x_2 \\ \frac{1}{2}, & \frac{1}{2}+x_1, & \frac{1}{2}+x_1-x_2 \end{matrix}\right.\right\} = \frac{\pi^2}{4}\sec\frac{\pi x_1}{2}\sec\frac{\pi x_2}{2}\sec\frac{\pi(x_2-x_1)}{2}.$$

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REFERENCES

1. Bateman, H. The Polynomials of Mittag-Leffler, Proc. Nat. Acad. Sci. (U.S.A.) 26 (1940) 491-496.

2. Baten, W.D. The Probability Law for the Sum of n Independent Variables, each subject to the Law $(1/(2h))\ \text{sech}(\pi x/(2h))$, Bull. Amer. Math. Soc. 40 (1934) 284-290.

3. Buchholz, H. The Confluente Hypergeometric Functions, Springer Verlag, New York, 1969.

4. Carlitz, L. Bernoulli and Euler Numbers and Orthogonal Polynomials, Duke Math. J. 26 (1959) 1-15.

5. Chihara, T.S. An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

6. Eagleson, G.K. and H.O. Lancaster. The Regression System of Sums with Random Elements in Common, Aust. J. Statist. 9 (1967) 119-125.

7. Erdélyi, A., A. Magnus, F. Oberhettinger and F. Tricomi. Higher Transcendental Functions, Vol. 1, McGraw-Hill, New York, 1953.

8. Erdélyi, A., A. Magnus, F. Oberhettinger and F. Tricomi. Higher Transcendental Functions, Vol. 2, McGraw-Hill, New York, 1953.

9. Hamden, M.A. Canonical Expansion of the Bivariate Binomial Distribution with Unequal Marginal Indices, Int. Statist. Rev. 40 (1972) 277-280.

10. Harkness, W.L. and M.L. Harkness. Generalized Hyperbolic Secant Distributions, J. Amer. Statist. Assoc. 63 (1968) 329-337.

11. Meixner, J. Orthogonale Polynomsysteme mit einem besonderen Gestalt der erzeugenden Funktion, J. London Math. Soc. 9 (1934) 6-13.

12. Pollaczek, F. Sur une famille de polynômes orthogonaux qui contient les polynômes d'Hermite et de Laguerre comme cas limites, C.R. Acad. Sci., Paris 230 (1950) 1563-1565.

13. Slater, L.J. Generalized Hypergeometric Functions, Cambridge University Press, London, 1966.