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ABELIAN THEOREMS FOR THE STIELTJES TRANSFORM OF FUNCTIONS, II

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ABSTRACT. An initial (final) value Abelian theorem concerning transforms of functions is a result in which known behavior of the function as its domain variable approaches zero (approaches ∞) is used to infer the behavior of the transform as its domain variable approaches zero (approaches ∞). We obtain such theorems in this paper concerning the Stieltjes transform. In our results all parameters are complex; the variable s of the transform is complex in the right half plane; and the initial (final) value Abelian theorems are obtained as $|s| \to 0$ $(|s| \to \infty)$ within an arbitrary wedge in the right half plane.

KEY WORDS AND PHRASES. Abelian theorem, Stieltjes transform.

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1. INTRODUCTION.

The analysis of [1] extended the Abelian theorems of Widder [2, pp. 183-185] and Misra [3, Theorems 3.1.1 and 4.1.1] for the generalized Stieltjes transform

$$F(s) = \int_0^\infty f(t)\ (s+t)^{-p-1}\ dt \tag{1.1}$$

of a function f(t) with p being real to the case that s is a complex variable. Recently Tiwari [4, pp. 52-57] obtained a further extension by considering the generalized Stieltjes transform as studied by Byrne and Love [5] in which case the parameter p is complex as well. In this paper we obtain initial and final value Abelian theorems for the generalized Stieltjes transform of functions which have as special cases all of the corresponding results of [1] - [4]; in addition to both s and p being complex here, a generalizing assumption is placed on the function f(t) in (1.1) in our present results. Further, we obtain completely new Abelian theorems by putting a new limit assumption on the function f(t) in (1.1) as $t \to 0+$ and as $t \to +\infty$.

Throughout this paper t will denote a real variable. $s = \sigma + i\omega$, $p = p_1 + ip_2$, and $\eta = \eta_1 + i\eta_2$ will be complex numbers. The set of all complex numbers will be denoted by C, and $\ell n(t)$ will denote the natural logarithm of $t > 0$. In this paper all powers will be principal value powers. Log(s) and Arg(s) will denote the usual principal value of the logarithm and argument of the complex variable s, $-\pi < \mathrm{Arg}(s) \leq \pi$, respectively. Following Carlson [6, p. 291] we put $C_> = \{s \in C : s \neq 0, |\mathrm{Arg}(s)| < \pi/2\}$ and $C_0 = \{s \in C : s \neq 0, |\mathrm{Arg}(s)| < \pi\}$. Thus $C_>$ is the open right half plane and C_0 is the complex plane cut along the non-positive real axis. The gamma and beta functions will be denoted by $\Gamma(x)$ and $B(x,y)$, respectively, for suitable variables x and y.

In this paper the generalized Stieltjes transform F(s) of a complex valued function f(t) of the real variable $t \geq 0$ refers to the transform defined in (1.1) with the stated values of the complex number p being given in the various results. Notice that we have taken the power on the (s+t) term in (1.1) to be -p-1; whereas Byrne and Love [5] and Tiwari [4], for example, take this power to be -p. We prefer to use the power -p-1 in (1.1) in this paper for typographic convenience with respect to our analysis in sections 3 and 4. If one preferred to take the power in (1.1) to be -p, which is usually done, the results of sections 3 and 4 remain valid by simply replacing p by p-1 in the hypotheses and conclusions of the results of these sections; for example the hypothesis in Theorem 3.1 that $(p - \eta) \in C_>$ would be changed to $(p - \eta - 1) \in C_>$; and p would be replaced by p-1 in the conclusion (3.2) of Theorem 3.1 with similar changes in Theorems 3.2, 4.1, and 4.2. The corresponding results are equivalent whether -p-1 or -p is chosen as the power on (s+t) in (1.1).

2. PRELIMINARIES.

In this section we obtain results which we shall use in the proofs of our Abelian theorems. The following result extends [1, Lemma] and [4, p. 40, Lemma 1.5]. The proof which we give here is based on properties of Carlson's R function [6, p. 97, Definition 5.9-1; p. 137, Corollary 6.3-4; and p. 153, Theorem 6.8-1].

LEMMA 2.1. Let p and η be complex numbers such that $p - \eta \in C_>$ and $\eta + 1 \in C_>$. Let $s \in C_0$. We have

$$s^{p-\eta} \int_0^\infty t^\eta (s+t)^{-p-1} \, dt = B(p-\eta,\eta+1) = \frac{\Gamma(p-\eta)\,\Gamma(\eta+1)}{\Gamma(p+1)} . \tag{2.1}$$

PROOF. In [6, p. 154, (6)] let $a = p - \eta$ and $a' = \eta + 1$; let $k = 2$ and $z = (z_1,z_2)$ where $z_1 = s$ and $z_2 = 1$. Further, let $b = (b_1,b_2)$ where $b_1 = a + a' = p + 1$ and $b_2 = 0$. For $t \geq 0$ and $z_1 = s \in C_0$ we have $|\mathrm{Arg}(t+s)| < \pi$; and obviously $|\mathrm{Arg}(t+z_2)| < \pi$ since $z_2 = 1$. Thus by [6, p. 154, (6)] we have

$$R_{\eta-p}(p+1,0;s,1) = \frac{1}{B(p-\eta,\eta+1)} \int_0^\infty t^\eta (t+s)^{-p-1} \, dt . \tag{2.2}$$

But because of [6, p. 136, (3)] and the definition of $R_t(b;z)$ for $k = 1$ [6, p. 97, Definition 5.9-1] we have

$$R_{\eta-p}(p+1,0;s,1) = R_{\eta-p}(p+1;s) = s^{\eta-p} . \tag{2.3}$$

The first equality in (2.1) is now obtained by combining (2.2) and (2.3). The second equality in (2.1) follows by [6, p. 60, Definition 4.2-1]. The proof is complete.

We correct an error and misprints in the proof of [1, Lemma]. In [1, Lemma] we assumed that ρ and η are real numbers such that $-1 < \eta < \rho$. Thus because of the possible values for ρ and η a more complicated contour should have been chosen with which to apply the Cauchy theorem in the proof. Under the assumption that $\omega = \mathrm{Im}(s) \geq 0$, $s \in C_0$, in the proof of [1, Lemma], we obtain the first equality of [1, (3)] as before. Now let $a = re^{i\theta'}$ and $A = Re^{i\theta'}$, $\theta' = \mathrm{Arg}(1/s)$, $0 < r < R$. Let Γ_1 denote the straight line segment from a to A; let Γ_2 be the arc of circumference $z = Re^{i\theta}$, $\theta' \leq \theta \leq 0$, from $z = A$ to $z = R$; let Γ_3 be the straight line segment along the real axis from $z = R$ to $z = r$; and let Γ_4 be the arc of circumference $z = re^{i\theta}$, $\theta' \leq \theta \leq 0$, from $z = r$ to $z = a$. Finally let Γ be the union of Γ_1, Γ_2, Γ_3, and Γ_4. Cauchy's theorem can now be applied with respect to Γ and the integrand $(z^{\eta}(1+z)^{-\rho-1})$, and we obtain

$$\int_r^R x^{\eta}\,(1+x)^{-\rho-1}\,dx = \left(\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_4}\right) z^{\eta}\,(1+z)^{-\rho-1}\,dz\ ,\quad z = x+iy. \tag{2.4}$$

Straightforward estimates show that

$$\lim_{r\to 0+} \int_{\Gamma_4} z^{\eta}\,(1+z)^{-\rho-1}\,dz = 0 = \lim_{R\to\infty} \int_{\Gamma_2} z^{\eta}\,(1+z)^{-\rho-1}\,dz\ .$$

Hence upon letting $r \to 0+$ and $R \to \infty$ in (2.4), the proof is completed for the case $\omega = \mathrm{Im}(s) \geq 0$, $s \in C_0$, as in [1, Lemma] using the first equality of [1, (3)]. The proof for the case $\omega = \mathrm{Im}(s) < 0$ in [1, Lemma] proceeds analogously.

The following two lemmas contain some inequalities which we shall use later. $p = p_1 + ip_2$, $\eta = \eta_1 + i\eta_2$, and $s = \sigma + i\omega$ are complex numbers in these lemmas and throughout the remainder of the paper.

LEMMA 2.2. Let $t \geq 0$ be a real number. Let p, η, and s be complex numbers such that $p - \eta \in C_>$, $\eta + 1 \in C_>$, and $s \in C_>$. We have

$$|t^{\eta}| = t^{\eta_1}\ ; \tag{2.5}$$

$$\begin{aligned} |(s+t)^{-p-1}| &\leq |s+t|^{-p_1-1} \exp(\pi|p_2|/2) \\ &\leq (\sigma+t)^{-p_1-1} \exp(\pi|p_2|/2)\ ; \end{aligned} \tag{2.6}$$

$$\begin{aligned} |(s+t)^{-p-1}| &\leq |s+t|^{-p_1-1} \exp(\pi|p_2|/2) \\ &\leq |s|^{-p_1-1} \exp(\pi|p_2|/2)\ ; \end{aligned} \tag{2.7}$$

$$|s^{p-\eta}| \leq |s|^{p_1-\eta_1} \exp(\pi|p_2-\eta_2|/2)\ . \tag{2.8}$$

Further, if $t \geq y > 0$ for fixed $y > 0$ then

$$|s+t|^{-p_1-1} \leq t^{-p_1-1}\ . \tag{2.9}$$

PROOF. All of the proofs follow easily by using the properties of the principal value power. As an example we prove (2.8) and leave the proofs for (2.5) - (2.7) and (2.9) to the interested reader. We have

$$\begin{aligned} |s^{p-\eta}| &= |\exp((p-\eta)\ \mathrm{Log}(s))| \\ &= \exp((p_1-\eta_1)\ \ell n|s|)\ \exp(-(p_2-\eta_2)\ \mathrm{Arg}(s)) \\ &\leq |s|^{p_1-\eta_1}\ \exp(|p_2-\eta_2|\ |\mathrm{Arg}(s)|) \\ &\leq |s|^{p_1-\eta_1}\ \exp(\pi|p_2-\eta_2|/2) \end{aligned}$$

for $s \in C_>$, and (2.8) is obtained.

LEMMA 2.3.

I. Let $s \in C_>$ and let η be any complex number. We have

$$|s^\eta|^{-1} \leq |s|^{-\eta_1}\ \exp(\pi|\eta_2|/2)\ . \tag{2.10}$$

II. Let η be a fixed complex number such that $-1 < \eta_1 = \mathrm{Re}(\eta) < 0$ and let $s \in C_>$ such that $\sigma = \mathrm{Re}(s) > 1$ and $\ell n(\sigma) > |\pi\ \mathrm{ctn}(\eta\pi)|$. We have

$$|\mathrm{Log}(s) - \pi\ \mathrm{ctn}(\eta\pi)|^{-1} \leq (\ell n(\sigma) - |\pi\ \mathrm{ctn}(\eta\pi)|)^{-1}\ . \tag{2.11}$$

III. Let η be a fixed complex number such that $-1 < \eta_1 = \mathrm{Re}(\eta) < 0$ and let $s \in C_>$ such that $0 < |s| < 1$ and $\left|\ell n|s|\right| > |\pi\ \mathrm{ctn}(\eta\pi)|$. We have

$$|\mathrm{Log}(s) - \pi\ \mathrm{ctn}(\eta\pi)|^{-1} \leq (\left|\ell n|s|\right| - |\pi\ \mathrm{ctn}(\eta\pi)|)^{-1}\ . \tag{2.12}$$

IV. Let η and s satisfy the assumptions of III and in addition assume that for a given fixed $K \geq 0$, $s \in P_K = \{s : s = \sigma + i\omega,\ \sigma > 0,\ |\omega| \leq K\sigma\}$, $(\sigma(1 + K^2)^{1/2}) < 1$, and $|\ell n(\sigma(1 + K^2)^{1/2})| > |\pi\ \mathrm{ctn}(\eta\pi)|$. Then (2.12) can be continued as

$$\begin{aligned} |\mathrm{Log}(s) - \pi\ \mathrm{ctn}(\eta\pi)|^{-1} &\leq (\left|\ell n|s|\right| - |\pi\ \mathrm{ctn}(\eta\pi)|)^{-1} \\ &\leq (|\ell n(\sigma(1 + K^2)^{1/2})| - |\pi\ \mathrm{ctn}(\eta\pi)|)^{-1}\ . \end{aligned} \tag{2.13}$$

PROOF. For real $r > 0$ we know that $|\ell n(r)| \to \infty$ as $r \to 0+$ or as $r \to \infty$. Thus the assumptions $\ell n(\sigma) > |\pi\ \mathrm{ctn}(\eta\pi)|$ in II, $\left|\ell n|s|\right| > |\pi\ \mathrm{ctn}(\eta\pi)|$ in III, and $|\ell n(\sigma(1 + K^2)^{1/2})| > |\pi\ \mathrm{ctn}(\eta\pi)|$ in IV are meaningful assumptions on $\sigma = \mathrm{Re}(s)$ and s for fixed η. Now the proof of (2.10) is like those used to prove Lemma 2.2 and is left to the reader. To prove (2.11) we note that under the stated hypotheses in II we have

$$\begin{aligned} |\mathrm{Log}(s) - \pi\ \mathrm{ctn}(\eta\pi)| &\geq |\mathrm{Log}(s)| - |\pi\ \mathrm{ctn}(\eta\pi)| \\ &\geq \ln|s| - |\pi\ \mathrm{ctn}(\eta\pi)| \geq \ln(\sigma) - |\pi\ \mathrm{ctn}(\eta\pi)| , \end{aligned} \tag{2.14}$$

and all differences are positive. (2.11) follows by taking reciprocals in (2.14). To prove (2.12) we note that under the assumptions in III we have

$$|\mathrm{Log}(s) - \pi\ \mathrm{ctn}(\eta\pi)| \geq |\mathrm{Log}(s)| - |\pi\ \mathrm{ctn}(\eta\pi)| \geq \Big|\ln|s|\Big| - |\pi\ \mathrm{ctn}(\eta\pi)| \tag{2.15}$$

with both differences being positive. (2.12) follows by taking reciprocals in (2.15). We now prove (2.13). Under the assumptions in IV, s is in the wedge P_K. Thus $|s| \leq \sigma(1 + K^2)^{1/2}$. Since $0 < |s| < 1$ and $(\sigma(1 + K^2)^{1/2}) < 1$ then $\ln|s| \leq \ln(\sigma(1 + K^2)^{1/2})$ and hence

$$\Big|\ln|s|\Big| \geq |\ln(\sigma(1 + K^2)^{1/2})| . \tag{2.16}$$

The second inequality of (2.13) now follows by subtracting $|\pi\ \mathrm{ctn}(\eta\pi)|$ from both sides of (2.16) and then taking reciprocals. The proof of Lemma 2.3 is complete.

We shall need the Stieltjes transform formula contained in [7, p. 218, (28)] in section 4 of this paper, and we state this formula in the following lemma.

LEMMA 2.4. Let η be a complex number such that $-1 < \mathrm{Re}(\eta) < 0$. Let s be a complex number such that $|\mathrm{Arg}(s)| < \pi$. Then

$$\int_0^\infty t^\eta \ln(t)\ (s+t)^{-1}\,dt = -\pi\ s^\eta \csc(\eta\pi)\ (\mathrm{Log}(s) - \pi\ \mathrm{ctn}(\eta\pi)) .$$

The next lemma contains representations of two improper Riemann integrals which we shall need in our analysis in section 4.

LEMMA 2.5. For $0 < \sigma < 1$ and $-1 < \beta < 0$ we have

$$\begin{aligned} \int_0^1 t^\beta \ln(t)\ (\sigma+t)^{-1}\,dt &= \sigma^\beta \ln(\sigma) \sum_{k=0}^\infty (-1)^k\ (\beta+k+1)^{-1} \\ &- \sigma^\beta \sum_{k=0}^\infty (-1)^k\ (\beta+k+1)^{-2} - \sum_{k=0}^\infty (-1)^k\ \sigma^k\ (\beta-k)^{-2} \\ &- \sigma^\beta \ln(\sigma) \sum_{k=0}^\infty (-1)^k\ (\beta-k)^{-1} + \sigma^\beta \sum_{k=0}^\infty (-1)^k\ (\beta-k)^{-2} . \end{aligned} \tag{2.17}$$

For $1 < \sigma < \infty$ and $-1 < \beta < 0$ we have

$$\int_1^\infty t^\beta \ln(t)\ (\sigma+t)^{-1}\, dt = \sigma^\beta \ln(\sigma) \sum_{k=0}^{\infty} (-1)^k (\beta+k+1)^{-1}$$

$$+ \sigma^\beta \sum_{k=0}^{\infty} (-1)^{k+1} (\beta+k+1)^{-2} + \sum_{k=0}^{\infty} (-1)^k \sigma^{-k-1} (\beta+k+1)^{-2} \tag{2.18}$$

$$+ \sigma^\beta \sum_{k=0}^{\infty} (-1)^k (\beta-k)^{-2} + \sigma^\beta \ln(\sigma) \sum_{k=0}^{\infty} (-1)^{k+1} (\beta-k)^{-1} .$$

PROOF. First note that all series on the right of (2.17) and (2.18) are convergent. We prove (2.17) now. The improper integral $\int_0^\sigma t^\beta \ln(t)\ (\sigma+t)^{-1}\, dt$, $0 < \sigma < 1$, is defined to be the value of

$$\lim_{\delta\to 0+} \int_\delta^\sigma t^\beta \ln(t)\ (\sigma+t)^{-1}\, dt .$$

Furthermore, since the integral of a Riemann integrable function is a continuous function of its upper limit of integration [8, Theorem 7.32, pp. 161 - 162], we have

$$\int_\delta^\sigma \frac{t^\beta \ln(t)}{\sigma+t}\, dt = \lim_{\epsilon\to 0+} \int_\delta^{\sigma-\epsilon} \frac{t^\beta \ln(t)}{\sigma+t}\, dt , \quad 0 < \delta < \sigma < 1. \tag{2.19}$$

Hence consider the integral over the interval $\delta \le t \le \sigma - \epsilon$ in (2.19). For such t we note that $|t/\sigma| \le (\sigma - \epsilon)/\sigma < 1$; hence the series

$$\sum_{k=0}^{\infty} (-1)^k t^{\beta+k} \sigma^{-k} \ln(t)$$

converges uniformly for $\delta \le t \le \sigma - \epsilon$ by the Weierstrass M-test [8, Theorem 9.6, p. 223]. The interchange of integration and summation in the second line of the following computation is therefore justified [8, Theorem 9.9, p. 226].

$$\int_\delta^{\sigma-\epsilon} \frac{t^\beta \ln(t)}{\sigma+t}\, dt = \sigma^{-1} \int_\delta^{\sigma-\epsilon} \frac{t^\beta \ln(t)}{1+(t/\sigma)}\, dt$$

$$= \sigma^{-1} \int_\delta^{\sigma-\epsilon} t^\beta \ln(t) \sum_{k=0}^{\infty} (-t/\sigma)^k\, dt = \sum_{k=0}^{\infty} (-1)^k \sigma^{-k-1} \int_\delta^{\sigma-\epsilon} t^{\beta+k} \ln(t)\, dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \sigma^{-k-1} [((\beta+k+1)^{-1} \ln(t) - (\beta+k+1)^{-2}) t^{\beta+k+1}]\Big|_{\delta}^{\sigma-\epsilon}$$

$$\begin{aligned} = (\sigma-\epsilon)^{\beta} \Big(&\sum_{k=0}^{\infty} (-1)^k ((\sigma-\epsilon)/\sigma)^{k+1} (\beta+k+1)^{-1} \ln(\sigma-\epsilon) \\ &- \sum_{k=0}^{\infty} (-1)^k ((\sigma-\epsilon)/\sigma)^{k+1} (\beta+k+1)^{-2}\Big) \\ &- \sigma^{-1} \delta^{\beta+1} \Big(\sum_{k=0}^{\infty} (-1)^k (\delta/\sigma)^k (\beta+k+1)^{-1} \ln(\delta) \\ &- \sum_{k=0}^{\infty} (-1)^k (\delta/\sigma)^k (\beta+k+1)^{-2}\Big) . \end{aligned} \tag{2.20}$$

The last step above is justified since all four series are convergent by the alternating series test [8, Theorem 8.16, p. 188]. By Abel's theorem [8, Theorem 9.31, p. 245] we know that we can evaluate the limit as $\epsilon \to 0+$ in (2.20) by merely setting $\epsilon = 0$ in the last equality. Also since the last two series in (2.20) converge uniformly in δ, say for $0 \leq \delta \leq \sigma/2$, and since $\beta + 1 > 0$, then the product involving the last two series in (2.20) converges to zero as $\delta \to 0+$. Thus first letting $\epsilon \to 0+$ and then letting $\delta \to 0+$ in (2.20) we obtain

$$\int_0^{\sigma} \frac{t^{\beta} \ln(t)}{\sigma+t} dt = \sigma^{\beta} \ln(\sigma) \sum_{k=0}^{\infty} (-1)^k (\beta+k+1)^{-1} - \sigma^{\beta} \sum_{k=0}^{\infty} (-1)^k (\beta+k+1)^{-2} . \tag{2.21}$$

Using the same type of argument as in obtaining (2.21), the following computation is also valid.

$$\begin{aligned} \int_{\sigma}^{1} \frac{t^{\beta} \ln(t)}{\sigma+t} dt &= \lim_{\epsilon \to 0+} \int_{\sigma+\epsilon}^{1} \frac{t^{\beta} \ln(t)}{\sigma+t} dt \\ &= \lim_{\epsilon \to 0+} \int_{\sigma+\epsilon}^{1} t^{\beta-1} \ln(t) \sum_{k=0}^{\infty} (-\sigma/t)^k dt \\ &= \lim_{\epsilon \to 0+} \sum_{k=0}^{\infty} (-1)^k \sigma^k \int_{\sigma+\epsilon}^{1} t^{\beta-k-1} \ln(t) dt \qquad (2.22) \\ &= \lim_{\epsilon \to 0+} \sum_{k=0}^{\infty} (-1)^k \sigma^k (-(\beta-k)^{-2} - (\sigma+\epsilon)^{\beta-k} ((\beta-k)^{-1} \ln(\sigma+\epsilon) - (\beta-k)^{-2})) \end{aligned}$$

$$= \lim_{\epsilon \to 0+} \Big(-\sum_{k=0}^{\infty} (-1)^k \sigma^k (\beta-k)^{-2} - (\sigma+\epsilon)^{\beta} \ln(\sigma+\epsilon) \sum_{k=0}^{\infty} (-1)^k (\sigma/(\sigma+\epsilon))^k (\beta-k)^{-1}$$

$$+ (\sigma+\epsilon)^{\beta} \sum_{k=0}^{\infty} (-1)^k (\sigma/(\sigma+\epsilon))^k (\beta-k)^{-2}\Big)$$

$$= -\sum_{k=0}^{\infty} (-1)^k \sigma^k (\beta-k)^{-2} - \sigma^{\beta} \ln(\sigma) \sum_{k=0}^{\infty} (-1)^k (\beta-k)^{-1} + \sigma^{\beta} \sum_{k=0}^{\infty} (-1)^k (\beta-k)^{-2} .$$

The desired equality (2.17) is now obtained by combining (2.21) and (2.22).

The proof of (2.18) is completely analogous to that of (2.17). We split the integral in (2.18) at σ, $1 < \sigma < \infty$, and proceed similarly as in the proof of (2.17). We leave the now straightforward details to the interested reader. The proof of Lemma 2.5 is complete.

3. ABELIAN THEOREMS GENERALIZING THOSE OF [1] - [4].

In the Abelian theorems for functions of [1] - [4], hypotheses are placed on the quotient $f(t)/t^{\eta}$ for certain specified real η and then limit properties are obtained for the generalized Stieltjes transform of $f(t)$. In this section we allow η to be complex in the assumptions on the quotient $f(t)/t^{\eta}$; the Abelian theorems for functions of [1] - [4] become special cases of the results presented in this section. We note that there exist complex valued functions $f(t)$ of the real variable $t \geq 0$ which satisfy the hypotheses stated in both Theorem 3.1 and Theorem 3.2 below.

Let $K \geq 0$ be an arbitrary but fixed real number. We recall the wedge $P_K = \{s : s = \sigma + i\omega, \sigma > 0, |\omega| \leq K\sigma\}$ in $C_>$ as defined in Lemma 2.3 (IV). Our initial value Abelian theorem is as follows.

THEOREM 3.1. Let p and η be complex numbers such that $p - \eta \in C_>$ and $\eta + 1 \in C_>$. Let $f(t)$ be a complex valued function of the real variable $t \geq 0$ such that the generalized Stieltjes transform $F(s)$ of $f(t)$ exists for $s \in C_>$ and such that $(f(t)/t^{\eta})$ is bounded on $y \leq t < \infty$ for all $y > 0$. Let α be a complex number such that

$$\lim_{t \to 0+} \frac{f(t)}{t^{\eta}} = \alpha . \tag{3.1}$$

Then for each fixed $K \geq 0$,

$$\lim_{\substack{|s|\to 0 \\ s\varepsilon P_K}} \frac{s^{p-\eta} F(s)}{B(p-\eta,\eta+1)} = \alpha . \tag{3.2}$$

PROOF. Using (1.1) and Lemma 2.1 we have for any $y > 0$ that

$$\begin{aligned} |s^{p-\eta} F(s) - \alpha B(p-\eta,\eta+1)| &\leq |s^{p-\eta}| \int_0^y \left| \frac{f(t) - \alpha t^\eta}{(s+t)^{p+1}} \right| dt \\ &+ |s^{p-\eta}| \int_y^\infty \left| \frac{f(t) - \alpha t^\eta}{(s+t)^{p+1}} \right| dt = I_1 + I_2 . \end{aligned} \tag{3.3}$$

We first estimate I_1. Let $\epsilon > 0$ be arbitrary. Applying the hypothesis (3.1), there exists a $\delta > 0$ such that

$$\left| \frac{f(t)}{t^\eta} - \alpha \right| < \epsilon \text{ if } 0 < t < \delta .$$

We now fix y in (3.3) such that $0 < y < \delta$ and obtain

$$I_1 \leq \epsilon |s^{p-\eta}| \int_0^y |t^\eta| \, |(s+t)^{-p-1}| \, dt . \tag{3.4}$$

Using (2.5), the second inequality of (2.6), (2.8), and Lemma 2.1, we obtain from (3.4) that

$$\begin{aligned} I_1 &\leq \epsilon \exp(\pi|p_2|/2) \exp(\pi|p_2-\eta_2|/2) \, |s|^{p_1-\eta_1} \int_0^y t^{\eta_1} (\sigma+t)^{-p_1-1} dt \\ &\leq \epsilon \exp(\pi|p_2|/2) \exp(\pi|p_2-\eta_2|/2) \, B(p_1-\eta_1,\eta_1+1) \left(\frac{|s|}{\sigma}\right)^{p_1-\eta_1} . \end{aligned} \tag{3.5}$$

Now restrict $s \varepsilon C_>$ to P_K; for $s = \sigma + i\omega \varepsilon P_K$

$$\left(\frac{|s|}{\sigma}\right)^{p_1-\eta_1} \leq \left\{\frac{(\sigma^2+K^2\sigma^2)^{1/2}}{\sigma}\right\}^{p_1-\eta_1} = (1 + K^2)^{(p_1-\eta_1)/2} . \tag{3.6}$$

(3.5) and (3.6) yield

$$I_1 \leq \epsilon \exp(\pi|p_2|/2) \exp(\pi|p_2-\eta_2|/2) \, B(p_1-\eta_1,\eta_1+1) \, (1 + K^2)^{(p_1-\eta_1)/2} \tag{3.7}$$

for arbitrary $\epsilon > 0$ where $s \varepsilon P_K$.

For I_2 we have

$$I_2 \leq \left(\sup_{y\leq t<\infty} \left| \frac{f(t)}{t^\eta} - \alpha \right|\right) |s^{p-\eta}| \int_y^\infty |t^\eta| \, |(s+t)^{-p-1}| \, dt \tag{3.8}$$

and the supremum is finite since by hypothesis $(f(t)/t^\eta)$ is bounded on $y \leq t < \infty$. Now use (2.5), the first inequality of (2.6), and (2.8) in (3.8) to obtain

$$I_2 \leq \left(\sup_{y\leq t<\infty} \left| \frac{f(t)}{t^\eta} - \alpha \right|\right) \exp(\pi|p_2|/2) \cdot \\ \cdot \exp(\pi|p_2-\eta_2|/2) \, |s|^{p_1-\eta_1} \int_y^\infty t^{\eta_1} \, |s+t|^{-p_1-1} \, dt \, . \tag{3.9}$$

Recall that $p - \eta \in C_>$ so that $p_1 - \eta_1 > 0$. Using (2.9) in (3.9) and then performing the integration we have

$$I_2 \leq \left(\sup_{y\leq t<\infty} \left| \frac{f(t)}{t^\eta} - \alpha \right|\right) \exp(\pi|p_2|/2) \cdot \\ \cdot \exp(\pi|p_2-\eta_2|/2) \left(\frac{y^{\eta_1-p_1}}{p_1-\eta_1}\right) |s|^{p_1-\eta_1} \, . \tag{3.10}$$

(3.10) is valid for all $s \in C_>$ and shows that $I_2 \to 0$ as $|s| \to 0$ in any manner in $C_>$. We now combine (3.3), (3.7), and (3.10) to obtain

$$\limsup_{\substack{|s|\to 0 \\ s\in P_K}} |s^{p-\eta} F(s) - \alpha B(p-\eta,\eta+1)|$$

$$\leq \epsilon \exp(\pi|p_2|/2) \exp(\pi|p_2-\eta_2|/2) B(p_1-\eta_1,\eta_1+1) (1 + K^2)^{(p_1-\eta_1)/2}$$

where $\epsilon > 0$ is arbitrary. The desired result (3.2) follows immediately, and the proof is complete.

The initial value Abelian theorems [1, Theorem 1], [2, pp. 183 - 185], [3, Theorem 3.1.1], and [4, Theorem 4.1] are all special cases of Theorem 3.1.

The approach of s to zero inside the wedge P_K for arbitrary but fixed $K \geq 0$ in Theorem 3.1 is a sufficient condition for the desired conclusion (3.2) to hold but is not a necessary condition. The example of [1, p. 51] shows this.

We now prove our final value Abelian theorem.

THEOREM 3.2. Let p and η be complex numbers such that $p - \eta \in C_>$ and $\eta + 1 \in C_>$. Let $f(t)$ be a complex valued function of the real variable $t \geq 0$

such that the generalized Stieltjes transform $F(s)$ of $f(t)$ exists for $s \in C_>$ and such that $(f(t)/t^{\eta})$ is bounded on $0 < t \leq y$ for all $y > 0$. Let α be a complex number such that

$$\lim_{t\to\infty} \frac{f(t)}{t^{\eta}} = \alpha . \tag{3.11}$$

Then for each fixed $K \geq 0$

$$\lim_{\substack{|s|\to\infty \\ s\in P_K}} \frac{s^{p-\eta} F(s)}{B(p-\eta,\eta+1)} = \alpha . \tag{3.12}$$

PROOF. We begin with (3.3) exactly as in the proof of Theorem 3.1 where I_1 and I_2 are exactly as in (3.3). For arbitrary $\epsilon > 0$ we apply the hypothesis (3.11) and choose a fixed $y > 0$ large enough to obtain

$$I_2 \leq \epsilon\, |s^{p-\eta}| \int_y^{\infty} |t^{\eta}|\ |(s+t)^{-p-1}|\ dt . \tag{3.13}$$

Using (2.5), the second inequality of (2.6), and (2.8) in (3.13) we get

$$\begin{aligned} I_2 &\leq \epsilon \exp(\pi|p_2|/2) \exp(\pi|p_2-\eta_2|/2)\ |s|^{p_1-\eta_1} \int_y^{\infty} t^{\eta_1} (\sigma+t)^{-p_1-1}\ dt \\ &\leq \epsilon \exp(\pi|p_2|/2) \exp(\pi|p_2-\eta_2|/2)\ (\sigma+|\omega|)^{p_1-\eta_1} \int_0^{\infty} t^{\eta_1} (\sigma+t)^{-p_1-1}\ dt . \end{aligned} \tag{3.14}$$

By the change of variable $u = (t/\sigma)$ and Lemma 2.1 we have

$$\begin{aligned} \int_0^{\infty} t^{\eta_1} (\sigma+t)^{-p_1-1}\ dt &= \sigma^{-p_1-1} \int_0^{\infty} t^{\eta_1} (1 + (t/\sigma))^{-p_1-1}\ dt \\ &= \sigma^{\eta_1-p_1}\ B(p_1-\eta_1,\eta_1+1) . \end{aligned} \tag{3.15}$$

Putting (3.15) into (3.14) and restricting $s \in C_>$ to P_K , $K \geq 0$ being arbitrary but fixed, we have

$$I_2 \leq \epsilon \exp(\pi|p_2|/2) \exp(\pi|p_2-\eta_2|/2)\ (1 + K)^{p_1-\eta_1}\ B(p_1-\eta_1,\eta_1+1) \tag{3.16}$$

and $\epsilon > 0$ is arbitrary here.

Using the boundedness hypothesis of $(f(t)/t^{\eta})$, (2.5), the second inequality of (2.7), and (2.8) in I_1 we obtain

$$I_1 \leq \left(\sup_{0<t\leq y} \left| \frac{f(t)}{t^{\eta}} - \alpha \right|\right) \exp(\pi|p_2|/2) \cdot \exp(\pi|p_2-\eta_2|/2)\, |s|^{p_1-\eta_1}\, |s|^{-p_1-1} \int_0^y t^{\eta_1}\, dt \tag{3.17}$$

with the supremum being finite. The integral in (3.17) is a Riemann integral if $\eta_1 \geq 0$ and an improper integral if $-1 < \eta_1 < 0$. In either case the value is

$$\int_0^y t^{\eta_1}\, dt = \frac{y^{1+\eta_1}}{1+\eta_1} \tag{3.18}$$

for our fixed $y > 0$ since $\eta + 1 \in C_>$. A combination of (3.17) and (3.18) yields

$$I_1 \leq \left(\sup_{0<t\leq y} \left| \frac{f(t)}{t^{\eta}} - \alpha \right|\right) \exp(\pi|p_2|/2) \cdot \exp(\pi|p_2-\eta_2|/2) \left(\frac{y^{1+\eta_1}}{1+\eta_1}\right) |s|^{-\eta_1-1} . \tag{3.19}$$

(3.19) shows that $I_1 \to 0$ as $|s| \to \infty$ in any manner in $C_>$. The desired result (3.12) now follows by combining (3.3), (3.16), and (3.19) and using the same limit superior argument as in the proof of Theorem 3.1. The proof is complete.

The final value Abelian theorems [1, Theorem 2], [2, pp. 183 - 185], [3, Theorem 4.1.1], and [4, Theorem 4.2] are special cases of Theorem 3.2. Notice that the conclusion of [1, Theorem 2] was obtained for $|s| \to \infty$, $s \in S_K = \{s : s = \sigma + i\omega,\ \sigma > 0,\ |\omega| \leq K\}$, where $K \geq 0$ is arbitrary but fixed; that is s was allowed to get big in absolute value within a strip centered about the real axis in $C_>$. The conclusion of our present Theorem 3.2 allows $|s|$ to get big within a wedge P_K in $C_>$, which is a more general situation than [1, Theorem 2]. For the special case of Theorem 3.2 considered [4, Theorem 4.2], Tiwari also recognized that S_K could be replaced by P_K in his result.

4. FURTHER ABELIAN THEOREMS.

In a private communication to one of us, R.D.C., Lavoine [9] suggested that an attempt be made to replace the assumption of the type $(f(t)/t^{\eta}) \to \alpha$ as $t \to 0+$ or as $t \to \infty$ in Abelian theorems for the Stieltjes transform by the more general assumption that

$$\lim_{\substack{t\to 0+ \\ (\text{or } t\to\infty)}} \frac{f(t)}{t^{\eta}(\ell n(t))^{j}} = \alpha \tag{4.1}$$

where j is a positive integer and $\ell n(t)$ is the natural logarithm. In a recent paper Lavoine and Misra [10] have obtained Abelian theorems for the distributional Stieltjes transform in which an assumption of the type (4.1) is made on the distributions under consideration. The Stieltjes transform of the distributions is considered for the case that the variable s of the transform is real, and the Abelian theorems are then obtained as $s \to 0+$ or $s \to \infty$. In this section we shall obtain Abelian theorems for functions under an assumption like (4.1) in which the variable s of the Stieltjes transform is in $C_>$, and the results are obtained as $|s| \to 0+$ or $|s| \to \infty$ in a wedge P_K . We note that there exist functions f(t) which satisfy the hypotheses of Theorems 4.1 and 4.2.

Our initial value result is as follows.

THEOREM 4.1. Let η be a complex number such that $-1 < \eta_1 = \operatorname{Re}(\eta) < 0$. Let f(t) be a complex valued function of the real variable $t \geq 0$ such that the Stieltjes transform

$$F(s) = \int_0^\infty f(t)\ (s+t)^{-1}\, dt$$

exists for $s \in C_>$ and such that $(f(t)/t^{\eta}\ \ell n(t))$ is bounded on $y \leq t < \infty$ for all $y > 0$. Let α be a complex number such that

$$\lim_{t\to 0+} \frac{f(t)}{t^{\eta}\ \ell n(t)} = \alpha\ . \tag{4.2}$$

Then

$$\lim_{\substack{|s|\to 0 \\ s\in P_K}} \frac{F(s)}{s^{\eta}(\operatorname{Log}(s) - \pi \operatorname{ctn}(\eta\pi))} = -\alpha\pi \csc(\eta\pi)\ . \tag{4.3}$$

PROOF. Using Lemma 2.4 we have for any $y > 0$ that

$$\left| \frac{F(s)}{s^{\eta}(\operatorname{Log}(s) - \pi \operatorname{ctn}(\eta\pi))} + \alpha\pi \csc(\eta\pi) \right| \leq I_1 + I_2 \tag{4.4}$$

where

$$I_1 = \frac{1}{|s^\eta||\mathrm{Log}(s) - \pi \,\mathrm{ctn}(\eta\pi)|} \int_0^y \frac{\left|\frac{f(t)}{t^\eta \,\ell n(t)} - \alpha\right| |t^\eta \,\ell n(t)|}{|s+t|} dt$$

$$I_2 = \frac{1}{|s^\eta||\mathrm{Log}(s) - \pi \,\mathrm{ctn}(\eta\pi)|} \int_y^\infty \frac{\left|\frac{f(t)}{t^\eta \,\ell n(t)} - \alpha\right| |t^\eta \,\ell n(t)|}{|s+t|} dt \,. \tag{4.5}$$

We first estimate I_1. Let $\epsilon > 0$ be arbitrary; by (4.2) there exists a $\delta > 0$ such that

$$\left| \frac{f(t)}{t^\eta \,\ell n(t)} - \alpha \right| < \epsilon \text{ if } 0 < t < \delta \,. \tag{4.6}$$

Now fix $y > 0$ such that $0 < y < \min\{1,\delta\}$. In this result we are letting $|s| \to 0$, $s \in P_K$, for $K \geq 0$ arbitrary but **fixed.** Hence to obtain our result it suffices to assume that $0 < |s| < 1$ and $\left|\ell n|s|\right| > |\pi \,\mathrm{ctn}(\eta\pi)|$ and that for $K \geq 0$ fixed, $(\sigma(1 + K^2)^{1/2}) < 1$ and $|\ell n(\sigma(1 + K^2)^{1/2})| > |\pi \,\mathrm{ctn}(\eta\pi)|$, where $\sigma = \mathrm{Re}(s) > 0$. We emphasize that we are making the assumptions on $s \in P_K$ as stated in the preceding sentence throughout the remainder of this proof. Thus for $s \in P_K$ and the above fixed $y > 0$ we apply (4.6), (2.5), the second inequality of (2.6) with $p = 0$, (2.10), and (2.13) to obtain

$$I_1 \leq \frac{\epsilon \exp(\pi|\eta_2|/2)}{|s|^{\eta_1} (|\ell n(\sigma(1+K^2)^{1/2})| - |\pi \,\mathrm{ctn}(\eta\pi)|)} \int_0^1 \frac{t^{\eta_1} |\ell n(t)|}{\sigma+t} dt \,. \tag{4.7}$$

Now $0 < |s| < 1$ implies $0 < \sigma = \mathrm{Re}(s) < 1$; and we know that $|\ell n(t)| = -\ell n(t)$, $0 < t < 1$. Thus we use (2.17) of Lemma 2.5 in (4.7) to obtain

$$I_1 \leq \frac{\epsilon \exp(\pi|\eta_2|/2)}{|s|^{\eta_1} (|\ell n(\sigma(1+K^2)^{1/2})| - |\pi \,\mathrm{ctn}(\eta\pi)|)} [\sigma^{\eta_1} \ell n(\sigma) \sum_{k=0}^\infty \frac{(-1)^{k+1}}{1+k+\eta_1}$$

$$+ \sigma^{\eta_1} \sum_{k=0}^\infty \frac{(-1)^k}{(1+k+\eta_1)^2} + \sum_{k=0}^\infty \frac{(-1)^k \sigma^k}{(\eta_1-k)^2}$$

$$+ \sigma^{\eta_1} \ln(\sigma) \sum_{k=0}^{\infty} \frac{(-1)^k}{\eta_1 - k} + \sigma^{\eta_1} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(\eta_1 - k)^2}] \tag{4.8}$$

$$= \frac{\epsilon \exp(\pi|\eta_2|/2)}{|s|^{\eta_1} (|\ln(\sigma(1+K^2)^{1/2})| - |\pi \operatorname{ctn}(\eta\pi)|)} [J_1 + J_2 + J_3 + J_4 + J_5]$$

and the sum $[J_1 + J_2 + J_3 + J_4 + J_5]$ is positive since the integral which the sum equals is positive. For $s \varepsilon P_K$ and $p = 0$, (3.6) yields

$$\left(\frac{\sigma}{|s|}\right)^{\eta_1} = \left(\frac{|s|}{\sigma}\right)^{-\eta_1} \leq (1 + K^2)^{-\eta_1/2}, \quad s \varepsilon P_K, \tag{4.9}$$

where $-1 < \eta_1 = \mathrm{Re}(\eta) < 0$ here. Using (4.9) we have

$$\frac{|J_1|}{|s|^{\eta_1} (|\ln(\sigma(1 + K^2)^{1/2})| - |\pi \operatorname{ctn}(\eta\pi)|)}$$

$$\leq (1 + K^2)^{-\eta_1/2} \frac{|\ln(\sigma)|}{|\ln(\sigma(1+K^2)^{1/2})| - |\pi \operatorname{ctn}(\eta\pi)|} \left| \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{1+k+\eta_1} \right|$$

from which we conclude with the aid of L'Hospital's rule (and the fact that as $|s| \to 0$, $s \varepsilon P_K$, then $\sigma = \mathrm{Re}(s)$ must tend to zero also) that

$$\limsup_{\substack{|s|\to 0 \\ s\varepsilon P_K}} \frac{|J_1|}{|s|^{\eta_1}(|\ln(\sigma(1+K^2)^{1/2})| - |\pi \operatorname{ctn}(\eta\pi)|)} \leq (1+K^2)^{-\eta_1/2} \left| \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{1+k+\eta_1} \right| . \tag{4.10}$$

By similar analysis we have

$$\limsup_{\substack{|s|\to 0 \\ s\varepsilon P_K}} \frac{|J_4|}{|s|^{\eta_1}(|\ln(\sigma(1+K^2)^{1/2})| - |\pi \operatorname{ctn}(\eta\pi)|)} \leq (1+K^2)^{-\eta_1/2} \left| \sum_{k=0}^{\infty} \frac{(-1)^{k}}{\eta_1 - k} \right| . \tag{4.11}$$

Further, it is easy to see that the absolute value of J_2, J_3, and J_5 when divided by the denominator on the left of (4.10) and (4.11) all tend to zero in the limit as $|s| \to 0$, $s \varepsilon P_K$. Using this fact together with (4.8), (4.10), and

(4.11) we obtain

$$\limsup_{\substack{|s|\to 0\\ s\in P_K}} I_1 \leq \epsilon \exp(\pi|\eta_2|/2)\,(1+K^2)^{-\eta_1/2}\left[\left|\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{1+k+\eta_1}\right| + \left|\sum_{k=0}^{\infty}\frac{(-1)^k}{\eta_1-k}\right|\right] \tag{4.12}$$

and $\epsilon > 0$ is arbitrary on the right of (4.12).

Let us now estimate I_2 in (4.5). Take $y > 0$ to be fixed as in the sentence succeeding (4.6) so that $0 < y < \min\{1,\delta\}$. Applying the boundedness hypothesis for $(f(t)/t^{\eta}\,\ell n(t))$ on $y \leq t < \infty$, (2.5), and (2.9) for $p = 0$ we have

$$I_2 \leq \frac{\sup\limits_{y\leq t<\infty}\left|\dfrac{f(t)}{t^{\eta}\,\ell n(t)} - \alpha\right|}{|s^{\eta}|\,|\mathrm{Log}(s) - \pi\,\mathrm{ctn}(\eta\pi)|}\int_y^{\infty} t^{\eta_1-1}\,|\ell n(t)|\,dt . \tag{4.13}$$

Since $0 < y < \min\{1,\delta\}$ and $-1 < \eta_1 < 0$ then

$$\int_y^{\infty} t^{\eta_1-1}\,|\ell n(t)|\,dt = \frac{y^{\eta_1}\,\ell n(y)}{\eta_1} + \frac{2}{(\eta_1)^2} - \frac{y^{\eta_1}}{(\eta_1)^2} . \tag{4.14}$$

Recall that we are assuming the conditions on $s \in P_K$ under which (2.13) holds without loss of generality in this proof. Thus using (2.13), (2.10), and (4.14) in (4.13) we get

$$I_2 \leq \left(\sup_{y\leq t<\infty}\left|\frac{f(t)}{t^{\eta}\,\ell n(t)} - \alpha\right|\right)\left(\frac{y^{\eta_1}\,\ell n(y)}{\eta_1} + \frac{2}{(\eta_1)^2} - \frac{y^{\eta_1}}{(\eta_1)^2}\right) \cdot$$
$$\cdot \exp(\pi|\eta_2|/2)\,|s|^{-\eta_1}\,(|\ell n(\sigma(1+K^2)^{1/2})| - |\pi\,\mathrm{ctn}(\eta\pi)|)^{-1} \tag{4.15}$$

from which the fact

$$\lim_{\substack{|s|\to 0\\ s\in P_K}} I_2 = 0 \tag{4.16}$$

follows. We now combine (4.4), (4.12), and (4.16) to conclude that for any fixed $K \geq 0$

$$\limsup_{\substack{|s|\to 0 \\ s\in P_K}} \left| \frac{F(s)}{s^\eta(\mathrm{Log}(s) - \pi \operatorname{ctn}(\eta\pi))} + \alpha\pi \csc(\eta\pi) \right| \tag{4.17}$$

$$\leq \epsilon \exp(\pi|\eta_2|/2)\ (1+K^2)^{-\eta_1/2}\ \left[\left| \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{1+k+\eta_1} \right| + \left| \sum_{k=0}^{\infty} \frac{(-1)^k}{\eta_1 - k} \right| \right] .$$

Since $\epsilon > 0$ is arbitrary on the right of (4.17), the conclusion (4.3) follows. The proof is complete.

We now obtain a similar final value Abelian theorem.

THEOREM 4.2. Let η be a complex number such that $-1 < \eta_1 = \mathrm{Re}(\eta) < 0$. Let $f(t)$ be a complex valued function of the real variable $t \geq 0$ such that the Stieltjes transform

$$F(s) = \int_0^\infty f(t)\ (s+t)^{-1}\ dt$$

exists for $s \in C_>$ and such that $(f(t)/t^\eta \ell n(t))$ is bounded on $0 < t \leq y$ for all $y > 0$. Let α be a complex number such that

$$\lim_{t\to\infty} \frac{f(t)}{t^\eta\ \ell n(t)} = \alpha. \tag{4.18}$$

Then

$$\lim_{\substack{|s|\to\infty \\ s\in P_K}} \frac{F(s)}{s^\eta(\mathrm{Log}(s) - \pi \operatorname{ctn}(\eta\pi))} = -\alpha\pi \csc(\eta\pi) . \tag{4.19}$$

PROOF. Let $K \geq 0$ be arbitrary but fixed. In this result we are letting $|s| \to \infty$, $s \in P_K$, to obtain (4.19). As $|s| \to \infty$, $s \in P_K$, $\sigma = \mathrm{Re}(s)$ must tend to ∞ also. Thus without loss of generality we assume throughout this proof that $\sigma = \mathrm{Re}(s) > 1$ and $\ell n(\sigma) > |\pi \operatorname{ctn}(\eta\pi)|$. Now proceeding as in the proof of Theorem 4.1 we obtain (4.4) where I_1 and I_2 are defined in (4.5). To estimate I_2 we first take an arbitrary $\epsilon > 0$ and apply hypothesis (4.18) to obtain a fixed $y > 1$ such that

$$\left| \frac{f(t)}{t^\eta\ \ell n(t)} - \alpha \right| < \epsilon \text{ if } t > y > 1. \tag{4.20}$$

For $y > 1$ so fixed we apply (4.20), (2.5), the second inequality of (2.6) for $p = 0$, (2.10), and (2.11) to obtain

$$I_2 \leq \frac{\epsilon \exp(\pi|\eta_2|/2)}{|s|^{\eta_1} (\ln(\sigma) - |\pi \operatorname{ctn}(\eta\pi)|)} \int_1^\infty \frac{t^{\eta_1} \ln(t)}{\sigma+t} dt . \tag{4.21}$$

Using (2.18) of Lemma 2.5 in (4.21) we get

$$\begin{aligned} I_2 \leq \frac{\epsilon \exp(\pi|\eta_2|/2)}{|s|^{\eta_1} (\ln(\sigma) - |\pi \operatorname{ctn}(\eta\pi)|)} &[\sigma^{\eta_1} \ln(\sigma) \sum_{k=0}^{\infty} \frac{(-1)^k}{1+k+\eta_1} \\ &+ \sigma^{\eta_1} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(1+k+\eta_1)^2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{\sigma^{k+1} (1+k+\eta_1)^2} \\ &+ \sigma^{\eta_1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\eta_1-k)^2} + \sigma^{\eta_1} \ln(\sigma) \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\eta_1-k}] \\ = \frac{\epsilon \exp(\pi|\eta_2|/2)}{|s|^{\eta_1} (\ln(\sigma) - |\pi \operatorname{ctn}(\eta\pi)|)} &[K_1 + K_2 + K_3 + K_4 + K_5] \end{aligned} \tag{4.22}$$

and $[K_1 + K_2 + K_3 + K_4 + K_5]$ is positive. For $s \in P_K$ we use (4.9) and L'Hospital's rule to obtain (recall (4.10))

$$\limsup_{\substack{|s|\to\infty \\ s\in P_K}} \frac{|K_1|}{|s|^{\eta_1} (\ln(\sigma) - |\pi \operatorname{ctn}(\eta\pi)|)} \leq (1+K^2)^{-\eta_1/2} \left| \sum_{k=0}^{\infty} \frac{(-1)^k}{1+k+\eta_1} \right| . \tag{4.23}$$

Similarly we get

$$\limsup_{\substack{|s|\to\infty \\ s\in P_K}} \frac{|K_5|}{|s|^{\eta_1} (\ln(\sigma) - |\pi \operatorname{ctn}(\eta\pi)|)} \leq (1+K^2)^{-\eta_1/2} \left| \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\eta_1-k} \right| . \tag{4.24}$$

Using (4.9) it is easy to see that the absolute value of K_2 and K_4 when divided by the denominator on the left of (4.23) and (4.24) both tend to zero as $|s| \to \infty$,

$s \in P_K$. To analyze the K_3 term we note that

$$\frac{|s|^{-\eta_1}}{\sigma} \leq \frac{(1+K^2)^{-\eta_1/2}}{\sigma^{1+\eta_1}}, \quad s = \sigma+i\omega \in P_K, \tag{4.25}$$

and hence

$$\begin{aligned}\frac{|K_3|}{|s|^{\eta_1}(\ln(\sigma) - |\pi \operatorname{ctn}(\eta\pi)|)} &= \frac{|s|^{-\eta_1}}{\sigma(\ln(\sigma) - |\pi \operatorname{ctn}(\eta\pi)|)} \left|\sum_{k=0}^{\infty} \frac{(-1)^k}{\sigma^k(1+k+\eta_1)^2}\right| \\ &\leq \frac{(1+K^2)^{-\eta_1/2}}{\sigma^{1+\eta_1}(\ln(\sigma) - |\pi \operatorname{ctn}(\eta\pi)|)} \sum_{k=0}^{\infty} \frac{1}{(1+k+\eta_1)^2}\end{aligned} \tag{4.26}$$

since we assumed at the beginning of this proof that $\sigma > 1$. Recalling that $-1 < \eta_1 < 0$, (4.26) proves that the left side of (4.26) tends to zero as $|s| \to \infty$, $s \in P_K$. We thus conclude from (4.22), (4.23), (4.24), (4.26), and the facts stated above concerning K_2 and K_4 that

$$\limsup_{\substack{|s|\to\infty \\ s\in P_K}} I_2 \leq \epsilon \exp(\pi|\eta_2|/2)\ (1+K^2)^{-\eta_1/2} \left[\left|\sum_{k=0}^{\infty} \frac{(-1)^k}{1+k+\eta_1}\right| + \left|\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\eta_1-k}\right|\right]. \tag{4.27}$$

We now estimate I_1. For the $y > 1$ fixed in (4.20) and any δ such that $0 < \delta < y$ we have

$$\begin{aligned}\int_\delta^y \frac{t^{\eta_1}\,|\ln(t)|}{|s+t|}\,dt &\leq \frac{1}{|s|}\left[- \int_\delta^1 t^{\eta_1} \ln(t)\,dt + \int_1^y t^{\eta_1} \ln(t)\,dt \right] \\ &= \frac{1}{|s|}\left[\frac{\delta^{1+\eta_1}\ln(\delta)}{1+\eta_1} - \frac{\delta^{1+\eta_1}}{(1+\eta_1)^2} + \frac{y^{1+\eta_1}\ln(y)}{1+\eta_1} - \frac{y^{1+\eta_1}}{(1+\eta_1)^2} + \frac{2}{(1+\eta_1)^2}\right].\end{aligned} \tag{4.28}$$

Using the boundedness hypothesis for $(f(t)/t^{\eta} \ln(t))$, (2.5), a proof as in obtaining (2.10), (2.11), and taking the limit in (4.28) as $\delta \to 0+$ we obtain

$$I_1 \leq \frac{\sup\limits_{0<t\leq y} \left| \frac{f(t)}{t^{\eta}\, \ell n(t)} - \alpha \right| \exp(\pi |\eta_2|/2)}{|s|^{1+\eta_1} (\ell n(\sigma) - |\pi \operatorname{ctn}(\eta\pi)|)} \cdot \left[\frac{2}{(1+\eta_1)^2} + \frac{y^{1+\eta_1}\, \ell n(y)}{1+\eta_1} - \frac{y^{1+\eta_1}}{(1+\eta_1)^2} \right] . \tag{4.29}$$

(4.29) shows that $I_1 \to 0$ as $|s| \to \infty$, $s \in P_K$, since $1 + \eta_1 > 0$. Using this fact, the estimate (4.27), and (4.4), the desired result (4.19) follows by the same reasoning as at the conclusion of the proof of Theorem 4.1. The proof of Theorem 4.2 is complete.

It is our goal to obtain results like Theorems 4.1 and 4.2 and like those of Lavoine and Misra [10] under an assumption like (4.1) for arbitrary $j = 1, 2, 3, \cdots$ and for functions and distributions with the parameters p and η and the variable $s \in C_>$ all being complex. Further, it is our goal to extend the results of Carmichael and Milton [11] and Tiwari [4, p. 42, Theorem 2.1; p. 49, Theorem 3.3] to the case that the variable and all parameters are complex.

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