

A NOTE ON SUFFICIENCY IN COHERENT MODELS

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ABSTRACT. Partly of an expository nature, this article brings together a number of notions related to sufficiency in an abstract measure theoretic setting. The notion of a coherent statistical model, as introduced by Hasegawa and Perlman [6], is studied in some details. A few results are generalized and their earlier proofs simplified. Among other things, it is shown that a coherent model can be connected in the sense of Basu [2] if and only if no splitting set (Koehn and Thomas, [7]) exists.

KEY WORDS AND PHRASES. Sufficiency, coherent model, splitting set.

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1. INTRODUCTION.

This article is partly of an expository nature and is written mainly for its pedagogical interest. Given a mathematical model $(X, \mathcal{A}, \mathcal{P})$ for a statistical experiment, it is of some theoretical interest to inquire whether the family of sufficient sub- σ -fields (subfields) \mathcal{C} has a minimum element in some sense. It is now known that if the model is coherent in the sense of Hasegawa and Perlman [6] then such a minimum sufficient subfield exists. The case of a coherent model is

studied in some details in this article. Among other things it is demonstrated that for a coherent model the notion of connectedness (Basu, [2]) and that of the nonexistence of a splitting set (Koehn and Thomas, [7]) coincide.

2. NOTATION AND DEFINITIONS.

The basic statistical model is denoted by (X, A, P) , where X is the sample space, A a σ -field of subsets of X , and $P = \{P_\theta : \theta \in \Theta\}$ a family of probability measures on A . By an A -measurable function we mean a measurable map of (X, A) into (R_1, B_1) . Any sub- σ -field C of A will be referred to as a subfield. The function f is C -measurable if $f^{-1}B_1$ is contained in C . Given a family $\{A_t : t \in T\}$ of measurable sets, we write $\sigma\{A_t : t \in T\}$ for the subfield generated by the family of sets. Likewise, $\sigma\{f_t : t \in T\}$ will stand for the smallest subfield C such that each f_t is C -measurable. By $C \vee D$ we denote $\sigma(C \cup D)$, that is, the smallest subfield containing both C and D .

A set N in A is P -null if $P_\theta(N) = 0$ for all $\theta \in \Theta$. Let N denote the class of all P -null sets. For $A, B \in A$, the statement " $A = B [P]$ " means that the symmetric difference $A \Delta B$ is P -null. Similarly, for any two A -measurable functions f and g , we write $f = g [P]$ to indicate that $\{x : f(x) \neq g(x)\} \in N$.

The completion \bar{C} of a subfield C is defined as $\bar{C} = C \vee \sigma(N)$. Accordingly, a subfield C is called complete if $N \subset C$. Let C and D be two subfields. We write $C \subset D[P]$ if $C \subset \bar{D}$, that is, corresponding to each set C in C , there exists a set $D \in D$ such that $C = D[P]$. If $C \subset D[P]$ and $D \subset C[P]$, then $C = D[P]$. By a P -essentially \bar{C} -measurable function we mean a function f such that $f^{-1}B_1 \in C [P]$, i.e., f is \bar{C} -measurable. This is also equivalent to the statement that there exists a C -measurable function g such that $f = g [P]$. An A -measurable function is P -integrable if $\int_X |f| dP_\theta < \infty$ for all $\theta \in \Theta$.

DEFINITION 1. (Halmos and Savage, [5]). The statistical model (X, A, P) is called dominated if every probability measure in P is absolutely continuous with respect to a fixed σ -finite measure λ on A .

In this case, we say that the family is dominated by λ and write $P \ll \lambda$.

DEFINITION 2. (Basu and Ghosh, [3]). The statistical model (X, A, P) is called discrete if

- (i) each P_θ is a discrete probability measure,
- (ii) A is the class of all subsets of X , and
- (iii) for each $x \in X$, there exists a $\theta \in \Theta$ such that $P_\theta(\{x\}) > 0$.

Condition (iii) implies that the empty set is the only P -null set. A discrete model with a countable sample space X is clearly dominated. If P is countable, then the model is dominated. We assume that both X and P are uncountable. In this case, the model will be undominated. As we shall see in the next section, dominated and discrete models are particular cases of what Hasegawa and Perlman [6] called a coherent model.

Finally, let us state the notion of sufficiency as follows. A subfield C is sufficient with respect to the model (X, A, P) if, corresponding to each A in A , there exists a C -measurable function I_A^+ such that $I_A^+ = E_\theta(I_A | C) [P_\theta]$ for all $\theta \in \Theta$. A subfield C is pairwise sufficient with respect to (X, A, P) if, for each A in A and each pair $\theta_1, \theta_2 \in \Theta$, there exists a C -measurable function I_A^* such that $I_A^* = E_{\theta_i}(I_A^* | C) [P_{\theta_i}]$ for $i = 1, 2$. (The function I_A^* may depend on θ_1 and θ_2 .)

3. COHERENT STATISTICAL MODEL.

Let F denote the class of all measurable functions $f: X \rightarrow [0, 1]$ and let

$$s = \{f_\theta: f_\theta \in F, \theta \in \Theta\}$$

be a collection of members of F that is indexed by θ . Let $S = \{s\}$ be the family of all such collections s .

DEFINITION 3. A member $\{f_\theta\}$ of S is said to be pairwise coherent if, for every pair θ_1, θ_2 in Θ , there exists a function f_{12} in F such that $f_{\theta_i} = f_{12} [P_{\theta_i}]$ for $i = 1, 2$.

DEFINITION 4. A member $\{f_\theta\}$ of S is said to be countably coherent if, for every countable subfamily $\Theta_0 = \{\theta_1, \theta_2, \dots\}$ of Θ , there exists a function f_0 in F such that $f_{\theta_i} = f_0 [P_{\theta_i}]$ for all $i = 1, 2, \dots$.

DEFINITION 5. A member $\{f_\theta\}$ of S is said to be coherent if there exists a function f in F such that $f_\theta = f [P_\theta]$ for all $\theta \in \Theta$.

DEFINITION 6. (Hasegawa and Perlman, [6]). The statistical model (X, A, P) is said to be coherent if every countably coherent member of S is coherent.

In the following lemma, we show that the notions of pairwise coherence and countable coherence do coincide.

LEMMA 1. If $\{f_\theta\}$ is a pairwise coherent member of S , then it is countably coherent.

PROOF. Choose and fix a countable subfamily $\Theta_0 = \{\theta_1, \theta_2, \dots\}$ of Θ . For each pair θ_i, θ_j in Θ_0 , there exists a function f_{ij} in F such that

$$f_{ij} = f_{\theta_i} [P_{\theta_i}] \text{ and } f_{ij} = f_{\theta_j} [P_{\theta_j}].$$

Let

$$g_i = \sup_j f_{ij}, h_j = \inf_i f_{ij}, \text{ and } f = \inf_i \sup_j f_{ij}.$$

For each fixed i , the functions f_{i1}, f_{i2}, \dots are P_{θ_i} -equivalent to f_{θ_i} . The supremum of a countable number of P_{θ_i} -equivalent functions is also P_{θ_i} -equivalent to those functions. Thus, $g_i = f_{\theta_i} [P_{\theta_i}]$. Likewise, for each fixed j , we have $h_j = f_{\theta_j} [P_{\theta_j}]$. Therefore, it follows that $g_n = h_n = f_{\theta_n} [P_{\theta_n}]$ for $n = 1, 2, \dots$. Observe that $h_n \leq f \leq g_n$ for all n . It then follows that

$$f = g_n = h_n = f_{\theta_n} [P_{\theta_n}] \text{ for } n = 1, 2, \dots$$

Hence, $\{f_\theta\}$ is countably coherent.

In general, the notions of pairwise coherence and coherence do not coincide as the following example shows.

EXAMPLE 1. (Pitcher, [10]). Let X be the unit interval $[0, 1]$, A the σ -field of Borel subsets of X , and P the family of all probability measures on A which are either degenerate at a single point of X or else are absolutely continuous with respect to the Lebesgue measure. For each P_θ in P and each $x \in X$, define $f_\theta(x) = P_\theta(\{x\})$. Then $\{f_\theta\}$ is pairwise coherent, but not coherent.

In this model, no proper subfield of A can be sufficient. To see this, let C be an arbitrary sufficient subfield and let P_x be the probability measure degenerate at x . Then it follows from the sufficiency of C that, for all A in

A, there exists a C-measurable function I_A^\dagger such that

$$I_A^\dagger(x) = \int_X I_A^\dagger dP_x = \int_X I_A dP_x = I_A(x).$$

Let $C = \{x: I_A^\dagger(x) = 1\}$. Then $A = C \in C$. Hence, $C = A$.

We now show that the model (X, A, P) is coherent if it is either dominated or discrete.

LEMMA 2. (Hasegawa and Perlman, [6]). If (X, A, P) is dominated, then it is coherent.

PROOF. Since P is dominated, it follows that there is a countable subfamily $\Theta_0 = \{\theta_1, \theta_2, \dots\}$ of Θ such that $P_0 = \{P_\theta: \theta \in \Theta_0\}$ is equivalent to P . Suppose that $\{f_\theta\}$ is countably coherent. Then there exists a function $f_0 \in F$ such that $f_0 = f_\theta [P_\theta]$ for $\theta \in \Theta_0$. We now show that $f_0 = f_\theta [P_\theta]$ for all $\theta \in \Theta$, so $\{f_\theta\}$ is coherent. For each $\theta \in \Theta$, consider $\Theta_1 = \Theta_0 \cup \{\theta\}$. Then there exists a function $f_1 \in F$ such that $f_1 = f_\theta [P_\theta]$ for $\theta \in \Theta_1$. Thus, $f_0 = f_1 [P_0]$ and so $f_0 = f_1 [P]$. Hence, $f_0 = f_1 = f_\theta [P_\theta]$ as required.

LEMMA 3. If (X, A, P) is discrete, then it is coherent.

PROOF. Let $\{f_\theta\}$ be pairwise coherent. We must show that it is also coherent. For each $\theta \in \Theta$, let $S_\theta = \{x \in X: P_\theta(\{x\}) > 0\}$ denote the countable support of P_θ . For each pair θ_1, θ_2 in Θ , there exists a function f_{12} in F such that $f_{12} = f_{\theta_i} [P_{\theta_i}]$ for $i = 1, 2$. Thus, we have

$$f_{\theta_i}(x) = f_{12}(x) \text{ for all } x \in S_{\theta_i}, i = 1, 2. \tag{1}$$

Hence, $f_{\theta_1}(x) = f_{\theta_2}(x)$ for all $x \in S_{\theta_1} \cap S_{\theta_2}$. Now, choose and fix $x \in X$. Let $\Theta_x = \{\theta \in \Theta: x \in S_\theta\}$. Let θ_0 be a member of Θ_x . Note that $x \in S_{\theta_0} \cap S_\theta$ for all $\theta \in \Theta_x$. In view of (1), we have $f_{\theta_0}(x) = f_\theta(x)$ for all $\theta \in \Theta_x$, that is, $f_\theta(x)$ is constant in $\theta \in \Theta_x$ (for this prefixed x). Let c_x be the common value of f_θ for $\theta \in \Theta_x$, evaluated at x , that is, $f_\theta(x) = c_x$ for all $x \in X$. Since x is arbitrary, define a function f by $f(x) = c_x$ for all $x \in X$. Clearly, $f = f_\theta [P_\theta]$ for all $\theta \in \Theta$ as required.

That the coherent case is not exhausted by the dominated and discrete cases is shown in the following example.

EXAMPLE 2. Let (X_1, A_1, P_1) be a non-discrete dominated model, let (X_2, A_2, P_2) be an undominated discrete model, where X_1 and X_2 are disjoint, $P_1 = \{P_\theta: \theta \in \Theta_1\}$, and $P_2 = \{P_\theta: \theta \in \Theta_2\}$. Let $X = X_1 \cup X_2$, $A = \{A \subset X: A \cap X_i \in A_i \text{ for } i = 1, 2\}$, and extend P_1 and P_2 to A by defining

$$P_\theta(A) = P_\theta(A \cap X_i) \text{ for all } \theta \in \Theta_i, i = 1, 2.$$

It follows from Lemmas 2 and 3 that both (X_1, A_1, P_1) and (X_2, A_2, P_2) are coherent. Now we claim that (X, A, P) is also coherent, where $P = P_1 \cup P_2 = \{P_\theta: \theta \in \Theta_1 \cup \Theta_2 = \Theta\}$. Suppose that $\{f_\theta: \theta \in \Theta\}$ is pairwise coherent with respect to (X, A, P) . Each f_θ , being an A -measurable function, can be written as

$$f_\theta = \begin{cases} f_\theta^1 & \text{on } X_1 \\ f_\theta^2 & \text{on } X_2, \end{cases}$$

where f_θ^i is A_i -measurable for $i = 1, 2$. Since $\{f_\theta^i: \theta \in \Theta_i\}$, being pairwise coherent with respect to (X_i, A_i, P_i) , is coherent, it follows that there exists an A_i -measurable function $0 \leq f_i \leq 1$ such that

$$f_i = f_\theta^i [P_\theta] \text{ for all } \theta \in \Theta_i.$$

Define

$$f = \begin{cases} f_1 & \text{on } X_1 \\ f_2 & \text{on } X_2 \end{cases}$$

Clearly, f is A -measurable. Observe that $f_\theta = f_\theta^i [P_\theta]$ for all $\theta \in \Theta_i, i = 1, 2$. Therefore, $f = f_\theta [P_\theta]$ for all $\theta \in \Theta$. However, it should be noted that (X, A, P) is neither dominated nor discrete.

4. SUFFICIENCY IN THE COHERENT CASE.

For each $\theta \in \Theta$, let N_θ denote the class of all P_θ -null sets, that is, $N_\theta = \{N \in A: P_\theta(N) = 0\}$. For each subfield C of A , define

$$\tilde{C} = \bigcap_{\theta \in \Theta} [C \vee \sigma(N_\theta)].$$

Then \tilde{C} is also a subfield of A . It is easy to see that a function f is \tilde{C} -measurable

if and only if, for each $\theta \in \Theta$, there exists a C -measurable function f_θ such that $f = f_\theta[P_\theta]$. Clearly, $C \subset \bar{C} \subset \tilde{C}$.

Let (X, A, P) be a coherent model. Then, so is (X, A, P_θ) for any subfamily P_θ of P . However, it is not true that (X, C, P) is coherent for every subfield C of A . Such an example is explicitly included in Pitcher's [9] example.

EXAMPLE 3. Let X be the real line and let \mathcal{B} be the σ -field of all Borel subsets of X . Choose and fix a non-empty, non-Borel set E that excludes the origin but is symmetric about the origin, i.e., $E = -E = \{x: -x \in E\}$. Let Θ be also the real line, and define a family $P = \{P_\theta: \theta \in \Theta\}$ of probability measures as follows: If $\theta \in E$, then P_θ is the discrete measure allotting probabilities $1/2$ and $1/2$ to the two points $-\theta$ and θ . If $\theta \notin E$, then P_θ is degenerate at θ . We claim that (X, \mathcal{B}, P) cannot be coherent. To see this, for each $\theta \in \Theta$ and each $x \in X$, define

$$f_\theta(x) = P_\theta(\{x\}).$$

Then $\{f_\theta: \theta \in \Theta\}$ is pairwise coherent with respect to (X, \mathcal{B}, P) . Suppose, on the contrary, that $\{f_\theta: \theta \in \Theta\}$ is coherent. Then there exists a \mathcal{B} -measurable function $0 \leq f \leq 1$ such that $f = f_\theta[P_\theta]$ for all $\theta \in \Theta$. This implies that

$$f(x) = \begin{cases} 1/2 & \text{if } x \in E \\ 1 & \text{if } x \notin E, \end{cases}$$

which obviously contradicts the initial supposition that $E \notin \mathcal{B}$. However, if we consider the class A of all subsets of X , then (X, A, P) is a discrete model and hence is coherent.

In the following lemma, however, we show that if (X, A, P) is a coherent model, then so is (X, \tilde{C}, P) for any subfield C of A .

LEMMA 4. If (X, A, P) is coherent and C is a subfield of A , then (X, \tilde{C}, P) is coherent.

PROOF. Let $\{f_\theta: \theta \in \Theta\}$ be pairwise coherent with respect to (X, \tilde{C}, P) . Then it is also pairwise coherent with respect to (X, A, P) . Since (X, A, P) is coherent, there exists an A -measurable function $0 \leq f \leq 1$ such that $f = f_\theta[P_\theta]$ for all $\theta \in \Theta$. Since each f_θ is \tilde{C} -measurable, it is seen that f must be

measurable with respect to $\tilde{C} = \tilde{C}$. This proves that (X, \tilde{C}, P) is coherent.

Let \tilde{P} denote the convex hull of P , that is, $\tilde{P} = \{Q: Q = \sum a_i P_{\theta_i}, a_i \geq 0, \sum a_i = 1, \theta_i \in \Theta\}$.

LEMMA 5. (X, A, P) is coherent if and only if (X, A, \tilde{P}) is coherent.

PROOF. The "only if" part is what needs to be proved. Let $\{f_Q: Q \in \tilde{P}\}$ be pairwise coherent with respect to (X, A, \tilde{P}) . Then the subset $\{f_\theta: \theta \in \Theta\}$ of $\{f_Q: Q \in \tilde{P}\}$ is pairwise coherent with respect to (X, A, P) . Since (X, A, P) is coherent, there exists an A -measurable function $0 \leq f \leq 1$ such that $f = f_\theta[P_\theta]$ for all $\theta \in \Theta$. We now claim that $f = f_Q[Q]$ for all $Q \in \tilde{P}$. Choose and fix $Q \in \tilde{P}$. Then $Q = \sum a_i P_{\theta_i}$, where $\theta_i \in \Theta$, $a_i \geq 0$, and $\sum a_i = 1$. For each pair Q, P_{θ_i} , there exists an A -measurable function $0 \leq f_i \leq 1$ such that

$$f_i = f_Q[Q]$$

and

$$f_i = f_{\theta_i} = f[P_{\theta_i}].$$

Since $\{P_{\theta_i} : i = 1, 2, \dots\} \equiv Q$, it follows that

$$f = f_i = f_Q[P_{\theta_i}] \text{ for } i = 1, 2, \dots$$

Hence, $f = f_Q[Q]$ as required.

LEMMA 6. Let C be a subfield of A such that (X, C, P) is coherent. If P is closed for countable convex combinations (i.e., $P = \tilde{P}$), then $\bar{C} = \tilde{C}$.

PROOF. Since $\bar{C} \subset \tilde{C}$, it suffices to show that $\tilde{C} \subset \bar{C}$. Let f be a \tilde{C} -measurable function such that $0 \leq f \leq 1$. Then, for each $\theta \in \Theta$, there exists a C -measurable function $0 \leq f_\theta \leq 1$ such that $f = f_\theta[P_\theta]$. For each pair $\theta_1, \theta_2 \in \Theta$, let $Q = (P_{\theta_1} + P_{\theta_2})/2$. Then $Q \in \tilde{P} = P$ and so $f = f_Q[Q]$. Since $\{P_{\theta_1}, P_{\theta_2}\} \equiv Q$, we have $f_Q = f = f_{\theta_i}[P_{\theta_i}]$ for $i = 1, 2$. Thus, we have shown that $\{f_\theta: \theta \in \Theta\}$ is pairwise coherent with respect to (X, C, P) . Therefore, there exists a C -measurable function $0 \leq f_0 \leq 1$ such that $f_0 = f_\theta[P_\theta]$ for all $\theta \in \Theta$, and hence $f = f_0[P]$. This shows that $\tilde{C} \subset \bar{C}$.

LEMMA 7. (X, C, P) is coherent if and only if (X, \bar{C}, P) is coherent.

PROOF. Let us first prove the "only if" part. Let $\{f_\theta: \theta \in \Theta\}$ be pairwise coherent with respect to (X, \bar{C}, P) . For each $\theta \in \Theta$, there exists a C -measurable function $0 \leq g_\theta \leq 1$ such that $f_\theta = g_\theta[P_\theta]$. For each pair $\theta_1, \theta_2 \in \Theta$, there exists a \bar{C} -measurable function $0 \leq f_{12} \leq 1$ such that $f_{12} = f_{\theta_i}[P_{\theta_i}]$ for $i = 1, 2$. Since there is a C -measurable function $0 \leq g_{12} \leq 1$ such that $f_{12} = g_{12}[P]$, it follows that $g_{12} = g_{\theta_i}[P_{\theta_i}]$ for $i = 1, 2$. Hence, $\{g_\theta: \theta \in \Theta\}$ is pairwise coherent with respect to (X, C, P) . Since (X, C, P) is coherent, there exists a C -measurable function $0 \leq f \leq 1$ such that $f = g_\theta = f_\theta[P_\theta]$ for all $\theta \in \Theta$. Therefore, (X, \bar{C}, P) is coherent.

To prove the "if" part, let $\{f_\theta: \theta \in \Theta\}$ be pairwise coherent with respect to (X, C, P) . Then, it is also pairwise coherent with respect to (X, \bar{C}, P) . Thus, there exists a \bar{C} -measurable function $0 \leq \bar{f} \leq 1$ such that $\bar{f} = f_\theta[P_\theta]$ for all $\theta \in \Theta$. Since there is a C -measurable function $0 \leq f \leq 1$ such that $f = \bar{f}[P]$, we have $f = f_\theta[P_\theta]$ for all $\theta \in \Theta$ as required.

Under the assumption of coherence, we now prove a number of results on sufficiency.

PROPOSITION 1. Suppose that (X, C, P) is coherent. If C is pairwise sufficient, then C is sufficient.

PROOF. Choose and fix $A \in \mathcal{A}$. For each $\theta \in \Theta$, let $0 \leq f_\theta \leq 1$ be a version of $E_\theta(I_A | C)$. Since C is pairwise sufficient, for each pair $\theta_1, \theta_2 \in \Theta$, there exists a C -measurable function $0 \leq f_{12} \leq 1$ such that $f_{12} = f_{\theta_i}[P_{\theta_i}]$ for $i = 1, 2$. Thus, $\{f_\theta: \theta \in \Theta\}$ is pairwise coherent with respect to (X, C, P) . Since (X, C, P) is coherent, there exists a C -measurable function $0 \leq f \leq 1$ such that $f = f_\theta[P_\theta]$ for all $\theta \in \Theta$. That is, C is sufficient.

COROLLARY 1. Suppose that (X, A, P) is coherent and $\bar{C} = \tilde{C}$. If C is pairwise sufficient, then C is sufficient.

PROOF. Since (X, A, P) is coherent, it follows from Lemma 4 that (X, \tilde{C}, P) is coherent. Since $\bar{C} = \tilde{C}$, (X, \bar{C}, P) is coherent. In view of Lemma 7, (X, C, P) is coherent. Consequently, the result follows from Proposition 1.

COROLLARY 2. Let $C \subset \mathcal{D}[P_{\theta_1}, P_{\theta_2}]$ for all pairs $\theta_1, \theta_2 \in \Theta$. If C is pairwise sufficient and (X, \mathcal{D}, P) is coherent, then \mathcal{D} is sufficient.

PROOF. Since C is pairwise sufficient and $C \subset \mathcal{D}[P_{\theta_1}, P_{\theta_2}]$ for all $\theta_1, \theta_2 \in \Theta$, it is easy to verify that \mathcal{D} is also pairwise sufficient. Since (X, \mathcal{D}, P) is coherent, it follows from Proposition 1 that \mathcal{D} is sufficient.

COROLLARY 3. Suppose that (X, A, P) is coherent and C is sufficient. If $C \subset \mathcal{D}[P_{\theta_1}, P_{\theta_2}]$ for all $\theta_1, \theta_2 \in \Theta$ and $\bar{\mathcal{D}} = \tilde{\mathcal{D}}$, then \mathcal{D} is sufficient.

PROOF. Since (X, A, P) is coherent and $\bar{\mathcal{D}} = \tilde{\mathcal{D}}$, it follows from Lemmas 4 and 7 that (X, \mathcal{D}, P) is coherent. In view of Corollary 2, we therefore conclude that \mathcal{D} is sufficient.

We now show that if (X, A, P) is coherent, then so is (X, C, P) for any sufficient subfield C . To this end, we shall need the following lemma.

LEMMA 8. (Pitcher, 1965). If C is sufficient, then $\bar{C} = \tilde{C}$.

PROOF. Let $A \in \tilde{C}$. Since C is sufficient, there exists a C -measurable function I_A^\dagger such that $I_A^\dagger = E_\theta(I_A | C) [P_\theta]$ for all $\theta \in \Theta$. Note that $I_A - I_A^\dagger$ is \tilde{C} -measurable. Thus, for each $\theta \in \Theta$, there exists a C -measurable function f_θ such that $I_A - I_A^\dagger = f_\theta [P_\theta]$ and so

$$\begin{aligned} \int_X (I_A - I_A^\dagger)^2 dP_\theta &= \int_X (I_A - I_A^\dagger) f_\theta dP_\theta \\ &= \int_X E_\theta[(I_A - I_A^\dagger) f_\theta | C] dP_\theta \\ &= \int_X f_\theta E_\theta(I_A - I_A^\dagger | C) dP \\ &= 0. \end{aligned}$$

Hence, $I_A = I_A^\dagger [P_\theta]$ for all $\theta \in \Theta$. Since I_A^\dagger is C -measurable, it follows that $A \in \bar{C}$. Therefore, $\bar{C} = \tilde{C}$.

PROPOSITION 2. If (X, A, P) is coherent and C is sufficient, then (X, C, P) is coherent.

PROOF. Since C is sufficient, $\bar{C} = \tilde{C}$ in view of Lemma 8. Since (X, A, P) is coherent, it follows from Lemma 4 and 7 that (X, C, P) is coherent.

5. BASU'S THEOREM.

As before, (X, A, P) is our basic model, where $P = \{P_\theta: \theta \in \Theta\}$. Two

probability measures P_{θ_1} and P_{θ_2} in \mathcal{P} are said to be overlapping if, for any set A in \mathcal{A} , $P_{\theta_1}(A) = 1$ implies that $P_{\theta_2}(A) > 0$. We write $\theta_1 \Leftrightarrow \theta_2$ if P_{θ_1} and P_{θ_2} overlap. If there exists a finite number of parameter points $\theta_1, \theta_2, \dots, \theta_k$ such that

$$\theta_1 \Leftrightarrow \theta_2 \Leftrightarrow \dots \Leftrightarrow \theta_k \Leftrightarrow \theta',$$

then we say P_θ and $P_{\theta'}$ are connected. The family \mathcal{P} is called connected if every pair of probability measures in the family are connected.

THEOREM 1. (Basu, [2]). If T is a sufficient statistic, \mathcal{P} is connected, and V is a statistic which is independent of T for all $\theta \in \Theta$, then the distribution of V does not depend on θ .

A set A in \mathcal{A} is called splitting set if there is a partition Θ_0, Θ_1 , of Θ such that

$$P_\theta(A) = \begin{cases} 0 & \text{if } \theta \in \Theta_0 \\ 1 & \text{if } \theta \in \Theta_1 \end{cases}$$

The above theorem has been generalized by Koehn and Thomas [7] as follows:

THEOREM 2. Let T be a sufficient statistic. There exists a statistic V , independent of T for all $\theta \in \Theta$, whose distribution depends on θ if and only if there exists a splitting set.

We now demonstrate that the two notions of connectedness of \mathcal{P} and the nonexistence of a splitting set are equivalent in the coherent case.

LEMMA 9. Let $(X, \mathcal{A}, \mathcal{P})$ be coherent. Then \mathcal{P} is connected if and only if there exists no splitting set.

PROOF. Clearly, the connectedness of \mathcal{P} implies the nonexistence of a splitting set. For each $\theta \in \Theta$, let A_θ be a set such that $P_\theta(A_\theta) = 1$ and $P_\theta(B) > 0$ if $\phi \neq B \subset A_\theta$. Suppose that \mathcal{P} is not connected. Choose and fix $\theta_0 \in \Theta$. Let $\Theta_0 = \{\theta \in \Theta: P_\theta \text{ and } P_{\theta_0} \text{ are connected}\}$. Since \mathcal{P} is not connected, $\Theta_1 = \Theta \setminus \Theta_0$ is not empty. For each $\theta \in \Theta$, let $f_\theta = I_{A_\theta}$. It is easy to show that $\{f_\theta: \theta \in \Theta\}$ is pairwise coherent. Since $(X, \mathcal{A}, \mathcal{P})$ is coherent, there exists an \mathcal{A} -measurable

function $0 \leq f \leq 1$ such that $I_{A_\theta} = f[P_\theta]$ for all $\theta \in \Theta$. Let $A = \bigcup_{\theta \in \Theta_0} A_\theta$. Since A_θ is the support of P_θ , it is easily seen that

$$P_\theta(A) = P_\theta(A_\theta) = 1 \text{ for all } \theta \in \Theta_0.$$

Note that $A \cap A_\theta = \phi$ if $\theta \in \Theta_1$. Thus,

$$P_\theta(A) = 0 \text{ for all } \theta \in \Theta_1.$$

That nonexistence of a splitting set is weaker than the connectedness property is seen from the following example.

EXAMPLE 4. Let \mathcal{P} be a family consisting of all two-point distributions and the standard normal distribution. Then \mathcal{P} is not connected and this does not possess a splitting set.

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