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INTEGRAL OPERATORS IN THE THEORY OF INDUCED BANACH REPRESENTATIONS II. THE BUNDLE APPROACH

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ABSTRACT. Let G be a locally compact group, H a closed subgroup and L a Banach representation of H. Suppose U is a Banach representation of G which is induced by L. Here, we continue our program of showing that certain operators of the integrated form of U can be written as integral operators with continuous kernels. Specifically, we show that: (1) the representation space of a Banach bundle; (2) the above operators become integral operators on this space with kernels which are continuous cross-sections of an associated kernel bundle.

KEY WORDS AND PHRASES. *Locally compact group, Banach representation, induced representation, integrated form, integral operator, vector field, cross-section, continuity structure, Banach bundle, quotient topology, kernel bundle.*

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1. INTRODUCTION.

Let G be a locally compact group with right Haar measure dx, H a closed subgroup of G with right Haar measure dt, and $\pi: G \to X$, the canonical projection onto the right coset space X = G/H. Suppose L is a (strongly continuous) representation of H on the Banach space E. Suppose also that U is a representation of G on a certain Banach space F_U which is induced by L in the sense of [1, sec. 3]. The integrated form of U is the representation of $L^1(G)$ determined by the bounded operators $U(\varphi)$ on F_U, where

$$U(\varphi) = \int_G \varphi(x) U(x) dx ,$$

for φ a continuous function on G with compact support, i.e. $\varphi \in C_c(G)$. It is well-known [2] that these operators can be written as integral operators with continuous kernels in the following sense: for f in a certain dense subspace of F_U, we have

$$U(\varphi)f(x) = \int_X I_\varphi(x,y)f(y)d(\pi(y)) , \quad x \in G ,$$

where $d(\pi(y))$ is a quasi-invariant measure on X and I_φ is a continuous mapping of $G \times G$ into the bounded operators $\mathrm{Hom}(E)$ on E. The existence of the integral is determined by the fact that for each x in G, the mapping $I_\varphi(x,\cdot)f(\cdot)$ is constant on cosets. Thus, the kernel I_φ for $U(\varphi)$ is defined on $G \times G$, while the integration is over X. Moreover, the mapping I_φ is not constant on cosets in general.

Our primary objective in [2] was to represent the operators $U(\varphi)$ as integral operators with continuous kernels in a consistent fashion, i.e. where the kernels and integration are defined over the same space. There are two canonical choices for this space -- namely G and X. In Chapter II of [2] we acomplished our objective over each of these spaces.

In section 3 of [2], we constructed a representation V of G on a Banach function space F_V which was isometrically equivalent to U. In particular, each operator $V(\varphi)$ was written in the following form: for each continuous g in a certain dense subspace of F_V, we have

$$V(\varphi)g(x) = \int_G J_\varphi(x,y)g(y)dy , \quad x \in G ,$$

where J_φ is a continuous mapping from $G \times G$ into $\mathrm{Hom}(E)$. Although this result is satisfactory from the consistency viewpoint, there is a significant shortcoming. The space F_V is a proper closed subspace of a vector-valued L^p-space. Thus, many of the important existing results for integral operators cannot be used with this model of the integrated form of U.

In section 4 of [2], we next turned our attention to the quotient space X. We constructed a representation W of G on a continuous sum F_W of Banach spaces $\{E_\xi : \xi \in X\}$ which was also isometrically equivalent to U. Each operator $W(\varphi)$

was written as follows: for each "continuous" vector field h in a certain dense subspace of F_W , we have

$$W(\varphi)h(\xi) = \int_X K_\varphi(\xi,\eta)h(\eta)d\eta \quad , \ \xi \in X \ ,$$

where K_φ is a "continuous" field of bounded operators in $\Pi \mathrm{Hom}(E_\eta,E_\xi)$ and $K_\varphi(\xi,\eta) \in \mathrm{Hom}(E_\eta,E_\xi)$. This model requires the knowledge of continuity structures and L^p-theory for vector fields [3,4], as well as the theory of kernels and integral operators for such Lebesque spaces [4]. (The latter was developed by the author expressly for this purpose.) Since the space F_W is a full L^p-space, this model of U does not have the shortcoming that V has. In fact, it proved to be quite useful in studying certain compactness properties of the integrated form of U [2, Ch. IV]. However, W has two minor shortcomings which are more akin to mathematical utility and aesthetics than to mathematical substance. First, a continuity structure (and its implications) is quite complicated and is a less intuitive object to work with than is the more fundamental and familiar notion of topological continuity. Second, although the kernels K_φ do belong to a continuity structure in $\Pi \mathrm{Hom}(E_\eta,E_\xi)$, this structure partially loses a desirable property in the transition from G to X . (See pp. 25-29 of [2] for a rigorous explanation.)

Since the writing of [2], it has been discovered [5] that the theory of continuity structures is equivalent to the theory of Banach bundles [6]. Moreover, in the bundle context, the appropriate mappings are cross-sections, so that continuity is simply topological continuity. These facts suggest:

(1) The representation W can be reconstructed in the setting of Banach bundles.

(2) The kernels K_φ should be continuous cross-sections for a suitable bundle.

The main objectives of this paper are to show exactly how to accomplish (1) and (2).

Section 2 is devoted to recalling the necessary preliminaries. In sections 3 and 4, we construct the Banach bundles corresponding to the continuity structures in ΠE_ξ and $\Pi \mathrm{Hom}(E_\eta,E_\xi)$ respectively. Finally, in section 5, we show that the kernels K_φ are continuous cross-sections for the bundle of section 4.

2. PRELIMINARIES.

The following is a brief summary of sections 1 and 2 of [2].

Let Δ_G and Δ_H be the modular functions for G and H respectively. The quotient function $\Delta_H/(\Delta_G|H)$ is a continuous homomorphism of H into the positive reals which we shall denote by δ. Let ρ be a continuous, non-negative function on G satisfying $\rho(tx) = \delta(t)\rho(x)$, $t \in H, x \in G$. Also let $d\xi = d(\pi(x))$ be the quasi-invariant measure on X corresponding to ρ.

The representation L of H on E is "Banach inducible up to G" if there exists a pair (p,q) such that $1 \leq p \leq \infty$, $0 < q \leq \infty$ and

$$\delta(t)^{1/q}\|L(t)\| \leq b\delta(t)^{1/p}, \quad t \in H,$$

for some $b \geq 1$. Given such a pair, we can construct [1] an isometric representation of G on a certain Banach function space as follows. Let $C_q(G,L)$ denote the linear space of all continuous mappings $f:G \to E$ such that: (i) f has compact support modulo H, i.e. $\pi(\mathrm{supp}(f))$ is compact. (ii) f is (L,q)-homogeneous, i.e.

$$f(tx) = \delta(t)^{1/q}L(t)f(x), \quad t \in H, x \in G.$$

For a suitable L^p-norm on $C_q(G,L)$ [1, sec. 3], we find that right translation $f \to fx$ by an element x of G is an isometry. Hence, its extension to the completion F_U of $C_q(G,L)$ is also an isometry which we denote by $U(x)$. The resulting mapping U is then an isometric representation of G on F_U which we call the induced Banach representation corresponding to (L,p,q). We refer the reader to [1] for a thorough development of such representations.

In this setting, the operators $U(\varphi), \varphi \in C_c(G)$, may be written as follows:

$$U(\varphi)f(x) = \Delta_G(x)^{-1} \int_X \rho(y)^{-1}\left[\int_H \delta(t)^{q:1}\varphi(x^{-1}ty)L(t)dt\right] f(y)d(\pi(y)),$$

for f in $C_q(G,L)$ (where $q:p = 1/q - 1/p$). Thus, the mapping I_φ referred to in the introduction is given by

$$I_\varphi(x,y) = \Delta_G(x)^{-1}\rho(y)^{-1} \int_H \delta(t)^{q:1}\varphi(x^{-1}ty)L(t)dt, \quad x,y \in G.$$

Before concluding this section, observe that if we let $M = \delta^{q:p}L$, then M is a bounded representation of H on E with bound b. Hence, we may renorm E by defining

$$\|v\|_M = \sup\{\|M(t)v\| : t \in H\}, \quad v \in E.$$

We thus obtain a Banach space E_M which is equivalent to E since

$$\|v\| \leq \|v\|_M \leq b\|v\|, \quad v \in E.$$

The Banach spaces $\mathrm{Hom}(E)$ and $\mathrm{Hom}(E_M)$ are also equivalent with

$$\|T\| \leq \|T\|_M \leq b\|T\|, \quad T \in \mathrm{Hom}(E).$$

The following result will be useful in what follows:

LEMMA 2.1. The mapping $(t,v) \to L(t)v$ of $H \times E$ into E is continuous.

PROOF. This follows from the fact that L is strongly continuous and locally bounded [1,3.3].

It will also be convenient to fix in advance a compact neighborhood Z of the identity e in G for use later on.

3. THE VECTOR FIELD BUNDLE

In order to construct the bundle version of W, we begin as in section 4 of [2].

Consider the space $G \times E$ with equivalence relation $\sim$ defined as follows: if $x,y \in G$ and $v,w \in E$, then $(x,v) \sim (y,w)$ if there exists t in H such that $y = tx$ and $w = L(t)v$. Denote the resulting space of equivalence classes (with quotient topology) by $\mathcal{E}$ and let $\sigma: G \times E \to \mathcal{E}$ be the canonical projection. Also let $\tau : \mathcal{E} \to X$ be the (well-defined) projection given by $\tau(\sigma(x,v)) = \pi(x)$. We then have the following composition:

$$G \times E \xrightarrow{\sigma} \mathcal{E} \xrightarrow{\tau} X.$$

LEMMA 3.1. The mapping σ is continuous and open.

PROOF. The openness of σ follows from the fact that the saturation $\sigma^{-1}(\sigma(A \times B))$ of a basic open subset $A \times B$ of $G \times E$ is of the form

$$\cup \{(tA) \times (L(t)B) : T \in H\} ,$$

which is open in $G \times E$.

LEMMA 3.2. The mapping τ is continuous and open.

PROOF. The continuity and openness of τ follow from the continuity and openness of σ and π .

PROPOSITION 3.3. The space $\mathcal{E}$ is Hausdorff.

PROOF. Let $\{\theta_i\}$ be a net in $\mathcal{E}$ converging to θ and θ' in $\mathcal{E}$, where $\theta = \sigma(x,v)$ and $\theta' = \sigma(x',v')$. Since σ is open, passing to a subnet if necessary, we may assume there exists a corresponding net $\{(x_i,v_i)\}$ in $G \times E$ such that $\sigma(x_i,v_i) = \theta_i$, $i = 1,2,$ and $(x_i,v_i) \to (x,v)$ in $G \times E$. Similarly, we may assume there exists a corresponding net $\{(x_i',v_i')\}$ in $G \times E$ such that $\sigma(x_i',v_i') = \theta_i$ and $(x_i',v_i') \to (x',v')$. Since $\sigma(x_i,v_i) = \sigma(x_i',v_i')$, there exists $t_i \in H$ such that $x_i' = t_i x_i$ and $v_i' = L(t_i)v_i$. Consequently, $t = x'x^{-1}$ is an element of H since the net $\{t_i = x_i' x_i^{-1}\}$ converges to $x'x^{-1}$ and H is closed in G . Moreover, by 1.1 and the triangle inequality, we see that $v' = L(t)v$. Thus, $\theta = \theta'$ and $\mathcal{E}$ is Hausdorff.

The next step is to make $(\mathcal{E},X,\tau)$ into a Banach bundle. As on p. 12 of [2], define $E_\xi = \tau^{-1}(\xi)$, $\xi \in X$. For fixed x in "$\xi = \pi^{-1}(\xi)$" , each element of E_ξ is uniquely of the form $\sigma(x,v)$, $v \in E$. Thus, E_ξ becomes a Banach space equivalent to E and E_M under the following (well-defined) operations:

$$\sigma(x,v) + \sigma(x,w) = \sigma(x,v+w)$$

$$c\sigma(x,v) = \sigma(x,cv)$$

$$\|\sigma(x,v)\| = \rho(x)^{q:p}\|v\|_M , \quad v,w \in E .$$

LEMMA 3.4. For each $x \in G$, $v \in E$,

$$\|\sigma(x,v)\| = \sup\{\rho(y)^{q:p}\|w\| : (y,w) \sim (x,v)\}$$

PROPOSITION 3.5. The bundle $(\mathcal{E},X,\tau)$ is a Banach bundle (as in section 1 of [6]).

Let $S(X,E)$ denote the linear space of cross-sections from X into E, $CS(X,E)$ those that are continuous and $CS_c(X,E)$ the continuous cross-sections with compact support. Analogously, recall [2, sec. 4] that there is a continuity structure Λ in ΠE_ξ given by $\Lambda = \{Sf\}$, where

$$Sf(\pi(x)) = Sf(x) = \sigma(x,\rho(x)^{-1/q}f(x)) \ , \ x \in G \ ,$$

for f in $H_q(G,L) \cap C(G,E)$, where $H_q(G,L)$ is the space of (L,q)-homogeneous functions on G. (In [2], Λ was denoted by Λ_q.) As in sections 1 and 5 of [4], we then have the space $C(\Lambda)$ of Λ-continuous vector fields and the subspace $C_c(\Lambda)$ of compactly supported such fields. Note that $S(X,E) = \Pi E_\xi$, which in turn is in bijective correspondence with $H_q(G,L)$ via the above mapping $f \to Sf$. Also, $C(\Lambda) = \Lambda$ [2, 4.15], so that

$$C(\Lambda) = S(H_q(G,L) \cap C(G,E)) \ .$$

The continuity structure Λ yields a topology (the Λ-topology) on E making (E,X,τ) a Banach bundle having the property that the elements of $C(\Lambda)$ are the continuous cross-sections relative to the Λ-topology [6, Prop. 1.6]. Actually, the space $C(\Lambda)$ is also the space of quotient-continuous cross-sections. [7, sec. 1]

THEOREM 3.6. The quotient and Λ-topologies are the same.

Before proving this theorem, observe that we then have $CS(X,E) = C(\Lambda)$ and $CS_c(X,E) = C_c(\Lambda)$. Furthermore, if μ is the measure $d\xi$ on X, then $L^p(\Lambda,\mu)$ [4, sec. 6] is the same as the space $L^p((E,X,\tau), \mu)$ [6, sec. 2], which is the L^p-completion of $CS_c(X,E)$. However, $L^p(\Lambda,\mu)$ is isometrically isomorphic to the representation space F_U of U and is the representation space F_W of W. Hence, the representation space of the bundle version of W will be $L^p((E,X,\tau), \mu)$.

PROOF OF 3.6. A basic Λ-open subset of E is of the form

$$W(h,A,\varepsilon) = \{\theta \in E : \tau(\theta) \in A \ , \ \|\theta - Sh(\tau(\theta))\| < \varepsilon\} \ ,$$

where $h \in H_q(G,L) \cap C(G,E)$, A is an open subset of X and $\varepsilon > 0$. Let $\theta = \sigma(x,v)$ be an element of $W(h,A,\varepsilon)$, so that $\tau(\theta) = \pi(x) \in A$ and

$$\begin{aligned}\|\theta - Sh(\tau(\theta))\| &= \|\sigma(x,v) - Sh(x)\| \\ &= \|\sigma(x,v - \rho(x)^{-1/q}h(x))\| \\ &= \rho(x)^{q:p}\|v - \rho(x)^{-1/q}h(x)\|_M \\ &< \varepsilon \ .\end{aligned}$$

The function $y \to \rho(y)^{q:p}\|v - \rho(y)^{-1/q}h(y)\|_M$ is continuous on G. Hence, the set N of y in G where this function is less than $\varepsilon/2$ is open in G. Also, $x \in N$, so that $x \in N \cap Zx \cap \pi^{-1}(A)$, which is a neighborhood of x in G. Thus, this set contains an open neighborhood B of x. For

$$a = \max\{\rho(y)^{q:p} : y \in Zx\} \ ,$$

define

$$C = \{w \in E : \|v - w\| < \varepsilon/2ab\} \ .$$

Then C is an open subset of E containing v. Thus, $\sigma(B \times C)$ is a basic quotient-open neighborhood of $\sigma(x,v) = \theta$ which can be shown to be in $W(h,A,\varepsilon)$.

Conversely, let $\sigma(B \times C)$ be a basic quotient-open subset of $\mathcal{E}$, for B open in G and C open in E. Let $\theta \in \sigma(B \times C)$. Then $\theta = \sigma(x,v)$, for some $x \in B$, $v \in C$. There exists $r > 0$ such that the ball $\{w \in E : \|w - v\| < r/a\}$ is contained in C. Choose c sufficiently large so that $c > 2a/m$, where

$$m = \min\{\rho(y)^{q:p} : y \in Zx\} > 0 \ .$$

Since $\{h(y) : h \in C_q(G,L)\}$ is dense in E (and hence in E_M) for each y in G [1, 3.8] and $h \to \rho^{-1/q}h$ is a bijection of $C_q(G,L)$, there exists h in $C_q(G,L)$ such that

$$\|v - \rho(x)^{-1/q}h(x)\|_M < r/ac \ .$$

By the continuity of h, there exists an open neighborhood N of x in G such that $N \subseteq Zx \cap V$ and

$$\|v - \rho(y)^{-1/q}h(y)\|_M < r/ac \ , \quad y \in N \ .$$

Let $A = \pi(N)$ and $\varepsilon = r/c$. Then $W(h,A,\varepsilon)$ is a basic Λ-open neighborhood of $\sigma(x,v) = \theta$ in $\mathcal{E}$ which is contained in $\sigma(B \times C)$. Therefore, the two topologies are the same.

4. THE KERNEL FIELD BUNDLE

Our next objective is to construct another Banach bundle for which the kernel fields K_φ , $\varphi \in C_c(G)$, are continuous cross-sections.

Let $\approx$ be the equivalence relation on $G \times G \times \mathrm{Hom}(E)$ defined by: $(x,y,T) \approx (x',y',T')$ if there exist r,s in H such that $x' = rx$, $y' = sy$ and $T' = L(r)TL(s)^{-1}$. Let $\mathcal{H}$ denote the space of equivalence classes (with quotient topology) and $\alpha : G \times G \times \mathrm{Hom}(E) \to \mathcal{H}$ the canonical projection. Also, let $\beta : \mathcal{H} \to X \times X$ be the (well-defined) projection given by $\beta(\alpha(x,y,T)) = (\pi(x),\pi(y))$, $x,y \in G$, $T \in \mathrm{Hom}(E)$. We then have the following composition:

$$G \times G \times \mathrm{Hom}(E) \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} X \times X .$$

LEMMA 4.1. The mapping α is continuous and open.

PROOF. The openness of α follows from the fact that the saturation $\alpha^{-1}(\alpha(A \times B \times C))$ of a basic open subset $A \times B \times C$ of $G \times G \times \mathrm{Hom}(E)$ is of the form

$$\cup \{rA \times sB \times L(r)CL(s)^{-1} : r,s \in H\} ,$$

which is open in $G \times G \times \mathrm{Hom}(E)$.

LEMMA 4.2. The mapping $\beta : \mathcal{H} \to X \times X$ is continuous and open.

PROOF. To see that β is open, observe that $\beta(A) = (\pi \times \pi)(\alpha^{-1}(A))$ for any subset A of $\mathcal{H}$. The lemma then follows from the facts that α is continuous and $\pi \times \pi$ is open.

PROPOSITION 4.3. The space $\mathcal{H}$ is Hausdorff.

PROOF. Similar to that of 3.3.

The next step is to make $(\mathcal{H}, X \times X, \beta)$ into a Banach bundle. For $\xi,\eta \in X$, define $\mathcal{H}_{\xi,\eta} = \beta^{-1}(\xi,\eta)$. For fixed x in ξ , y in η , each element of $\mathcal{H}_{\xi,\eta}$ is uniquely of the form $\alpha(x,y,T)$, for T in $\mathrm{Hom}(E)$. Define:

$$\alpha(x,y,T) + \alpha(x,y,T') = \alpha(x,y,T + T') ,$$
$$c\alpha(x,y,T) = \alpha(x,y,cT) ,$$

and

$$\|\alpha(x,y,T)\| = (\rho(x)/\rho(y))^{q:p}\|T\|_M , \quad T,T' \in \mathrm{Hom}(E) , \quad c \in \mathbb{C} ,$$

where the norm is well-defined. Under these operations $H_{\xi,\eta}$ is a Banach space. In fact:

THEOREM 4.4. The space $H_{\xi,\eta}$ is isometrically isomorphic to $\mathrm{Hom}(E_\eta,E_\xi)$.

PROOF. Fix $\alpha(x,y,T)$ in $H_{\xi,\eta}$ and define $T':E_\eta \to E_\xi$ by

$$T'(\sigma(y,v)) = \sigma(x,Tv) , \quad v \in E .$$

Then T' is well-defined, linear and $\|T'\| = \|\alpha(x,y,T)\|$ as we shall next verify. We have:

$$\|T'\| = \sup\{\|T'(\alpha(y,v))\| : v \in E , \ \|\alpha(y,v)\| = 1\} ,$$

where $\|\sigma(y,v)\| = \rho(y)^{q:p}\|v\|_M$. Thus, $\|\sigma(y,v)\| = 1$ if and only if $\|v\|_M = \rho(y)^{p:q}$. Also,

$$\begin{aligned}\|T'(\sigma(y,v))\| &= \|\sigma(x,Tv)\| \\ &= \rho(x)^{q:p}\|Tv\|_M \\ &= (\rho(x)/\rho(y))^{q:p}\|T(\rho(y)^{q:p}v)\|_M .\end{aligned}$$

Therefore,

$$\begin{aligned}\|T'\| &= \sup\{\|T'(\sigma(y,v))\| : v \in E , \ \|v\|_M = \rho(y)^{p:q}\} \\ &= \sup\{(\rho(x)/\rho(y))^{q:p}\|T(\rho(y)^{q:p}v)\|_M : v \in E , \ \|v\|_M = \rho(y)^{p:q}\} \\ &= \sup\{(\rho(x)/\rho(y))^{q:p}\|Tw\|_M : w \in E , \ \|w\|_M = 1\} \\ &= (\rho(x)/\rho(y))^{q:p}\|T\|_M \\ &= \|\alpha(x,y,T)\| .\end{aligned}$$

Hence, $T' \in \mathrm{Hom}(E_\eta,E_\xi)$ and we have a mapping $\alpha(x,y,T) \to T'$ of $H_{\xi,\eta}$ into $\mathrm{Hom}(E_\eta,E_\xi)$, which is clearly a linear isometry. We will be done once we show that this mapping is onto.

Recall that E, E_M and the E_ξ, $\xi \in X$, are all equivalent Banach spaces. Thus, the same is true of $\mathrm{Hom}(E)$, $\mathrm{Hom}(E_M)$ and the $\mathrm{Hom}(E_\eta, E_\xi)$, $\xi, \eta \in X$. In particular, if $T \in \mathrm{Hom}(E)$, then the corresponding element of $\mathrm{Hom}(E_\eta, E_\xi)$ is the operator T' which is the image of $\alpha(x,y,T)$ under our identification, $x \in \xi$, $y \in \eta$.

Consequently, modulo the isomorphisms, the field $\{\mathrm{Hom}(E_\eta, E_\xi) : \xi, \eta \in X\}$ of Banach spaces is the same as the field $\{\mathcal{H}_{\xi,\eta} : \xi, \eta \in X\}$.

REMARK 4.5. Recall that in 3.4 we verified that

$$\|\sigma(x,v)\| = \sup\{\rho(y)^{q:p}\|w\| : (x,v) \sim (y,w)\} .$$

The analogous question here is: does

$$\|\alpha(x,y,T)\| = \sup\{(\rho(x')/\rho(y'))^{q:p}\|T'\| : (x,y,T) \approx (x',y',T')\} \ ?$$

The answer depends on the question: does

$$\|T\|_M = \sup\{\|M(r)TM(s)^{-1}\| : r,s \in H\} , \quad T \in \mathrm{Hom}(E) \quad ?$$

We believe the answer to both questions is yes; however, we have been able to only partially verify the latter.

PROPOSITION 4.6. The bundle $(\mathcal{H}, X \times X, \beta)$ is a Banach bundle.

As in section 3, we obtain the spaces $S(X \times X, \mathcal{H})$ and $CS(X \times X, \mathcal{H})$ of cross-sections and continuous cross-sections respectively. The space $CS(X \times X, \mathcal{H})$ is then the space of continuous kernels.

The vector field analogue of this space requires the existence of a continuity structure in $\Pi\mathrm{Hom}(E_\eta, E_\xi)$. This was essentially accomplished in section 4 of [2] as follows. Let $H_L(G \times G, \mathrm{Hom}(E))$ denote the linear space of mappings $B: G \times G \to \mathrm{Hom}(E)$ satisfying

$$B(rx,sy) = L(r)B(x,y)L(s)^{-1} , \quad r,s \in H , \quad x,y \in G .$$

For each such B, define $RB(\xi,\eta) : E_\eta \to E_\xi$, $\xi, \eta \in X$, by

$$RB(\xi,\eta)(\sigma(y,v)) = \sigma(x,B(x,y)v) , \quad x \in \xi, y \in \eta , \quad v \in E .$$

Then $RB(\xi,\eta) \in Hom(E_\eta,E_\xi)$, so that $RB \in \Pi Hom(E_\eta,E_\xi)$ and we thus have a mapping

$$R:H_L(G \times G,Hom(E)) \to \Pi Hom(E_\eta,E_\xi) .$$

Note that $RB(\xi,\eta) = B(x,y)'$, which is the element corresponding to $\alpha(x,y,B(x,y))$ under the bijection between $\mathcal{H}_{\xi,\eta}$ and $Hom(E_\eta,E_\xi)$ (4.4).

LEMMA 4.7. The mapping R is a bijection.

PROOF. The onto property is proved using a (possibly non-measurable) cross-section.

Now consider the linear space

$$C_L(G \times G,Hom(E)) = H_L(G \times G,Hom(E)) \cap C(G \times G,Hom(E)) .$$

The set $\Omega = \{RB:B \in C_L(G \times G,Hom(E))\}$ was observed to be a precontinuity structure in $\Pi Hom(E_\eta,E_\xi)$ [2, 4.32]. This means that we were unable to verify that $\{RB(\xi,\eta):RB \in \Omega\}$ is dense in $Hom(E_\eta,E_\xi)$, $\xi,\eta \in X$. If this is not the case, then the Ω-topology on $\mathcal{H}$ is not the same as the quotient topology - the latter is Hausdorff while the former is not. Moreover, our proof in 3.6 (which we would like to duplicate here) required this density property. This is not a significant problem since we can bypass it by simply restricting our attention to the appropriate portion of $\mathcal{H}$. Before doing so, the following result shows to what extent the density property does hold and (more importantly) suggests why it may not hold in general.

THEOREM 4.8. The subspace $\{RB(\xi,\eta) : RB \in \Omega\}$ is strongly dense in $Hom(E_\eta,E_\xi)$, $\xi,\eta \in X$.

PROOF. (After section 3 of [1]). Let φ,ψ be elements of $C_c(G)$ and T an element of Hom(E) . Define

$$F(x,y) = \varphi(x)\psi(y)T , \quad x,y \in G .$$

Then $F:G \times G \to Hom(E)$ is continuous and has compact support. For each x,y in G , consider the mapping

$$(r,s) \to L(r)^{-1}F(rx,sy)L(s)$$

of $H \times H$ into $\mathrm{Hom}(E)$. Since L is strongly continuous, so is this mapping. Hence, for v in E, the mapping

$$(r,s) \to [L(r)^{-1}F(rx,sy)L(s)](v)$$

of $H \times H$ into E is continuous and has compact support. Thus, it defines an element $F^L(x,y)(v)$ of E, where

$$F^L(x,y)(v) = \iint_{H\times H} [L(r)^{-1}F(rx,sy)L(s)](v)drds \quad .$$

The mapping $F^L(x,y)$ is easily seen to be an element of $\mathrm{Hom}(E)$; in fact, $F^L \in H_L(G \times G, \mathrm{Hom}(E))$. Moreover, since φ and ψ are uniformly continuous, we actually have that $F^L \in C_L(G \times G, \mathrm{Hom}(E))$. Unfortunately, however, we are able to verify only that $\{F^L(e,e) : F = \varphi \otimes \psi T\}$ is strongly dense in $\mathrm{Hom}(E)$, using the strong continuity of L. Hence, $\{B(e,e) : B \in C_L(G \times G, \mathrm{Hom}(E))\}$ is strongly dense in $\mathrm{Hom}(E)$. By right translating φ and ψ by x and y respectively, we see that $\{B(x,y) : B \in C_L(G \times G, \mathrm{Hom}(E))\}$ is strongly dense in $\mathrm{Hom}(E)$, $x,y \in G$. The theorem then follows from this fact by the definition of RB, for B as above.

It appears from the previous proof that Ω may not be a complete continuity structure since L is not norm-continuous in general. In any event, as we have indicated above, this difficulty is easily circumvented.

LEMMA 4.9. The space Ω is a continuity structure if and only if $\{B(x,y) : B \in C_L(G \times G, \mathrm{Hom}(E))\}$ is dense in $\mathrm{Hom}(E)$ for some (equivalently) all $(x,y) \in G \times G$.

Let $\mathrm{Hom}_L(E)$ denote the norm closure in $\mathrm{Hom}(E)$ of the span of the set

$$\{B(rx,sy) : r,s \in H, B \in \mathrm{Hom}(E)\}$$

(independent of x,y). Then $G \times G \times \mathrm{Hom}_L(E)$ is a saturated subset of $G \times G \times \mathrm{Hom}(E)$, i.e. it is a union of equivalence classes. Let

$$\mathcal{H}_L = \alpha(G \times G \times \mathrm{Hom}_L(E))$$

and $\beta_L = \beta | H_L$. Then H_L is equipped with the relativized quotient topology of H (or equivalently, the quotient topology of $G \times G \times \mathrm{Hom}_L(E)$) and $(H_L, X \times X, \beta_L)$ is a Banach bundle with

$$\beta_L^{-1}(\xi,\eta) \equiv (H_L)_{\xi,\eta} = H_L \cap H_{\xi,\eta} = H_L \cap \beta^{-1}(\xi,\eta) \ , \quad \xi,\eta \in X \ .$$

Similarly, we let $\mathrm{Hom}_L(E_\eta, E_\xi)$ denote the closed subspace of $\mathrm{Hom}(E_\eta, E_\xi)$ corresponding to $\mathrm{Hom}_L(E)$, so that

$$\mathrm{Hom}_L(E_\eta, E_\xi) \cong (H_L)_{\xi,\eta} \ , \quad \xi,\eta \in X \ .$$

It is clear that Ω is a (complete) continuity structure in $\Pi\mathrm{Hom}_L(E_\eta, E_\xi) \cong H_L$. Consequently, H_L is also equipped with the Ω-topology.

THEOREM 4.10. The quotient and Ω-topologies on H_L are the same.

PROOF. This is proved in the same way as 3.6 now that Ω has the density property.

We have thus verified that the continuous cross-sections of the kernel bundle $(H_L, X \times X, \beta_L)$ correspond exactly to the Ω-continuous kernel fields in $\Pi\mathrm{Hom}_L(E_\eta, E_\xi)$. Of course, if Ω has the desired density property, then the L-subscript may be dropped in 4.10, i.e. $C(\Omega) = CS(X \times X, H)$ - the ideal result.

5. THE CONTINUOUS KERNELS

We are now in a position to complete our project. Recall (section 2) that I_φ is a continuous mapping of $G \times G$ into $\mathrm{Hom}(E)$. For φ in $C_c(G)$, define

$$k_\varphi : G \times G \to G \times G \times \mathrm{Hom}(E)$$

by

$$k_\varphi(x,y) = (x,y,\rho(x)^{-1/q}\rho(y)^{1/q} I_\varphi(x,y)) \ , \quad x,y \in G \ .$$

Then k_φ is continuous and constant on cosets. Hence, it defines a continuous mapping of $X \times X$ into $G \times G \times \mathrm{Hom}(E)$. Furthermore, it follows that the following diagram commutes (modulo isomorphism):

$$\begin{array}{ccc} G \times G & \xrightarrow{k_\varphi} & G \times G \times \mathrm{Hom}(E) \\ {\scriptstyle \pi \times \pi}\downarrow & & \downarrow{\scriptstyle \alpha} \\ X \times X & \xrightarrow[K_\varphi]{} & \mathcal{H} \end{array}$$

where $K_\varphi(\xi,\eta) : E_\eta \to E_\xi$ is given by [2, p.20]

$$K_\varphi(\xi,\eta)(\sigma(y,v)) = \rho(x)^{-1/q}\rho(y)^{1/q}\sigma(x,I_\varphi(x,y)v) ,$$

for $\xi,\eta \in X$, $x \in \xi$, $y \in \eta$, $v \in E$. Hence, the kernel K_φ for the operator $W(\varphi)$ is a continuous cross-section. We had already seen [2, 4.33] that it is an Ω-continuous kernel field. Finally, since $K_\varphi = RB_\varphi$, for $B_\varphi = \rho^{-1/q} \otimes \rho^{1/q} I_\varphi$, we see that $K_\varphi : X \times X \to \mathcal{H}_L$, i.e. $K_\varphi \in \Pi\mathrm{Hom}_L(E_\eta,E_\xi)$, so that the two notions of continuity for K_φ are the same.

In conclusion, let us briefly summarize what we have shown. The induced representation U is isometrically equivalent to a representation W of G on the bundle space $L^p((\mathcal{E},X,\tau) , \mu)$, where $W(x)$ acts on $CS_c(X,\mathcal{E})$ by right translation by x , $x \in G$. The integrated form of W is given by

$$W(\varphi)g(\xi) = \int_X K_\varphi(\xi,\eta)g(\eta)d\eta , \quad \xi \in X , \ g \in CS_c(X,\mathcal{E}) ,$$

where $K_\varphi : X \times X \to \mathcal{H}_L \subseteq \mathcal{H}$ is a continuous kernel for $W(\varphi)$, $\varphi \in C_c(G)$.

Of the four ways we have of realizing the integrated form of the induced representation (U,V, together with the vector field and bundle versions of W) , the last one is the most satisfying and usable.

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