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ON THE PRODUCT OF SELF-ADJOINT OPERATORS

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ABSTRACT. A proof is given for the fact that the product of two self-adjoint operators, one of which is also positive, is again self-adjoint if and only if the product is normal. This theorem applies, in particular, if one operator is an orthogonal projection. In general, the positivity requirement cannot be dropped.

KEY WORDS AND PHRASES. Self-adjoint operators, normal operators, commutativity relations in quantum mechanics.

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1. INTRODUCTION.

Products of self-adjoint operators in Hilbert space play a role in several different areas of pure and applied mathematics. We shall give three examples:

a. In the simplified Hilbert space model of quantum mechanical systems, measurable quantities a,b,... (location, momentum, etc.) are represented by self-adjoint operators ("observables") A,B,... (Mackey [1,2]). The state of the system itself is given by the so-called "statistical operator" W, which is positive with trace (W) = 1 and also named the "density operator" of the system. This probabilistic parlance stems from the intrinsic stochastic nature of quantum mechanics: property a, say, with representing operator A, will be found in the system not with certainty, but with a probability given by

$$P_W(a) = \text{trace }(WA),$$

and by measuring a, the original system changes into a new one whose density or state is given by

$$W' = \frac{AWA}{\text{trace }(WA)}$$

(see Lüders [3], and for a recent discussion, Bub [4]). W' determines the conditional probability of "b given a" via

$$P_W(b|a) = \text{trace } (W'B).$$

If A and W commute, A is called "objective" with respect to W, and WA is a new observable of the system. If A and B commute, AB represents the property "a and b".

b. Every bounded operator T may be written $T = A + iB$ with A and B self-adjoint. If T is already known to be semi-normal, Putnam [5, p. 57] proved that normality and self-adjointedness of AB are the same.

c. Radjavi and Rosenthal [6] proved that the product of a positive and a self-adjoint operator always has a non-trivial invariant subspace. It has not yet been decided whether the product of two self-adjoint operators or, more generally, of a positive and a unitary operator has an invariant subspace (this is the famous "invariant subspace problem").

The starting point for the discussion in the present note is the following theorem (all operators are supposed bounded).

2. MAIN RESULTS.

THEOREM. Let A and B be self-adjoint, and A or B be positive. Then AB is self-adjoint if and only if AB is normal.

PROOF. Of course the "only if" implication is obvious. As to the converse, we use the well-known Fuglede-Putnam theorem [7,8] which states the following: For normal operators N_1 and N_2 and arbitrary operator A, if

$$AN_1 = N_2A \tag{2.1}$$

then

$$AN_1^* = N_2^*A. \tag{2.2}$$

To prove our result, set $N_1 = BA$ and $N_2 = AB = N_1^*$ in (2.1). Then, by (2.2), $A^2B = BA^2$; i.e., A^2 commutes with B. Since A is positive, A is the square root of A^2 and hence A commutes with B. (If B is positive, exchange the roles of A and B).

This theorem characterizes the self-adjoint operators in the class of normal operators; it is known that every self-adjoint operator T can be written in its polar decomposition as a product T = AB with A positive and B unitary. Here B is even self-adjoint, because T is (Rudin [9, p. 315] Proof (b) of Theorem 12.35). Our theorem states the converse: all operators $T = AB$ with A positive and B self-adjoint are already self-adjoint!

COROLLARY 1. Let $T = A + iB$ be a bounded operator in its canonical form with self-adjoint A and B. If AB is normal and A or B is positive, then T is normal.

COROLLARY 2. Let B be self-adjoint and A the orthogonal projection onto a closed subspace M. Then M reduces B; i.e., $BM \subset M$ and $BM^{\perp} \subset M^{\perp}$ if and only if AB is normal.

PROOF. M reduces B iff $AB = BA$, i.e., AB is self-adjoint. Since A is positive, our theorem applies.

COROLLARY 3. Let A and B be orthogonal projections. Then the commutativity relation $AB = BA$ is equivalent to $ABA = BAB$.

PROOF. $AB = BA$ means that AB is self-adjoint, whereas $ABA = BAB$ expresses normality of AB.

The fact that $ABA = BAB$ implies $AB = BA$ may also be seen directly by evaluating $(ABA - AB)^*(ABA - AB) = 0$.

We give now an example showing that the positivity requirement in the theorem cannot be dropped: the self-adjoint matrices

$$A = \begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

fulfill $AB = -BA$, so that AB is normal but not self-adjoint. The reason, according to our theorem, is that neither A nor B are positive. From this we conclude the following weakening of the theorem:

COROLLARY 4. Let A and B be self-adjoint, and A or B be positive. If $AB - BA \neq 0$, then also $AB + BA \neq 0$. (If the commutator of A and B is non-zero, then their anticommutator is also non-zero).

On the other hand, the assumptions of the theorem are not necessary: If B and AB are self-adjoint, it is not necessary that A even be normal: take B as above and

$$A = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}.$$

What about the other partial converse of the theorem? If A is positive and AB self-adjoint, does it follow that B is also self-adjoint? Not in general, but if A is invertible and B is normal: from self-adjointness of AB follows

$$AB = (AB)^* = B^*A, \tag{2.3}$$

therefore, by Fuglede-Putnam,

$$AB^* = BA, \tag{2.4}$$

hence $A^2B = AB^*A = BA^2$, and as above $AB = BA$.

Since A is invertible, we conclude $B = B^*$. We can even do without assumptions on AB (besides (2.3)) if instead of the positivity of A we require the following (Beck-Putnam [10]): If A is invertible, with the polar decomposition

$$A = PU \ (P \geq 0, \ U \text{ unitary}),$$

and if the spectrum of U is contained in some open semi-circle $\{e^{i\mu}: \ \alpha < \mu < \alpha + \mu\}$, then every normal operator B satisfying (2.3) must be self-adjoint. The proof uses the spectral resolution of U and the fact that the set $\{e^{in\lambda}\}$ is complete on the interval $0 \leq \lambda \leq 2\pi$ (for details see [10, p. 214]).

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