Internat. J. Math. & Math. Sci.
Vol. 6 No. 1 (1983) 125-136

FOREST DECOMPOSITIONS OF GRAPHS WITH CYCLOMATIC NUMBER 2

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(Received June 16, 1981 and in revised form June 28, 1982)

ABSTRACT. The tree polynomials [1] of the basic graphs with cyclomatic number 2 are derived. From these polynomials, results about forest decompositions are deduced. Explicit formulae are given for the number of decompositions of the basic graphs into forest with specified finite cardinalities.

KEY WORDS AND PHRASES. Tree polynomial, chain, forest decompositions, circuit, restricted graph, graphs with chains attached, cyclomatic number.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 05C05, 05A10.

1. INTRODUCTION.

The graphs considered here will be finite. Let G be such a graph. With every tree α in G, let us associate an indeterminate or weight w_α. With every spanning forest or cover C of G, let us associate the weight

$$w(C) = \Pi\, w_\alpha,$$

where the product is taken over all the elements of C. Then the tree polynomial of G is $\Sigma w(C)$, where the summation is taken over all the spanning forests in G.

In this paper, we will assign the weight w_n to each tree with n nodes. Therefore, the tree polynomial of G will be a polynomial in the indeterminates $w_1, w_2, w_3, \ldots$ We will denote it by $T(G;\underline{w})$, where $\underline{w} = (w_1, w_2, \ldots)$. If we put $w_i = w$, for all i, then the resulting polynomial, $T(G;w)$, will be called the simple tree polynomial of G. The basic properties of tree polynomials are given in Farrell [1].

Let H be a forest subgraph of G. We say that H is incorporated in G if H is required to belong to every cover of G that we consider. When G contains an incorporated subgraph, we indicate this by writing G^*. We call G^* a restricted graph. By attaching a chain (a tree with nodes of valencies 1 and 2 only) A to graph B, we will mean that identifying of an end node of A with a node of B. If both end nodes are attached to different nodes of B, then we say that the chain A has been added to B, (n.b. B must have at least two nodes).

We will use the term forest decomposition to mean a decomposition into spanning forests. If G has p nodes, q edges and k components, we define the cyclomatic number of G to be $q - p + k$. The basic graphs with cyclomatic number 2 is the minimum set of graphs with cyclomatic number 2 can be obtained from these graphs, by attaching trees to them. We refer the reader to Harary [4] for the basic definitions in Graph Theory.

We will derive the tree polynomials of the basic graphs with cyclomatic number 2, using some of the results for chains and circuits derived in Farrell [2]. The tree polynomials of the basic graphs will then be used to deduce results about forest decompositions of graphs with cyclomatic number 2.

2. SOME FUNDAMENTAL RESULTS.

The covers in G can be partitioned into two classes according to whether or not they contain a specified edge. This leads to the following theorem, which is also given in [1].

THEOREM 1. (The Fundamental Theorem). Let G be a graph (possibly restricted) containing an unincorporated edge xy. Let G' be the graph obtained from G by deleting xy and G* the graph obtained from G by incorporating xy. Then

$$T(G;\underline{w}) = T(G';\underline{w}) + T(G^*;\underline{w}).$$

As indicated in the Introduction, we will assign the same weight to all trees with the same number of nodes; we will also associate an integer to each incorporated subgraph, at the same time shrinking the entire subgraph to a single node which replaces it. This integer equals the number of edges in the subgraph. We will call such a node a compound node and speak of it as representing the incorporated subgraph. A convenient incorporation process would therefore be node identification (as for chromatic polynomials), see Read [5]. We will also omit loops formed during the identification process [5], since a loop could be identified with the final edge needed to complete a circuit in the incorporated subgraph. Thus, by deleting loops formed during identification, we are assured that all our incorporated subgraphs will be trees. We will therefore speak about "incorporated trees". The fundamental algorithm for tree polynomials, or the reduction process, for brevity, consist of repeated applications of Theorem 1, until we obtain graphs H_i for which $T(H_i;\underline{w})$ are known.

By a block we will mean a maximal nonseparable subgraph. The following results are also given in [1].

THEOREM 2. (The Component Theorem). Let G be a graph with components $H_1, H_2, \ldots, H_n$. Then

$$T(G;\underline{w}) = \prod_{i=1}^{n} T(H_i;\underline{w})$$

THEOREM 3. (The Cutnode Theorem). Let G be a graph consisting of n blocks $B_1, B_2, \ldots, B_n$. Then

$$T(G;w) = w^{-(n-1)} \prod_{i=1}^{n} T(B_i;w).$$

3. PRELIMINARY RESULTS.

We will now give some lemmas which will be useful in obtaining our main results. These lemmas were proved in [2]. We will therefore quote the results. The interested reader might wish to consult [2] for detailed proofs.

We will denote the chain with n nodes by P_n, and the circuit with n nodes, by C_n. P(n) will be written for $T(P_n;\underline{w})$. The tree polynomial of the chain P_n can be obtained by application of the reduction process, by successive deletion of the terminal edges incident to a fixed terminal node (and to its subsequent incorporated subchain).

LEMMA 1.

$$P(p) = \sum_{i=1}^{p} w_i \, P(p - i).$$

Since most of our results will be obtained in terms of the tree polynomials of chains, we will give a table of values of $T(P_p;\underline{w})$, for p = 1, up to p = 6. We define $T(P_0;\underline{w})$ to be 1.

Table 1

p	P(p)
1	w_1
2	$w_1^2 + w_2$
3	$w_1^3 + 2w_1w_2 + w_3$
4	$w_1^4 + 3w_1^2w_2 + 2w_1w_3 + w_2^2 + w_4$
5	$w_1^5 + 4w_1^3w_2 + 3w_1^2w_3 + 3w_1w_2^2 + 2w_1w_4 + 2w_2w_3 + w_5$
6	$w_1^6 + 5w_1^4w_2 + 4w_1^3w_3 + 6w_1^2w_2^2 + 3w_1^2w_4 + 6w_1w_2w_3 + 2w_1w_5 + w_2^3 + 2w_2w_4 + w_3^2 + w_6$

We will give the tree polynomials of two types of restricted chains: (i) those in which the compound node is an endnode of the chain, and (ii) those in which the compound node is not an endnode of the chain. The restricted graphs of type (i) will be represented by $P_p^*[r]$, where $p - 1$ is the number of edges in the unincorporated subchain, and r is the number of edges in the incorporated tree. The restricted graphs of type (ii) will be represented by $P_{p+q-1}^*[r]$, where $p - 1$ and $q - 1$ are the number of edges in the unincorporated subchains $(p,q > 0)$ and r is the number of edges in the incorporated tree (Note that the subscripts of P^* are formal and should not be computed). The results can be easily obtained by application of the reduction process.

LEMMA 2.

$$T(P_p^*[r];\underline{w}) = \sum_{i=1}^{p} w_{r+i}\, P(p - i).$$

LEMMA 3.

$$T(P_{p+q-1}^*[r];\underline{w}) = \sum_{s=1}^{p} \sum_{t=1}^{q} w_{r+s+t-1}\, P(p - s)P(q - t).$$

LEMMA 4.

$$T(C_p;\underline{w}) = \sum_{r=1}^{p} rw_r\, P(p - r).$$

PROOF. Apply the reduction to C_p in such a manner that Lemma 2 could be used. The result then follows. □

We will denote by $C_p^*[r]$, the restricted circuit with p nodes, one of which is a compound node representing a tree with r edges.

LEMMA 5.

$$T(C_p^*[r];\underline{w}) = \sum_{i=1}^{p-1} \sum_{j=1}^{p-i} w_{r+i+j}\ P(p - i - j).$$

PROOF. Apply the reduction process in such a manner that Lemma 2 could be used.□

We will denote by H_n, the graph obtained by attaching the chain P_n to a graph G. Let us apply the reduction process to H_n, by deleting the edge of P_n incident to the node of attachment. Then G' will consist of components P_{n-1} and G, and G^* will be $H^*_{n-1}[1]$, i.e. the graph H_{n-1} with a compound node of attachment representing a tree with one edge. By continuing (in a similar manner), the reduction process on $H^*_{n-1}[1]$ and on subsequent graphs $H^*_{n-k}[k]$, we obtain the following result, in which $G^*[r]$ denotes the graph G with a compound node representing a tree with r edges.

LEMMA 6.

$$T(H_n;\underline{w}) = \sum_{k=1}^{n} P(n - k)\ T(G^*[k - 1];\underline{w})\quad (G^*[0] \equiv G).$$

We will use the notation $G^*[r_1, r_2, \ldots, r_k]$, for a restricted graph G containing k compound nodes, representing trees with $r_1,r_2,\ldots,r_k$ edges. We will denote by $H^*_n[r_1,r_2,\ldots,r_k]$ the graph obtained by attaching the chain P_n to $G^*[r_1,r_2,\ldots,r_k]$. The following result can be easily proved.

LEMMA 7. If the node of attachment is a compound node representing r_i edges, then

$$T(H^*_n[r_1,r_2,\ldots,r_k];\underline{w}) = \sum_{s=1}^{n} P(n - s)\ T(G^*[r_1,r_2,\ldots,r_i + s - 1,\ldots,r_k];\underline{w}).$$

If it is an ordinary node, then

$$T(H^*_n[r_1,r_2,\ldots,r_k];\underline{w}) = \sum_{s=1}^{n} P(n - s)\ T(G^*[r_1,r_2,\ldots,r_k, s - 1];\underline{w})$$

Lemma 7 can be used to obtain a result for the graph J_n formed by adding the chain P_n to a graph G.

LEMMA 8.

$$T(J_n;\underline{w}) = \sum_{r=1}^{n-2} \sum_{s=1}^{n-r-1} P(n - r - s - 1)\ T(G^*[r,s-1];\underline{w}) + T(G^*[n-1];\underline{w}).$$

4. TREE POLYNOMIALS OF THE BASIC GRAPHS WITH CYCLOMATIC NUMBER 2.

The basic graphs with cyclomatic number 2 are shown below in Figure 1. The number of nodes in the subgraphs are indicated by p,q and r, where p,q,r > 2.

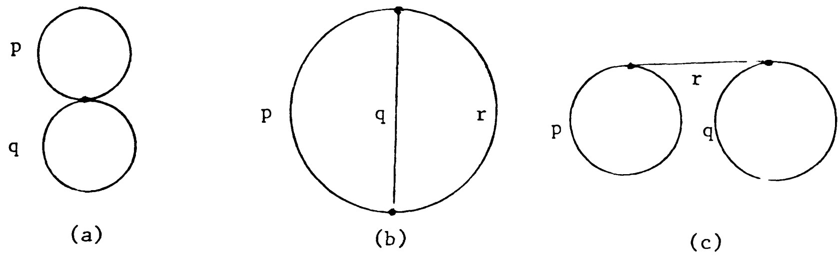

Figure 1

We will denote the graphs shown in (a), (b) and (c) by G(p,q), G(p,q,r) and G((p,q),r) respectively. G(p,q,r) is sometimes called a 'tetha' graph.

THEOREM.

$$T(G(p,q);\underline{w}) = \sum_{s=0}^{p-1} \sum_{m=1}^{p-s} P(p - s - m)[\sum_{i=0}^{q-1} \sum_{j=1}^{q-i} w_{s+m+i+j-1} P(q - i - j)].$$

PROOF. Apply the reduction process to G(p,q) by deleting an edge of the subgraph C_p, incident to the node of valency 4. In this case G' will be H_p and G* will be G*(p - 1,q) [1]. We can apply the reduction process to G*(p - 1,q) [1] in a similar manner, to obtain the graphs $H^*_{p-1}[1]$ and G*(p - 2,q) [2]. By continuing in this manner until a loop is formed in the restricted graph G*, we get

$$T(G(p,q);\underline{w}) = \sum_{i=0}^{p-1} H^*_{p-r}[r],$$

where

$$H^*_p[0] \equiv H_p \quad \text{and} \quad H^*_1[p - 1] \equiv C^*_q[p - 1].$$

The result then follows by using Lemmas 7 and 5.

In order to find the tree polynomial of G(p,q,r) we will establish two lemmas.

LEMMA 9. Let $P^*_{p+q-r}[r,s]$ denote the restricted chain $P^*_{p+q-r}[s]$ with one of its endnodes being a compound node representing a tree with r edges. Then

$$T(P^*_{p+q-1}[r,s];\underline{w}) = \sum_{m=1}^{p-1} \sum_{i=1}^{p-m} \sum_{j=1}^{q-1} w_{r+i}\, w_{s+m+j-1}\, P(p - m - i)\, P(q - j).$$

PROOF. Without loss in generality, we will assume that the compound endnode is on the subchain P_p. We can apply the reduction process to $P^*_{p+q-1}[r,s]$, by deleting the edge of the subchain P_p adjacent to the internal compound node. G' will consist of two components, $P^*_{p-1}[r]$ and $P^*_q[s]$, while G* will be $P^*_{p+q-2}[r,s+1]$. By continuing the reduction process on G* in a similar manner until the entire subchain P_p is incorporated, we get

$$T(P^*_{p+q-1}[r,s];\underline{w}) = \sum_{m=1}^{p-1} P^*_{p-m}[r]\ P^*_q[s + m - 1].$$

The result then follows by using Lemma 2. □

Let us denote by $C^*_{p+q-2}[r,s]$, the restricted (p + q - 2)-gon containing two compound nodes, representing trees with r and s edges, and separated by paths of lengths p - 1 and q - 1.

LEMMA 10.

$$T(C^*_{p+q-2}[r,s];\underline{w}) = \sum_{k=2}^{p} \sum_{m=1}^{p-k+1} \sum_{i=1}^{p-k-m+1} \sum_{j=1}^{q-1} w_{r+k+i-2}\ w_{s+m+j-1}$$
$$\times P(p + k - m - i - 1)\ P(q - j) + \sum_{i=1}^{q-2} \sum_{j=1}^{q-i-1} w_{r+s+p+i+j-1}$$
$$\times P(q - i - j - 1).$$

PROOF. Apply the reduction process to $C^*_{p+q-2}[r,s]$ by deleting an edge incident with the compound node representing the tree with r edges. G' will be $P^*_{p+q-2}[r,s]$. G* will be $C^*_{p+q-3}[r+1,s]$. By continuing in this manner (until the entire path of length p - 1 is incorporated), we get

$$T(C^*_{p+q-2}[r,s];\underline{w}) = \sum_{k=2}^{p} P^*_{p+q-k}[r + k - 2,s] + C^*_{q-1}[r + s + p - 1].$$

The result then follows from Lemmas 9 and 5.

We can view the graph G(p,q,r) as the circuit C_{p+q-2} with the chain P_r added to it. Therefore, Lemma 8 can be immediately applied. In this case, G*[r,s - 1] will be $C^*_{p+q-2}[a,b - 1]$ and G*[n - 1] will be $C^*_{p+q-2}[r - 1]$. Hence we get

$$T(G(p,q,r);\underline{w}) = \sum_{q=1}^{r-2} \sum_{b=1}^{r-q-1} P(r - a - b - 1)\ T(C^*_{p+q-2}[a,b - 1];\underline{w}) +$$
$$T(C^*_{p+q-2}[r - 1];\underline{w}).$$

By using Lemmas 10 and 5, we obtain the following theorem.

THEOREM 5.

$$T(G(p,q,r);\underline{w}) = \sum_{a=1}^{r-2} \sum_{b=1}^{r-a-1} P(r - a - b - 1)[\sum_{k=2}^{p} \sum_{m=1}^{p-k+1} \sum_{i=1}^{p-k-m+1} \sum_{j=1}^{q-1} w_{a+k+i-2}\, w_{b+m+j-2}\, P(p + k - m - i - 1)\, P(q - j)$$

$$+ \sum_{i=1}^{q-2} \sum_{j=1}^{q-i-1} w_{a+b+p+i+j-2}\, P(q - i - j - 1)$$

$$+ \sum_{i=1}^{p+q-3} \sum_{j=1}^{p+q-i-2} w_{r+i+j-1}\, P(p + q - i - j - 2).$$

The following corollary of Theorem 4 will be useful in finding the tree polynomial of G((p,q),r). Its proof is straightforward.

COROLLARY 4.1. Let G*(p,q)[r] be the restricted graph consisting of the graph G(p,q) with its node of valency 4 being a compound node, representing a tree with r edges. Then

$$T(G^*(p,q)[r];\underline{w}) = \sum_{s=0}^{p-1} \sum_{m=1}^{p-s} P(p - s - m)[\sum_{i=0}^{q-1} \sum_{j=1}^{q-i} w_{r+s+m+i+j-2}\, P(q - i - j)].$$

We can obtain the tree polynomial of G((p,q),r) by using Lemma 8. In this case, G will be the graph with two components C_p and C_q, and the added chain will be P_r. In the lemma, G*[r,s-1] will consist of two components $C_p^*[r]$ and $C_q^*[s-1]$, and G*[n-1] will be the graph G((p,q),r) with the entire path P_r incorporated. Notice that if we shrink this incorporated path to a node, then G*[n-1] will become G*(p,q)[r-1]. By using Lemmas 8 and 5, and Corollary 4.1, we get the following result.

THEOREM 6.

$$T(G((p,q),r);\underline{w}) = \sum_{k=1}^{r-2} \sum_{s=1}^{r-k-1} P(r - k - s - 1)[\sum_{i=1}^{p-1} \sum_{j=1}^{p-i} w_{k+i+j}\, P(p - i - j)]$$

$$[\sum_{i=1}^{q-1} \sum_{j=1}^{q-i} w_{s+i+j-1}\, P(q - i - j)]$$

$$+ \sum_{s=0}^{p-1} \sum_{m=1}^{p-s} P(p - s - m)\, [\sum_{i=0}^{q-1} \sum_{j=1}^{q-i} w_{r+s+m+i+j-3}\, P(q - i - j)].$$

5. SIMPLE TREE POLYNOMIALS OF GRAPHS WITH CYCLOMATIC NUMBER 2.

The deletion of any k of the p - 1 edges of the chain P_p yields a cover with cardinality k + 1. Hence we have

LEMMA 11. $$T(P_p;w) = w(1 + w)^{p-1}.$$

A similar argument yields

LEMMA 12. $$T(C_p;w) = (1 + w)^p - 1.$$

The expression $(1 + w)^p - 1$ will occur quite often in our results. We will therefore replace it by $\phi(p)$. A useful property of $\phi(p)$ is the following.

LEMMA 13. $$\phi(m + n) = \phi(m)\,\phi(n) + \phi(m) + \phi(n).$$

The Cutnode Theorem can be applied to $G(p,q)$ to obtain the following result.

THEOREM 7.

$$T(G(p,q);\underline{w}) = w^{-1}[\phi(p + q) - \phi(p) - \phi(q)].$$

THEOREM 8.

$$T(G(p,q,r);\underline{w}) = w^{-1}[\phi(p + q + r - 3) - \phi(p - 1) - \phi(q - 1) - \phi(r - 1)].$$

PROOF. Apply the reduction process to the tetha graph, by deleting an edge incident to the node of valency 3 and belonging to the path with r nodes. This yields

$$T(G(p,q,r);w) = T(H_{r-1};w) + T(G(p,q,r - 1);w).$$

By continuing the reduction process in the same manner on all subsequent theta graphs formed by incorporation of edges, we obtain

$$T(G(p,q,r);w) = \sum_{k=1}^{r-1} T(H_{r-k};w) = T(G(p - 1,q - 1);w) \tag{5.1}$$

The graph H_{r-k} consists of the blocks C_{p+q-2} and P_{r-k} with a common cutnode. Therefore, by using the Cutnode Theorem, we get

$$\begin{aligned} T(H_{r-k};w) &= w^{-1}\, T(C_{p+q-2};w)\; P(r - k) \\ &= \phi(p + q - 2)(1 + w)^{r-k-1}. \end{aligned}$$

By substituting into Equation (5.1), we get

$$\begin{aligned} T(G(p,q,r);w) &= \phi(p + q - 2) \sum_{k=1}^{r-1} (1 + w)^{r-k-1} + w^{-1}\phi(p - 1)\phi(q - 1) \\ &= w^{-1}\phi(p + q - 2)\phi(r - 1) + w^{-1}\phi(p - 1)\phi(q - 1). \end{aligned}$$

The result is then obtained by using Lemma 13.

$G((p,q),r)$ consists of three blocks C_p, C_q and P_r, with two cutnodes of valency 3. Hence Theorem 3 can be immediately applied. This yields

THEOREM 9. $$T(G((p,q),r);w) = w^{-1}[\phi(p + q) - \phi(p) - \phi(q)][\phi(r - 1) + 1].$$

The result given in Lemma 11 holds for all trees with p nodes. Thus, if T_p is a tree with p nodes,

$$T(T_p;w) = w(1 + w)^{p-1} \tag{5.2}$$

Let us denote by $G_{p_1p_2\dots p_n}$, the graph obtained by attaching n trees $T_{p_1}, T_{p_2}, \dots, T_{p_n}$ to a connected graph G. By applying the Cutnode Theorem, we obtain the following theorem.

THEOREM 10.

$$T(G_{p_1p_2\dots p_n};w) = (1 + w)^{N-n}T(G;w),$$

where

$$N = \sum_{i=1}^{n} p_i.$$

Since any graph with cyclomatic number 2 can be obtained from the basic graphs by attaching trees to them, it follows that Theorem 10, together with the results given in Section 5, is sufficient to obtain the tree polynomials of all graphs with cyclomatic number 2.

6. FOREST DECOMPOSITIONS OF GRAPHS WITH CYCLOMATIC NUMBER 2.

It is clear that the coefficient of the term $w_1^{n_1}w_2^{n_2}\dots w_k^{n_k}$ in $T(G;\underline{w})$, is the number of decompositions of G into n_1 nodes, n_2 edges, ..., n_k trees with k nodes. Therefore, Theorems 4, 5, and 6 provide information about the decompositions of the basic graphs with cyclomatic number 2 into spanning forests with specified trees.

The coefficient of w^k in $T(G;w)$ is the number of decompositions of G, with cardinality k. We will represent this by $T_k(G)$. Hence we have the following corollaries in which $\binom{n}{r} = 0$ for $r > n$, and $k' = k + 1$.

COROLLARY 7.1.

$$T_k(G(p,q)) = \binom{p+q}{k'} - \binom{p}{k'} - \binom{q}{k'}.$$

COROLLARY 8.1.

$$T_k(G(p,q,r)) = \binom{p+q+r-3}{k'} - \binom{p-1}{k'} - \binom{q-1}{k'} - \binom{r-1}{k'}.$$

COROLLARY 9.1.

$$T_k(G((p,q),r)) = \binom{p+q}{k'} - \binom{p}{k'} - \binom{q}{k'} + \sum_{i=0}^{k+1} \binom{r-1}{k'-i}\left[\binom{p+q}{i} - \binom{p}{i} - \binom{q}{i}\right].$$

In order to extend our results to all graphs with cyclomatic number 2, we add the following corollary of Theorem 10.

COROLLARY 10.1.

$$T_k(G_{p_1 p_2 \ldots p_n}) = \sum_{i=0}^{k} \binom{N-n}{k-i} T_i(G).$$

Let us denote by N(G), the number of spanning forests in G. Then N(G) is the sum of the coefficients of the terms in T(G;w). Hence it is clear that

$$N(G) = T(G;1).$$

The following theorem is immediate.

THEOREM 11.

(i) $N(G(p,q)) = (2^p - 1)(2^q - 1)$,

(ii) $N(G(p,q,r)) = (2^{p+q-2} - 1)(2^{r-1} - 1) + (2^{p-1} - 1)(2^{q-1} - 1)$,

(iii) $N(G((p,q),r)) = 2^{r-1}(2^p - 1)(2^q - 1)$.

Let us denote by $\Gamma(G)$, the number of spanning trees in G. Then $\Gamma(G)$ is the coefficient of w in T(G;w). The number of spanning trees in the basic graphs can be immediately obtained by trivial counting techniques. However, we could test our simple tree polynomials, by extracting the coefficients of w. For completeness, we add the following result.

THEOREM 12.

(i) $\Gamma(G(p,q)) = pq$,

(ii) $\Gamma(G(p,q,r)) = pq + pr + qr - 2(p + q + r) + 3$,

(iii) $\Gamma(G((p,q),r)) = pq$.

It should be noted that these explicit formulae agree with the results obtained in Farrell [3].

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