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ON THE RITT ORDER OF A CERTAIN CLASS OF FUNCTIONS

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ABSTRACT. The authors introduce the notions of Ritt order and lower order to functions defined by the series $\sum_1^\infty f_n(s) \exp(-\lambda_n s)$ where (λ_n) is a D-sequence and $f_n(s)$ are entire functions of bounded index.

KEY WORDS AND PHRASES. *Ritt order, entire functions of bounded index, Dirichlet series.*

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1. INTRODUCTION.

Let us consider an M-dirichletian element:

$$\{\phi\}: \sum_{n=1}^{\infty} f_n(s) \exp(-\lambda_n s), \quad s = \sigma + i\tau, \quad (\sigma,\tau) \in \mathbb{R}^2 \tag{1.1}$$

where (λ_n) is a D-sequence (a strictly increasing unbounded sequence of positive numbers) and $f_n(s)$ are entire functions of bounded index (defined below). Convergence properties of such elements were discussed by J.S.J. Mac Donnell in his doctoral dissertation [1] under the conditions $\lim_{n\to\infty} \frac{\log n}{\lambda_n} = 0$ and $\lim_{n\to\infty} \frac{m_n}{\lambda_n} = 0$ where m_n is the index of f_n. In this paper, we first study the convergence properties of these elements with less restrictions, namely,

$$L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n} < \infty \text{ and} \tag{1.2}$$

$$\beta = \limsup_{n\to\infty} \frac{m_n}{\lambda_n} < \infty \tag{1.3}$$

As the functions defined by these series are unbounded in the half-plane, it is not possible to define Ritt order directly. However, by making use of functions defined by associated intermediate series, we introduce the notions of Ritt order and lower order to these functions.

2. MAIN RESULTS.

DEFINITION 2.1. [2]. An entire function f is said to be of bounded index if there exists a non-negative integer N such that

$$\max_{0\le k\le N} \left\{ \frac{|f^{(k)}(s)|}{k!} \right\} \ge \frac{|f^{(j)}(s)|}{j!} \quad (f^{(0)}(s) = f(s))$$

for all j and for all s. The least such integer N is called the index of f.

We require the following lemma which shows that an entire function of bounded index is of exponential type.

LEMMA 2.2. [3], [2]. Let f be an entire function of bounded index N. Then

$$|f(s)| \le \left\{ \max_{0\le k\le N} \frac{|f^{(k)}(0)|}{(N+1)^k} \right\} \exp (N+1)\, |s| \, . \tag{2.1}$$

Let $f_n(s) = \sum_{j=0}^{\infty} a_{nj} s^j$ be an entire function of bounded index m_n;

$$A_n = \max \{ |a_{nj}| / j = 0,1,\dots,m_n \} = \max \left\{ \frac{|f^{(j)}(0)|}{j!}, \; j = 0,1,\dots,m_n \right\} \tag{2.2}$$

$\{X\}$: $\sum_{1}^{\infty} A_n \exp(-\lambda_n s)$ the associated dirichletian element whose abscissa of convergence is denoted by σ_c^X; $k = \lim_{n\to\infty} \dfrac{\log A_n}{\lambda_n}$.

REMARK 2.3. It can be easily seen from Lemma 2.2 that

$$|f_n(s)| \le \max_{0\le j\le m_n} \left\{ \frac{|f_n^{(j)}(0)|}{j!} \right\} \exp (m_n + 1)\, |s|$$

$$= A_n \exp (m_n + 1)\, |s| \, .$$

THEOREM 2.4. If $0 \le \beta < 1$, the region of absolute convergence of (1.1) is the exterior of the hyperbola centre $(k(1-\beta^2)^{-1}, 0)$ and eccentricity β^{-1} contained in the half-plane $\sigma > k$.

PROOF. Using Remark 2.3, we have

$$|f_n(s) \exp(-\lambda_n s)| \le A_n \exp(m_n + 1)\, |s| \exp(-\lambda_n \sigma) \, . \tag{2.3}$$

From the definitions of k and β it follows that for $\varepsilon > 0$

$$\exists_{n'} \quad \forall_{n \geq n'} \qquad A_n < \exp\,(k+\varepsilon)\lambda_n \text{ and}$$

$$\exists_{n''} \quad \forall_{n \geq n''} \qquad m_n + 1 < (\beta+\varepsilon)\lambda_n \;.$$

Hence

$$\forall_{n(\varepsilon) = \max(n', n'')} \qquad |f_n(s)\,\exp(-\lambda_n s)| \leq \exp(-\lambda_n(\sigma - k - \varepsilon - (\beta+\varepsilon)|s|)$$

and

$$\sum_{n(\varepsilon)}^{\infty} f_n(s)\,\exp(-\lambda_n s) \leq \sum_{n(\varepsilon)}^{\infty} \exp\,(-\lambda_n(\sigma - k - \varepsilon - (\beta+\varepsilon)\;|s|)\;.$$

The series in the right hand side converges provided

$$\sigma - k - \beta\;|s| > 0 \tag{2.4}$$

which is valid only if $\sigma > k$ and $0 \leq \beta < 1$.

Thus any point in the region of convergence of (1.1) must satisfy

$$(\sigma - k)^2 - \beta^2(\sigma^2 + \tau^2) > 0$$

which reduces to

$$(\sigma - \frac{k}{1-\beta^2})^2 - \frac{\beta^2\;\tau^2}{1-\beta^2} > \frac{k^2\;\beta^2}{(1-\beta^2)^2} \tag{2.5}$$

from which the theorem follows.

REMARK 2.5. If $\beta = 0$ the M-dirichletian element converges in the half-plane $\sigma > k$ (which coincides with the half-plane of convergence of the associated series $\{\chi\}$) thus giving the result of MacDonnell [1] as a particular case.

Next we proceed to introduce the notions of Ritt order and lower order for functions defined by (1.1). We need the following lemmas.

Let the M-dirichletian element given by (1.1) converge absolutely on E_a^{ϕ} and $D_o = \{s \in C \mid \sigma = 0,\ \tau \in \mathbb{R}\}$ denote the imaginary axis.

LEMMA 2.6. Under the conditions $\sigma_c^{\chi} = -\infty$ and $0 \leq \beta < \infty$, we have $E_a^{\phi} = \mathbb{C}$ and ϕ is holomorphic on $\mathbb{C}$.

PROOF. Using (2.3) we have

$$\forall_{n \in N - \{0\}} \;\; \forall_{s \in \mathbb{C} - D_o} \qquad |f_n(s)\,\exp(-\lambda_n s)| \leq A_n\,\exp[-\,\sigma\lambda_n(1 - \frac{m_n + 1}{\lambda_n}\,\theta_\sigma\,\frac{|s|}{|\sigma|}\,)]$$

where $\theta_\sigma = 1$ if $\sigma > 0$ and $\theta_\sigma = -1$ if $\sigma < 0$.

Since $0 \le \beta < \infty$ given $\varepsilon > 0$

$$\exists_{n'(=n_\epsilon)} \ \forall_{n \ge n'} \ \forall_{s \in ¢ - D_0} \ |f_n(s)\exp(-\lambda_n s)| \le A_n \exp[-\sigma \lambda_n(1 - (\beta + \varepsilon) \frac{|s|}{|\sigma|} \theta_\sigma)]. \tag{2.6}$$

For any point (on the imaginary axis) $s_o = i\tau_o$ of D_o we extend $\frac{\sigma |s| \theta_\sigma}{|\sigma|}$ by its limiting value as $s \to s_o$.

The function $¢ \ni s \to \sigma[1 - (\beta + \varepsilon) \frac{|s|}{|\sigma|} \theta_\sigma]$ is continuous on ¢. Let $E_\mu = \{s \in ¢/\sigma(1 - (\beta + \varepsilon) \frac{|s|}{|\sigma|} \theta_\sigma) \ge \mu\}$ indexed by μ on $\mathbb{R}$. Then $\{\phi_{n'}\}$ converges uniformly on each E_μ as $\mu > \sigma_c^\chi$, where

$$\{\phi_{n'}\}: \sum_{n=n'}^{\infty} f_n(s) \exp(-\lambda_n s).$$

Let G be any open subset of ¢. We put

$$\mu_G = \inf\{\sigma(1 - (\beta + \varepsilon) \frac{|s|}{|\sigma|} \theta_\sigma) \ |s \in G\}.$$

The number μ_G is finite and $\{\phi_{n'}\}$ converges absolutely in G if $\sigma_c^\chi = -\infty$; further, ϕ_G: $G \ni s \to \phi(s)$ is holomorphic on G. Since G is arbitrary on ¢, $\{\sigma\}$ converges absolutely on each point of ¢ and ϕ_G can be continued analytically on the totality of ¢. Let ϕ denote its analytic continuation. Now we put

$$\forall_{\sigma \in \mathbb{R}} \quad M^\phi(\sigma,B) = \sup\{|\phi(s')|/\sigma' \ge \sigma \quad s \in B(\tau_1,\ell)\}$$

where $B(\tau_1,\ell) = \{s \in ¢/|\tau - \tau_1| \le \ell\}$ is the horizontal strip with $\tau = \tau_1$ as axis and of width 2ℓ. Then

LEMMA 2.7. Under the conditions $\sigma_c^\chi = -\infty$ and $0 \le \beta < 1$ we have $\forall_{(\tau_1,\ell) \in \mathbb{R} \times \mathbb{R}_o^+}$ $M^\phi(\sigma,B)$ is bounded on each point $\sigma \in \mathbb{R}$ and $\lim M^\phi(\sigma,B) = 0$ as $\sigma \to +\infty$.

PROOF. Let (τ_1,ℓ) be fixed arbitrarily on $\mathbb{R} \times \mathbb{R}_o^+$. Then given $\varepsilon' > 0$, $\exists_{\sigma_{\varepsilon'}} \ \forall_{s \in \{\sigma \ge \sigma_{\varepsilon'}/s \in B(\tau_1,\ell)\}} \ \frac{|s|}{|\sigma|} < 1 + \varepsilon'$ and with $0 \le \beta < 1$. We have

$$\forall_{\sigma \geq \sigma_{\varepsilon'}} \quad \sigma(1 - (\beta + \varepsilon) \frac{|s|}{|\sigma|} \theta_\sigma) > \sigma(1 - (\beta + \varepsilon)(1 + \varepsilon')) \tag{2.7}$$

$$\geq \sigma_\varepsilon \; (1 - (\beta + \varepsilon)(1 + \varepsilon'));$$

as a result of(2.6)and (2.7)

$$M^{\phi_{n'}} (\sigma,B) < A_{n'} \exp (-\sigma_{\varepsilon'}(1 - (\beta + \varepsilon)(1 + \varepsilon'))$$

$$= \chi_{n'} (\sigma_{\varepsilon'}(1 - (\beta + \varepsilon) (1 + \varepsilon'))$$

and hence

$$\lim M^{\phi_{n'}} (\sigma,B) = 0 \text{ as } \sigma \to \infty;$$

finally, $\lim M^{\phi} (\sigma,B) = 0$ as $\sigma \to \infty$.

Further, we have

$$\forall_{\sigma \geq -\sigma_{\varepsilon'}} \quad \sigma(1 - (\beta + \varepsilon) \frac{|s|}{|\sigma|} \theta_\sigma) > \sigma(1 + (\beta + \varepsilon) (1 + \varepsilon'));$$

we put $M(\sigma_{\varepsilon'}) = \text{Max } \{|\phi_{n'}(s)|/\sigma \leq \sigma_{\varepsilon'} \wedge s \in B(\tau_1,\ell)$. As $\phi_{n'}$ is holomorphic on the compact set, $\{s \in B(\tau_1,\ell)/|\sigma| \leq \sigma_\varepsilon\}$, $M(\sigma_{\varepsilon'})$ is finite. Then we get

$$\forall_{\sigma \in \mathbb{R}} \quad M^{\phi_{n'}}(\sigma,B) \leq \text{Max } \{\chi_{n'}(\sigma(1 + (\beta + \varepsilon)(1 + \varepsilon')), M(\sigma_{\varepsilon'}), \chi_{n'}(\sigma_{\varepsilon'}(1-(\beta+\varepsilon)(1+\varepsilon'))\};$$

$\chi_{n'}$ is a strictly decreasing function in $\mathbb{R}$ and hence

$$\chi_{n'}(\sigma[1 + \beta + \varepsilon)(1 + \varepsilon')] > \chi_{n'} (\sigma(1 - (\beta + \varepsilon)(1 + \varepsilon')) \quad \text{if } \sigma < 0 \text{ and}$$

$$\chi_{n'}(\sigma(1 + (\beta + \varepsilon)(1 + \varepsilon')) < \chi_{n'} (\sigma(1 - (\beta + \varepsilon)(1 + \varepsilon')) \quad \text{if } \sigma > 0,$$

(equality holds for $\sigma = 0$) with $\chi_{n'}(\sigma) = 0$ as $\sigma \to \infty$.

As all M-dirichletian polynomials satisfy the two properties of the lemma, in each horizontal strip $B(\tau,\ell)$ $M^{\phi}(\sigma,B)$ is bounded for each $\sigma \in \mathbb{R}$ and hence the function $\sigma \to M^{\phi}(\sigma,B)$ is decreasing on $\mathbb{R}$ with $\lim_{\sigma \to \infty} M^{\phi}(\sigma,B) = 0$.

DEFINITION 2.8. We put

$$\rho_B^{\phi} = \limsup_{\sigma \to -\infty} \{ \frac{\log \log^+ M^{\phi}(\sigma,B)}{-\sigma} \} .$$

Then ρ_B^{ϕ} is called the Ritt order of ϕ on B. Let $\lambda_B^{\phi} = \liminf_{\sigma \to \infty} \{ \frac{\log \log^+ M^{\phi}(\sigma,B)}{-\sigma} \}$. Then λ_B^{ϕ} is called the lower order of ϕ on B.

THEOREM 2.9. Under the conditions $\sigma_c^{\chi} = -\infty$ and $0 \leq \beta < 1$, we have

$$\forall_{(\tau_1,\ell) \in \mathbb{R} \times \mathbb{R}_0^+} \quad \rho_B^{\phi} \leq \rho_R^{\chi} \text{ and } \lambda_B^{\phi} \leq \lambda_R^{\chi}$$

where ρ_R^χ and λ_R^χ are respectively the Ritt order and lower order of χ in the whole plane.

PROOF. Proceeding as in Lemma 2.6 for $0 \le \beta < 1$, we have by (2.6)

$$\forall_{\varepsilon \in \mathbb{R}_0^+} \ \exists_{n'(=n_\varepsilon)} \ \forall_{s \in B(\tau_1,\ell)} \quad |\phi_{n'}(s)| < \chi_{n'}\left[\sigma\left(1 - (\beta+\varepsilon)\frac{|s|}{|\sigma|}\theta_\sigma\right)\right] .$$

Now denoting by $\phi_{n'}$ and $\chi_{n'}$ the holomorphic functions on $\mathbb{C}$ defined by the elements $\{\phi_{n'}\}$ and $\{\chi_{n'}\}$, we have for σ negative with $|\sigma|$ sufficiently large:

$$M^{\phi_{n'}}(\sigma,B) \le \chi_{n'}\left[\sigma\left(1 - (\beta+\varepsilon)\frac{|s|}{|\sigma|}\theta_\sigma\right)\right]$$

which gives

$$\forall_{\varepsilon' \in \mathbb{R}_0^+} \quad \rho_B^{\phi}n' \le \rho_R^{\chi}n' \ [1 + (\beta+\varepsilon)\ \varepsilon']$$

and hence $\rho_B^{\phi}n' \le \rho_R^{\chi}n'$.

As adding finite number of terms to a holomorphic function defined by a classical Dirichlet series does not affect its Ritt order [4], we add $\sum_{n=0}^{n'-1} A_n \exp(-\lambda_n s)$ to $\{\chi_{n'}\}$ and then $\rho_R^{\chi}n' = \rho_R^{\chi}$.

Now

$$\forall_{\sigma \in \mathbb{R}} \quad M^{\phi}(\sigma,B) \le M^{\phi_{n'}}(\sigma,B) + M^{\phi^o_{n'}}(\sigma,B)$$

where $\{\phi^o_{n'}\}$: $\sum_{n=1}^{n-1} f_n(s)\exp(-\lambda_n s)$; then

$$\forall_{(\tau,\ell) \in \mathbb{R}\times\mathbb{R}_0^+} \quad \rho_B^{\phi} \le \max(\rho_B^{\phi}n',\ \rho_B^{\phi^o}n')$$

$$= \rho_B^{\phi}n' \quad \text{since} \quad \rho_B^{\phi^o}n' = 0.$$

Finally we have

$$\forall_{(\tau,\ell) \in \mathbb{R}\times\mathbb{R}_0^+} \quad \rho_B^{\phi} \le \rho_R^{\chi}$$

and similarly we can show:

$$\forall_{(\tau,\ell) \in \mathbb{R}\times\mathbb{R}_0^+} \quad \lambda_B^{\phi} \le \lambda_R^{\chi} .$$

Now we are in a position to define the Ritt order and lower order of ϕ in the whole plane $\mathbb{C}$.

DEFINITION 2.10. We put

$$\rho_R^{\phi} = \sup \{\rho_B^{\phi} / (\tau_1, \ell) \in \mathbb{R} \times \mathbb{R}_0^+\}$$

and

$$\lambda_R^{\phi} = \sup \{\lambda_B^{\phi} / (\tau_1, \ell) \in \mathbb{R} \times \mathbb{R}_o^+\}$$

Then ρ_R^{ϕ} is called the Ritt order of ϕ on $\mathbb{C}$ and λ_R^{ϕ} is called the lower order of ϕ on $\mathbb{C}$.

REFERENCES

1. MAC DONNELL, J.S.J. Doctoral dissertation, The Catholic University of America, (1957).

2. SHAH, S.M. Entire functions of bounded index, Proc. Amer. Math. Soc. 19, (1968) 1017-1022.

3. HAYMAN, W.K. Differential inequalities and local valency, Pacific J. Math. 44, (1973) 117-137.

4. BLAMBERT, M. Sur la comparison des ordes boreliens et (R) des functions entieres d'une certaene classe, Bull. Sc. Math. 89, (1965) 103-125.