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# ON CONTACT *CR*-SUBMANIFOLDS OF SASAKIAN MANIFOLDS

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ABSTRACT: Recently, K.Yano and M.Kon [5] have introduced the notion of a contact *CR*-submanifold of a Sasakian manifold which is closely similar to the one of a *CR*-submanifold of a Kaehlerian manifold defined by A.Bejancu [1].

In this paper, we shall obtain some fundamental properties of contact *CR*-submanifolds of a Sasakian manifold. Next, we shall calculate the length of the second fundamental form of a contact *CR*-product of a Sasakian space form (THEOREM 7.4). At last, we shall prove that a totally umbilical contact *CR*-submanifold satisfying certain conditions is totally geodesic in the ambient manifold (THEOREM 8.1).

KEY WARDS AND PHRASES : *Kaehlerian manifold, Sasakian space form, contact CR-product.*
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## 1. INTRODUCTION.

This paper is directed to specialist readers with background in the area and appreciative of its relation of this area of study.

Let $\tilde{M}$ be a $(2n + 1)$-dimensional Sasakian manifold with structure tensors $(\phi,\xi,\eta,\langle,\rangle)$ [4] and let $M$ be an $m$-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$ and let $\langle,\rangle$ be the induced metric on $M$. Let $\nabla$ and $\tilde{\nabla}$ be the covariant differentiations on $M$ and $\tilde{M}$, respectively. Then the Gauss and Weingarten's formulas for $M$ are respectively given by

$$\tilde{\nabla}_U V = \nabla_U V + \sigma(U,V), \tag{1.1}$$

$$\tilde{\nabla}_U \lambda = -A_\lambda U + \nabla^\perp_U \lambda \tag{1.2}$$

for any vector fields $U,V$ tangent to $M$ and any vector field $\lambda$ normal to $M$, where $\sigma$ denotes the second fundamental form and $\nabla^\perp$ is the normal connection. The second fundamental tensor $A_\lambda$ is related to $\sigma$ by

$$\langle A_\lambda U, V\rangle = \langle \sigma(U,V), \lambda\rangle. \tag{1.3}$$

The mean curvature vector $H$ is defined by

$$H = \frac{1}{m} \text{ trace } \sigma. \tag{1.4}$$

The submanifold $M$ is called a minimal submanifold of $\tilde{M}$ if $H = 0$ and $M$ is called a totally geodesic submanifold of $\tilde{M}$ if $\sigma = 0$.

For any vector field $U$ tangent to $M$, we put

$$\phi U = PU + FU, \tag{1.5}$$

where $PU$ and $FU$ are the tangential and the normal components of $\phi U$, respectively. Then $P$ is an endomorphism of the tangent bundle $TM$ of $M$ and $F$ is a normal-bundle-valued 1-form of $TM$.

For any vector field $\lambda$ normal to $M$, we put

$$\phi\lambda = t\lambda + f\lambda, \tag{1.6}$$

where $t\lambda$ and $f\lambda$ are the tangential and the normal components of $\phi\lambda$, respectively. Then $f$ is an endomorphism of the normal bundle $T^\perp M$ of $M$ and $t$ is a tangent-bundle-valued 1-form of $T^\perp M$.

We put

$$\xi = \xi_1 + \xi_2, \tag{1.7}$$

where $\xi_1$ and $\xi_2$ are the tangential and the normal components of $\xi$, respectively. Then we can put

$$\eta = \eta_1 + \eta_2, \tag{1.8}$$

where $\eta_1(U) = \langle \xi_1, U\rangle$ and $\eta_2(\lambda) = \langle \xi_2, \lambda\rangle$ for any vector field $U$ tangent to $M$ and any vector field $\lambda$ normal to $M$.

By virtue of (1.5),(1.6),(1.7) and (1.8), we get

$$P^2U + tFU = -U + \eta_1(U)\xi_1, \tag{1.9}$$

$$FPU + fFU = \eta_1(U)\xi_2, \tag{1.10}$$

$$\eta_2(\lambda)\xi_1 = Pt\lambda + tf\lambda, \tag{1.11}$$

$$Ft\lambda + f^2\lambda = -\lambda + \eta_2(\lambda)\xi_2 \tag{1.12}$$

for any vector field $U$ tangent to $M$ and any vector field $\lambda$ normal to $M$.

Let $\tilde{M}(k)$ be a Sasakian space form with constant $\phi$-holomorphic sectional curvature $k$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}(k)$ is given by

$$\tilde{R}(X,Y)Z = \frac{k+3}{4}\{<Y,Z>X - <X,Z>Y\} + \frac{k-1}{4}\{\eta(X)<Y,Z>\xi - \eta(Y)<X,Z>\xi$$

$$+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - <\phi Y,Z>\phi X + <\phi X,Z>\phi Y + 2<\phi X,Y>\phi Z\} \tag{1.13}$$

for any vector fields $X, Y$ and $Z$ in $\tilde{M}(k)$ [3].

For the second fundamental form $\sigma$, we define the covariant differentiation $\bar{\nabla}$ with respect to the connection on $TM \oplus T^{\perp}M$ by

$$(\bar{\nabla}_U\sigma)(V,W) = \nabla^{\perp}_U(\sigma(V,W)) - \sigma(\nabla_U V, W) - \sigma(V, \nabla_U W) \tag{1.14}$$

for any vector fields $U, V$ and $W$ tangent to $M$. We denote $R$ the curvature tensor associated with $\nabla$. Then the equations of Gauss and Codazzi are respectively given by

$$\tilde{R}(U,V;W,Z) = R(U,V;W,Z) + <\sigma(U,W),\sigma(V,Z)> - <\sigma(U,Z),\sigma(V,W)>, \tag{1.15}$$

$$(\tilde{R}(U,V)W)^{\perp} = (\bar{\nabla}_U\sigma)(V,W) - (\bar{\nabla}_V\sigma)(U,W) \tag{1.16}$$

for any vector fields $U,V,W$ and $Z$ tangent to $M$, where $\tilde{R}(U,V;W,Z) = <\tilde{R}(U,V)W,Z>$ and $(\tilde{R}(U,V)W)^{\perp}$ denotes the normal component of $\tilde{R}(U,V)W$.

## 2. CONTACT *CR*-SUBMANIFOLDS OF A SASAKIAN MANIFOLD.

DEFINITION 2.1: A submanifold $M$ of a Sasakian manifold $\tilde{M}$ with structure tensors $(\phi,\xi,\eta,<,>)$ is called a contact *CR*-submanifold if there is a differentiable distribution $D: x \longrightarrow D_x \subseteq T_xM$ on $M$ satisfying the following conditions:

(i) $\xi \in D$,

(ii) $\phi D_x \subset T_xM$ for each $x$ in $M$,

(iii) the complementary orthogonal distribution $D^{\perp}: x \longrightarrow D^{\perp}_x \subset T_xM$ satisfies $\phi D^{\perp}_x \subseteq T^{\perp}_xM$ for each point $x$ in $M$.

Let $M$ be a contact *CR*-submanifold of a Sasakian manifold $\tilde{M}$. Then $\xi \in D \subset TM$, so, the equations (1.9),(1.10),(1.11) and (1.12) can be written as

$$P^2U + tFU = -U + \eta(U)\xi, \tag{2.1}$$

$$FPU + fFU = 0, \tag{2.2}$$

$$Pt\lambda + tf\lambda = 0, \tag{2.3}$$

$$Ft\lambda + f^2\lambda = -\lambda \tag{2.4}$$

for any vector field $U$ tangent to $M$ and any vector field $\lambda$ normal to $M$, respectively. By virtue of (1.5), we have

PROPOSITION 2.1: In a contact $CR$-submanifold $M$ of a Sasakian manifold $\tilde{M}$, in order to a vector field $U$ tangent to $M$ belong to $D$ it is necessary and sufficient that $FU = 0$.

Taking account of (2.1) and PROPOSITION 2.1, we have

$$P^2X = -X + \eta(X)\xi \tag{2.5}$$

for any $X$ in $D$ and we have from (1.6)

$$P\xi = 0. \tag{2.6}$$

Furthermore, we obtain

$$\langle PX,PY\rangle = \langle\phi X,\phi Y\rangle = \langle X,Y\rangle - \eta(X)\eta(Y) \tag{2.7}$$

for any $X$ and $Y$ in $D$. Thus we have

PROPOSITION 2.2: In a contact $CR$-submanifold $M$ of a Sasakian manifold $\tilde{M}$, the distribution $D$ has an almost contact metric structure $(P,\xi,\eta,\langle,\rangle)$ and hence $\dim D_x =$ odd.

We denote by $\Pi$ the complementary orthogonal subbundle of $\phi D^{\perp}$ in $T^{\perp}M$. Then we have

$$T^{\perp}M = \phi D^{\perp} \oplus \Pi, \qquad \phi D^{\perp} \perp \Pi. \tag{2.8}$$

Thus we have

PROPOSITION 2.3: For a contact $CR$-submanifold $M$ of a Sasakian manifold $\tilde{M}$, the subbundle $\Pi$ has an almost complex structure $f$ and hence $\dim \Pi_x =$ even.

## 3. BASIC PROPERTIES.

Let $M$ be a contact $CR$-submanifold of a Sasakian manifold $M$. Then we have

$$\phi(\nabla_U Z + \sigma(U,Z)) = -A_{\phi Z}U + \nabla_U(\phi Z) - \langle U,Z\rangle\xi \tag{3.1}$$

for any vector field $U$ tangent to $M$ and $Z$ in $D^\perp$. From (3.1), we get

$$\langle\nabla_U Z,\phi X\rangle = \langle A_{\phi Z}U,X\rangle + \eta(X)\langle U,Z\rangle \tag{3.2}$$

for any vector field $U$ tangent to $M$, $X$ in $D$ and $Z$ in $D^\perp$. In (3.2), if we put $X = \phi X$, then (3.2) means

$$\langle\nabla_U Z,X\rangle = \langle\phi A_{\phi Z}U,X\rangle + \eta(X)\langle\nabla_U Z,\xi\rangle \tag{3.3}$$

for any vector field $U$ tangent to $M$, $X$ in $D$ and $Z$ in $D^\perp$. By virtue of (3.2), we obtain

$$\langle A_{\phi Z}X,\xi\rangle = 0 \qquad \text{(or equivalently } \langle\sigma(X,\xi),\phi Z\rangle = 0) \tag{3.4}$$

for any $X$ in $D$ and $Z$ in $D^\perp$.

On the other hand, we have

$$\langle\sigma(X,\xi),\lambda\rangle = \langle\tilde{\nabla}_X\xi - \nabla_X\xi,\lambda\rangle = \langle\phi X,\lambda\rangle = 0$$

for any $X$ in $D$ and $\lambda$ in $\Pi$. Thus we have from (3.4) and the above equation $\sigma(X,\xi) = 0$ for any $X$ in $D$. So, we have from (1.7) and the last equation

$$\nabla_X\xi = PX \tag{3.5}$$

for any $X$ in $D$. Thus we have

PROPOSITION 3.1: In a contact $CR$-submanifold $M$ of a Sasakian manifold $\tilde{M}$, the distribution $D$ has a $K$-contact metric structure $(P,\xi,\eta,\langle,\rangle)$.

In (3.1), if we put $U = W \in D^\perp$, then the equation (3.1) can be written as

$$\phi(\nabla_W Z + \sigma(Z,W)) = -A_{\phi Z}W + \nabla^\perp_W\phi Z - \langle W,Z\rangle\xi,$$

from which

$$\phi([Z,W]) = A_{\phi Z}W - A_{\phi W}Z + \nabla^\perp_Z\phi W - \nabla^\perp_W\phi Z, \tag{3.6}$$

where $[Z,W] = \nabla_Z W - \nabla_W Z$.

LEMMA 3.2: In a contact $CR$-submanifold $M$ of a Sasakian manifold $\tilde{M}$, we have

$$\nabla^\perp_W\phi Z - \nabla^\perp_Z\Phi W \in \phi D^\perp \tag{3.7}$$

for any $Z$ and $W$ in $D^\perp$.

PROOF: For any $Z,W$ in $D^{\perp}$ and $\lambda$ in $\Pi$, we obtain

$$<\nabla^{\perp}_W\phi Z - \nabla^{\perp}_Z\phi W,\lambda> = <\tilde{\nabla}_W\phi Z - \tilde{\nabla}_Z\phi W,\lambda> = <\phi(\tilde{\nabla}_W Z - \tilde{\nabla}_Z W) + (\tilde{\nabla}_W\phi)Z - (\tilde{\nabla}_Z\phi)W,\lambda>$$

$$= <\phi(\nabla_W Z - \nabla_Z W),\lambda> = <\nabla_Z W - \nabla_W Z,\phi\lambda> = 0.$$

On the other hand, we can easily have

$$A_{\phi Z}W = A_{\phi W}Z \tag{3.8}$$

for any $Z$ and $W$ in $D^{\perp}$. In fact, for any vector field $U$ tangent to $M$ and $Z$ and $W$ in $D^{\perp}$, we have from (3.1)

$$<\phi(\nabla_U Z + \sigma(U,Z)),W> = <\phi\nabla_U Z,W> + <\phi\sigma(U,Z),W> = -<\sigma(U,Z),\phi W>$$

$$= -<A_{\phi W}Z,U> = -<A_{\phi Z}W,U>,$$

from which, we have (3.8).

By virtue of (3.6) and (3.8) and LEMMA 3.2, we have

PROPOSITION 3.3: In a contact $CR$-submanifold $M$ of a Sasakian manifold $\tilde{M}$, the distribution $D^{\perp}$ is integrable.

For any $X$ in $D$ and $\lambda$ in $\Pi$, we have

$$A_{\lambda}\phi X = -A_{\phi\lambda}X. \tag{3.9}$$

Next, we assume that the distribution $D$ is integrable. Then for any $X$ and $Y$ in $D$, $\phi[X,Y]$ is an element of $D$, that is, $\phi[X,Y] \in TM$. Since we have

$$\phi[X,Y] = \phi(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X) = \{\nabla_X\phi Y - \nabla_Y\phi X + \eta(X)Y - \eta(Y)X\} + \{\sigma(X,\phi Y) - \sigma(\phi X,Y)\},$$

we get $\sigma(X,\phi Y) = \sigma(\phi X,Y)$. From which we obtain

$$<\sigma(X,\phi Y),\phi Z> = <\sigma(\phi X,Y),\phi Z> \tag{3.10}$$

for any $X$ and $Y$ in $D$ and $Z$ in $D^{\perp}$.

Conversely, if (3.10) is satisfied, we can easily show that the distribution $D$ is integrable. Thus we have

PROPOSITION 3.4: In a contact $CR$-submanifold $M$ of a Sasakian manifold $\tilde{M}$, the distribution $D$ is integrable if and only if the equation (3.10) is satisfied.

Next, we can prove

PROPOSITION 3.5: In a contact $CR$-submanifold $M$ of a Sasakian manifold $\tilde{M}$, the vector field $\xi$ is parallel along $D^{\perp}$.

PROOF: For any $Z$ in $D^{\perp}$, we have

$$\phi Z = \nabla_Z \xi + \sigma(Z,\xi).$$

Since the vector field $Z$ is an element of $D^\perp$, $\phi Z$ is in $T^\perp M$. Thus we have from the above equation $\nabla_Z \xi = 0$, that is, the vector field $\xi$ is parallel along any vector field in $D^\perp$.

## 4. SOME COVARIANT DIFFERENTIATIONS.

DEFINITION 4.1: In a contact $CR$-submanifold $M$ of a Sasakian manifold $\tilde{M}$, we define

$$(\bar{\nabla}_U P)V = \nabla_U(PV) - P\nabla_U V, \tag{4.1}$$

$$(\bar{\nabla}_U F)V = \nabla^\perp_U(FV) - F\nabla_U V, \tag{4.2}$$

$$(\bar{\nabla}_U t)\lambda = \nabla_U(t\lambda) - t\nabla^\perp_U \lambda, \tag{4.3}$$

$$(\bar{\nabla}_U f)\lambda = \nabla^\perp_U(f\lambda) - f\nabla^\perp_U \lambda, \tag{4.4}$$

for any vector fields $U$ and $V$ tangent to $M$ and any vector field $\lambda$ normal to $M$ [2].

DEFINITION 4.2: The endomorphism $P$ (resp. the endomorphism $f$, the 1-forms $F$ and $t$) is parallel if $\bar{\nabla}P = 0$ (resp. $\bar{\nabla}f = 0$, $\bar{\nabla}F = 0$ and $\bar{\nabla}t = 0$).

By virtue of (1.5) and (1.6), we can prove

PROPOSITION 4.1: For the covariant differentiations defined in DEFINITION 4.1, we have

$$(\bar{\nabla}_U P)V = \langle U,V\rangle\xi + \eta(V)U + t\sigma(U,V) + A_{FV}U, \tag{4.5}$$

$$(\bar{\nabla}_U F)V = f\sigma(U,V) - \sigma(U,PV), \tag{4.6}$$

$$(\bar{\nabla}_U t)\lambda = A_{f\lambda}U - PA_\lambda U, \tag{4.7}$$

$$(\nabla_U f)\lambda = -FA_\lambda U - \sigma(U,t\lambda) \tag{4.8}$$

for any vector fields $U$ and $V$ tangent to $M$ and any vector field $\lambda$ normal to $M$.

By virtue of (4.5), we get

$$(\bar{\nabla}_X P)Y = -\langle X,Y\rangle\xi + \eta(Y)X + t\sigma(X,Y) \tag{4.9}$$

for any $X$ and $Y$ in $D$. Thus we have

PROPOSITION 4.2: In a contact $CR$-submanifold $M$ of a Sasakian manifold $\tilde{M}$, the structure $(P,\xi,\eta,\langle,\rangle)$ is Sasakian if and only if $\sigma(X,Y)$ is in $\Pi$ for any $X$ and $Y$ in $D$.

COROLLARY 4.3: In a contact *CR*-submanifold $M$ of a Sasakian manifold $\tilde{M}$, if $\dim D_x^\perp = 0$, then the submanifold $M$ is a Sasakian submanifold.

Next, we assume that the endomorphism $P$ is parallel. Then we have from (4.9)

$$t\sigma(X,Y) = \langle X,Y\rangle\xi - \eta(Y)X \tag{4.10}$$

for any $X$ and $Y$ in $D$. By virtue of $\sigma(X,\xi) = 0$ and (4.10), we have $X = \alpha\xi$ for any $X$ in $D$, where $\alpha$ is a certain scalar field on $D$. Thus we have

PROPOSITION 4.4: In a contact *CR*-submanifold $M$ of a Sasakian manifold $\tilde{M}$, if the endomorphism $P$ is parallel, then $\dim D_x = 1$.

## 5. THE DISTRIBUTION $D^\perp$.

In a contact *CR*-submanifold $M$ of a Sasakian manifold $\tilde{M}$, we assume that the leaf $M^\perp$ of $D^\perp$ is totally geodesic in $M$, that is, $\nabla_Z W$ is in $D^\perp$ for any $Z$ and $W$ in $D^\perp$. This means

$$\langle \nabla_Z W, \phi X\rangle = 0 \tag{5.1}$$

for any $X$ in $D$ and $Z$ and $W$ in $D^\perp$. By virtue of PROPOSITION 3.4 and (5.1), we have

$$\langle \sigma(X,Z) - \eta(X)\phi Z, \phi W\rangle = 0 \tag{5.2}$$

for any $X$ in $D$ and $Z$ and $W$ in $D^\perp$.

Conversely, if the equation (5.2) is satisfied, then it is clear that the leaf $M^\perp$ of $D^\perp$ is totally geodesic in $M$. Thus we have

PROPOSITION 5.1: In a contact *CR*-submanifold $M$ of a Sasakian manifold $\tilde{M}$, the leaf $M^\perp$ of $D^\perp$ is totally geodesic in $M$ if and only if the equation (5.2) is satisfied.

Next, let us prove

THEOREM 5.2: In a contact *CR*-submanifold $M$ of a Sasakian manifold $\tilde{M}$, we assume that the leaf $M^\perp$ of $D^\perp$ is totally geodesic in $M$. If the endomorphism $P$ satisfies

$$(\bar{\nabla}_U P)V = \eta(V)U - \langle U,V\rangle\xi \tag{5.3}$$

for any vector fields $U$ and $V$ tangent to $M$, then $\dim D_x^\perp = 0$, that is, the submanifold $M$ is a Sasakian one.

PROOF: The equation (5.2) means

$$\langle A_{\phi W}X - \eta(X)W, Z\rangle = 0 \tag{5.4}$$

for any $X$ in $D$ and $Z$ and $W$ in $D^\perp$.

On the other hand, we have from (5.3)

$$t\sigma(U,V) + A_{FV}U = 0$$

for any vector fields $U$ and $V$ tangent to $M$. From this, we obtain $t\sigma(U,X) = 0$ for any vector field $U$ tangent to $M$ and $X$ in $D$. Thus we have

$$<\sigma(U,X),\phi W> = <A_{\phi W}X,U> = -<\phi\sigma(U,X),W> = 0,$$

that is, $A_{\phi W}X = 0$ for any $X$ in $D$ and $W$ in $D^{\perp}$. Substituting this equation into (5.4), we get $\dim D_x^{\perp} = 0$.

## 6. A CONTACT *CR*-PRODUCT OF A SASAKIAN MANIFOLD I.

In this section, we shall define a contact *CR*-product and give a necessary and sufficient condition that a contact *CR*-submanifold is a contact *CR*-product.

DEFINITION 6.1: A contact *CR*-submanifold $M$ of a Sasakian manifold $\tilde{M}$ is called a contact *CR*-product if it is locally product of $M^{\perp}$ and $M^{\top}$, where $M^{\top}$ denotes the leaf of the distribution $D$.

THEOREM 6.1: A contact *CR*-submanifold $M$ of a Sasakian manifold $\tilde{M}$ is a contact *CR*-product if and only if

$$A_{\phi W}X = \eta(X)W \tag{6.1}$$

for any $Y$ in $D$ and $W$ in $D^{\perp}$.

PROOF: Since (6.1) means

$$<\sigma(X,Z) - \eta(X)\phi Z,\phi W> = 0 \tag{6.2}$$

for any $X$ in $D$ and $Z$ and $W$ in $D^{\perp}$, the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic in $M$. Furthermore, we have

$$<\sigma(X,\phi Y),\phi Z> = \eta(X)<Z,\phi Y> = 0$$

for any $X$ and $Y$ in $D$ and $Z$ in $D^{\perp}$. So, by virtue of PROPOSITION 3.4, the distribution $D$ is integrable.

Let $M^{\top}$ be the leaf of the distribution $D$, then we have from (6.1)

$$<\nabla_X Y,Z> = <\tilde{\nabla}_X Y,Z> = <\phi\tilde{\nabla}_X Y,\phi Z> = <\tilde{\nabla}_X\phi Y - (\tilde{\nabla}_X\phi)Y,\phi Z> = <\sigma(X,\phi Y),\phi Z>$$

$$= <A_{\phi Z}X,Y> = 0$$

for any $X$ and $Y$ in $D$ and $Z$ in $D^{\perp}$, that is, the leaf $M^{\top}$ of $D$ is totally geodesic in $M$. Thus the submanifold $M$ is a contact *CR*-product.

Conversely, if the submanifold $M$ of a Sasakian manifold $\tilde{M}$ is a contact $CR$-product, then we have from (5.2)

$$A_{\phi W}X - \eta(X)W \in D \tag{6.3}$$

for any $X$ in $D$ and $W$ in $D^{\perp}$. So, it is sufficient to prove the following:

$$A_{\phi W}X - \eta(X)W \in D^{\perp} \tag{6.4}$$

for any $X$ in $D$ and $W$ in $D^{\perp}$. In fact, since the distribution $D$ is totally geodesic in $M$, we have

$$<A_{\phi W}X - \eta(X)W,Y> = <\sigma(X,Y),\phi W> = -<\phi\sigma(X,Y),W> = -<\phi(\tilde{\nabla}_X Y - \nabla_X Y),W>$$

$$= -<\phi\tilde{\nabla}_X Y,W> = <\nabla_X \phi Y,W> = 0$$

for any $X$ and $Y$ in $D$ and $W$ in $D^{\perp}$. This means (6.4). By virtue of (6.3) and (6.4), we have (6.1).

## 7. A CONTACT $CR$-PRODUCT OF A SASAKIAN MANIFOLD II.

In this section, we shall mainly study the second fundamental form of a contact $CR$-product.

Let $M$ be a contact $CR$-product of a Sasakian manifold $\tilde{M}$. In $M$, we shall calculate the $\tilde{H}_B(X,Z)$ for any unit vectors $X$ in $D$ and $Z$ in $D^{\perp}$, where $\tilde{H}_B(X,Z)$ is defined by

$$\tilde{H}_B(X,Z) = -<\tilde{R}(X,\phi X)Z,\phi Z>. \tag{7.1}$$

By virtue of (1.14) and (1.16), we get

$$<R(\tilde{X},\phi X)Z,\phi Z> = <\nabla^{\perp}_X\sigma(\phi X,Z),\phi Z> - <\nabla^{\perp}_{\phi X}\sigma(X,Z),\phi Z> - <A_{\phi Z}\nabla_X X,Z>$$

$$- <A_{\phi Z}X,\nabla_X Z> + <A_{\phi Z}\nabla_{\phi X}X,Z> + <A_{\phi Z}X,\nabla_{\phi X}Z>.$$

Since the leaves $M^{\top}$ and $M^{\perp}$ are both totally geodesic in $M$, we have

$$\nabla_U Y \in D \qquad \text{and} \qquad \nabla_U Z \in D^{\perp} \tag{7.2}$$

for any vector field $U$ tangent to $M$,$Y$ in $D$ and $Z$ in $D^{\perp}$. Thus we have from (6.1) and (7.2)

$$<\tilde{R}(X,\phi X)Z,\phi Z> = <\nabla^{\perp}_X\sigma(\phi X,Z),\phi Z> - <\nabla^{\perp}_{\phi X}\sigma(X,Z),\phi Z> - \eta(\nabla_X\phi X) + \eta(\nabla_{\phi X}X). \tag{7.3}$$

On the other hand, we obtain

$$\eta(\nabla_X \phi X) = <\nabla_X \phi X, \xi> = <\tilde{\nabla}_X \phi X, \xi> = -1 + \eta(X)^2.$$

So, (7.3) can be written as

$$<\tilde{R}(X,\phi X)Z,\phi Z> = <\nabla^{\perp}_X \sigma(\phi X,Z),\phi Z> - <\nabla^{\perp}_{\phi X}\sigma(X,Z),\phi Z> + 1 - \eta(X)^2 + \eta(\nabla_{\phi X} X). \tag{7.4}$$

Next, we have from (6.1)

$$<\sigma(X,W),\phi Z> = <\xi,X><Z,W>, \tag{7.5}$$

from which

$$<\sigma(X,Z),\phi Z> = <\xi,X>, \tag{7.6}$$

$$<\sigma(\phi X,Z),\phi Z> = 0. \tag{7.7}$$

Covariant differentiation of (7.6) and (7.7) along $\phi X$ and $X$ respectively give us

$$<\nabla^{\perp}_{\phi X}\sigma(X,Z),\phi Z> = -<\sigma(X,Z),\nabla^{\perp}_{\phi X}\phi Z> + <\nabla_{\phi X}\xi,X> + \eta(\nabla_{\phi X}X), \tag{7.8}$$

$$<\nabla^{\perp}_X \sigma(\phi X,Z),\phi Z> = -<\sigma(\phi X,Z),\nabla^{\perp}_X \phi Z>. \tag{7.9}$$

Substituting (7.8) and (7.9) into (7.4), we get

$$<\tilde{R}(X,\phi X)Z,\phi Z> = -<\sigma(X,Z),\nabla^{\perp}_{\phi X}\phi Z> + <\sigma(\phi X,Z),\nabla^{\perp}_X \phi Z> + 2(1 - \eta(X)^2). \tag{7.10}$$

By virtue of (3.9) and (6.1), we can calculate

$$<\sigma(\phi X,Z),\nabla^{\perp}_X \phi Z> - <\sigma(X,Z),\nabla^{\perp}_{\phi X}\phi Z> = -2\|\sigma(X,Z)\|^2 + 2\eta(X)^2. \tag{7.11}$$

Thus we have

PROPOSITION 7.1: In a contact *CR*-product of a Sasakian manifold, we have

$$\tilde{H}_B(X,Z) = 2(\|\sigma(X,Z)\|^2 - 1) \tag{7.12}$$

for any unit vectors $X$ in $D$ and $Z$ in $D^{\perp}$.

Especially, if the ambient manifold $\tilde{M}$ is a Sasakian space form $\tilde{M}(k)$, then we have from (1.15)

$$\tilde{R}(X,\phi X;\phi Z,Z) = \frac{k-1}{2}(1 - \eta(X)^2). \tag{7.13}$$

Thus we have

PROPOSITION 7.2: In a contact *CR*-product of a Sasakian space form $\tilde{M}(k)$, we have

$$\|\sigma(X,Z)\|^2 = \frac{k+3}{4} - \frac{k-1}{4}\eta(X)^2 \tag{7.14}$$

for any unit vectors $X$ in $D$ and $Z$ in $D^\perp$.

By virtue of PROPOSITION 3.4. and (7.14), we have

$$\left.\begin{aligned} &\|\sigma(X,Z)\|^2 = \frac{k+3}{4} \qquad \text{for } X \perp \xi, \\ &\|\sigma(\xi,Z)\|^2 = 1 \end{aligned}\right\} \tag{7.15}$$

Thus we have

COROLLARY 7.3: In a Sasakian space form $\tilde{M}(k)$ with constant $\phi$-holomorphic sectional curvature $k < -3$, there does not exist a contact $CR$-product of $\tilde{M}(k)$.

Next, we shall prove

THEOREM 7.4: Let $M$ be a contact $CR$-submanifold of a Sasakian space form $\tilde{M}(k)$. Then we have

$$\|\sigma\|^2 \geqq 2p(\frac{h(k+3)}{2} + 1), \tag{7.16}$$

where $p = \dim D^\perp$ and $2h = \dim D - 1$. If the equality sign of (7.16) holds, then $M^T$ and $M^\perp$ are both totally geodesic in $\tilde{M}(k)$.

PROOF: Let $A_1, A_2, \ldots, A_h, \phi A_1, \phi A_2, \ldots, \phi A_h, A_{2h+1}$ (= $\xi$) and $B_1, B_2, \ldots, B_p$ be orthogonal basis of $D_x$ and $D_x^\perp$, respectively. Then $\|\sigma\|^2$ is given by

$$\|\sigma\|^2 = \sum_{i,j=1}^{2h+1} \|\sigma(A_i,A_j)\|^2 + 2\sum_{i=1}^{2h+1}\sum_{\alpha=1}^{p} \|\sigma(A_i,B_\alpha)\|^2 + \sum_{\alpha,\beta=1}^{p} \|\sigma(B_\alpha,B_\beta)\|^2.$$

By virtue of (7.15), the above equation can be written as

$$\|\sigma\|^2 = 2p(\frac{h(k+3)}{2}) + 1) + \sum_{i,j=1}^{2h+1} \|\sigma(A_i,A_j)\|^2 + \sum_{\alpha,\beta=1}^{p} \|\sigma(B_\alpha,B_\beta)\|^2. \tag{7.17}$$

From the above equation, we have our theorem.

## 8. TOTALLY UMBILICAL CONTACT $CR$-SUBMANIFOLDS.

Let $M$ be a totally umbilical contact $CR$-submanifold of a Sasakian manifold $\tilde{M}$. Then by definition we have

$$\sigma(U,V) = \langle U,V\rangle H \tag{8.1}$$

for any vector fields $U$ and $V$ tangent to $M$. By virtue of (1.6), we can write

$$\phi H = tH + fH. \tag{8.2}$$

Since the vector field $tH$ is in $D^\perp$, we have from (3.8)

$$A_{\phi tH}W = A_{\phi W}tH \tag{8.3}$$

for any $W$ in $D^{\perp}$. From this, we obtain

$$-\langle W,W\rangle\langle tH,tH\rangle = \langle tH,W\rangle\langle H,\phi W\rangle. \tag{8.4}$$

We assume that $\dim D^{\perp} \geqq 2$. Then we can put $W$ as the orthogonal vector field of $tH$. The equation (8.4) means

$$tH = 0 \tag{8.5}$$

Next, let $Q_1$ and $Q_2$ be the projections of $TM$ to $D$ and $D^{\perp}$, respectively. Then for any vector field $U$ tangent to $M$ we can put

$$U = Q_1U + Q_2U \tag{8.6}$$

and

$$\phi Q_1U \in D, \qquad \phi Q_2U \in \phi D^{\perp} \subset T^{\perp}M. \tag{8.7}$$

The equation (8.7) and the covariant differentiation of $\phi\lambda = t\lambda + f\lambda$ teach us

$$-\phi Q_1A_{\lambda}U = Q_1\nabla_U t\lambda - Q_1A_{f\lambda}U \tag{8.8}$$

for any vector field $U$ tangent to $M$ and any vector field $\lambda$ normal to $M$. In (8.8), if we put $\lambda = H$ and taking account of (8.5), we have

$$\phi Q_1A_HU = Q_1A_{\phi H}U \tag{8.9}$$

for any vector field $U$ tangent to $M$. For any $X$ in $D$ and any vector field $U$ tangent to $M$, we get

$$\langle Q_1A_{\phi H}U,X\rangle = \langle A_{\phi H}U,X\rangle = \langle\sigma(U,X),\phi H\rangle = \langle U,X\rangle\langle H,\phi H\rangle = 0,$$

$$\langle\phi Q_1A_HU,X\rangle = -\langle Q_1A_HU,\phi X\rangle = -\langle A_HU,\phi X\rangle = -\langle\sigma(U,\phi X),H\rangle = -\langle U,\phi X\rangle\langle H,H\rangle.$$

By virtue of (8.9) and the above two equations, we have

$$\langle H,H\rangle\langle U,\phi X\rangle = 0 \tag{8.10}$$

for any vector field $U$ tangent to $M$ and any $X$ in $D$. We assume that $\dim D_x \geqq 2$ and if we take $U = \phi X$ such that the vector field $X$ is orthogonal to $\xi$, we have from (8.10) $H = 0$. Thus we have

THEOREM 8.1: Let $M$ be a totally umbilical contact $CR$-submanifold of a Sasakian manifold $\tilde{M}$. We assume that $\dim D_x \geqq 2$ and $\dim D_x^{\perp} \geqq 2$. Then the submanifold $M$ is totally geodesic in $\tilde{M}$.

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