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A HILLE-WINTNER TYPE COMPARISON THEOREM FOR SECOND ORDER DIFFERENCE EQUATIONS

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ABSTRACT. For the linear difference equation

$$\Delta(c_{n-1}\,\Delta\, x_{n-1}) + a_n x_n = 0 \quad \text{with } c_n > 0,$$

a non-oscillation comparison theorem given in terms of the coefficients c_n and the series $\sum_{n=k}^{\infty} a_n$, has been proved.

KEY WORDS AND PHRASES. Difference equations, oscillatory and non-oscillatory solutions, comparison theorems, Riccati transformation.

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1. INTRODUCTION.

We consider linear homogeneous second order difference equations of the form

$$\Delta(c_{n-1}\,\Delta\, x_{n-1}) + a_n x_n = 0, \quad n = 1,2,3,\ldots, \tag{1.1}$$

where Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$, and $a = \{a_n\}$ and $c = \{c_n\}$ are real-valued infinite sequences with $c_n > 0$ for $n = 0,1,2,\ldots$. (No assumption is made about the sign of a_n.)

Equation (1.1) is equivalent to the difference equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \tag{1.2}$$

where $b_n = c_n + c_{n-1} - a_n$, $n = 1,2,3,\ldots$. Recent papers ([1],[2], and [3]) have treated oscillation and comparison theorems for this equation.

The theorem to be considered here is a difference equation analogue of Taam's generalized version [4] of the well-known Hille-Wintner comparison theorem for second-order linear differential equations (see [5, Thm. 7, p. 245] and [6], or [7, p. 60-62]).

Let $x = \{x_n\}$, $n = 0,1,2,\ldots$, be a real, non-trivial solution of (1). Then x is said to be oscillatory if, for every N, there exists $n \geqq N$ such that $x_n x_{n+1} \leqq 0$. Since either all non-trivial real solutions of (1.1) are oscillatory or none are (see [8, p. 153]), equation (1.1) may be classified as oscillatory or non-oscillatory. Also, if x is a solution of (1.1), so is -x, and it then follows that (1.1) is non-oscillatory if and only if there exists a solution x with $x_n > 0$ for all $n \geqq N$, for some integer $N \geqq 0$. (The variables j, k, n, M, N will always be understood below to represent non-negative integers.)

2. MAIN RESULTS.

We will prove the following comparison result:

THEOREM 1. Given the difference equations

$$\Delta(C_{n-1} \Delta x_{n-1}) + A_n x_n = 0 \tag{2.1}$$

$$\Delta(c_{n-1} \Delta x_{n-1}) + a_n x_n = 0, \tag{2.2}$$

assume that

$$0 < C_n \leqq c_n \quad \text{and} \quad C_n \leqq K \tag{2.3}$$

for all $n \geqq 0$, for some constant $K > 0$, and

$$0 \leqq \sum_{n=k}^{\infty} a_n \leqq \sum_{n=k}^{\infty} A_n < \infty \tag{2.4}$$

for all sufficiently large k. Then, if (2.1) is non-oscillatory, (2.2) is non-oscillatory also.

Before proceeding to the proof of Theorem 1, we need two preliminary results. The first of these is an elementary property of real numbers:

LEMMA 1. If $0 \leqq a \leqq b$ and $c > 0$, then

$$0 \leq \frac{a^2}{a+c} \leq \frac{b^2}{b+c} \quad .$$

Our second lemma may be thought of as a discrete analogue of Theorem 4 of Hille [5, p. 243], which gives a necessary and sufficient condition for non-oscillation of solutions of $x'' + f(t)x = 0$ in terms of the existence of a solution of a related Riccati integral equation. Hille then used a successive approximations technique to show the existence of a solution of the integral equation. We will use a similar device here to prove Theorem 1. Our proof of Lemma 2 is a discrete version of a standard Riccati transformation argument, as used, for example, in the proof of the Hille-Wintner theorem presented by Swanson [7]. The resulting difference equation (2.7) below is quite different in form, however, from the Riccati differential equation.

LEMMA 2. Assume that

$$\sum_{n=1}^{\infty} A_n < \infty.$$

Then the difference equation (1.2) is non-oscillatory if and only if there exists a sequence v satisfying

$$v_k = \sum_{n=k}^{\infty} \frac{v_n^2}{v_n + C_n} + \sum_{n=k}^{\infty} A_{n+1} \tag{2.5}$$

for all sufficiently large k, say $k \geq M$, with $v_k + C_k > 0$ for $k \geq M$.

PROOF. Let (2.1) be non-oscillatory and let x be a solution of (2.1) with $x_n > 0$ for $n \geq M$. Let $v_n = C_n(\Delta x_n)/x_n$, $n \geq M$. Then

$$v_n = C_n \left(\frac{x_{n+1} - x_n}{x_n}\right) = C_n \left(\frac{x_{n+1}}{x_n} - 1\right) > -C_n,$$

so $v_n + C_n > 0$, $n \geq M$. From (2.1) we have

$$C_{n+1} \Delta x_{n+1} - C_n \Delta x_n + A_{n+1} x_{n+1} = 0.$$

Dividing by x_{n+1} and adding and subtracting v_n, one obtains

$$v_{n+1} - v_n + v_n - v_n x_n/x_{n+1} + A_{n+1} = 0,$$

or

$$\Delta v_n + v_n(1 - x_n/x_{n+1}) + A_{n+1} = 0. \tag{2.6}$$

Now

$$\frac{x_{n+1}}{x_n} = \frac{x_{n+1} - x_n + x_n}{x_n} = \frac{\Delta x_n}{x_n} + 1 = \frac{v_n}{C_n} + 1,$$

and substitution of this expression into (2.6) yields

$$\Delta v_n + \frac{v_n^2}{v_n + C_n} + A_{n+1} = 0, \; n \geqq M. \tag{2.7}$$

Summing from k to N, where $M \leqq k < N$, we obtain

$$v_{N+1} - v_k + \sum_{n=k}^{N} \frac{v_n^2}{v_n + C_n} = - \sum_{n=k}^{N} A_{n+1}. \tag{2.8}$$

By hypothesis, the right side of (2.8) has a finite limit as $N \to \infty$, so the left side also has such a limit.

As noted above, $v_n + C_n > 0$ for $n \geqq M$, so $v_n^2/(v_n + C_n) \geqq 0$ for $n \geqq M$. If $\sum_{n=k}^{\infty} v_n^2/(v_n + C_n) = + \infty$, then, from (2.8), $v_{N+1} \to -\infty$ as $N \to \infty$. But this is impossible, since $v_n \geqq -C_n$, and $-C_n \geqq -K$ by hypothesis. Thus, for every $k \geqq M$, we have

$$0 \leqq \sum_{n=k}^{\infty} \frac{v_n^2}{v_n + C_n} < \infty.$$

Therefore, $v_n^2/(v_n + C_n) \to 0$ as $n \to \infty$, from which it follows, since C_n is bounded, that $v_n \to 0$ as $n \to \infty$. Equation (2.5) then follows immediately from (2.8).

Conversely, if v is a sequence satisfying (2.5), with $v_n + C_n > 0$ for $n \geqq M$, then application of the forward difference operator Δ to both sides of (2.5) leads immediately to equation (2.7). We then define a sequence x inductively as

$$x_M = 1, \quad x_{n+1} = \left(\frac{v_n + C_n}{C_n}\right)x_n, \; n \geqq M.$$

Then $x_n > 0$ for $n \geqq M$, and $v_n = C_n(x_{n+1}/x_n - 1)$; hence, $v_n = C_n \, \Delta x_n/x_n$. Substitution of this expression into equation (2.7) then leads readily to equation (2.1), so x_n as defined above satisfies (2.1) for $n \geqq M$. We may then define $x_{M-1}, x_{M-2}, \ldots, x_0$ successively, using (2.1). The resulting sequence x is thus a non-oscillatory solution of (2.1), which completes the proof of Lemma 2.

Proceeding with the proof of Theorem 1, we assume that (2.1) is non-oscillatory. Then by Lemma 2 there exists a sequence $V = \{V_k\}$, $k \geqq M$, for some $M \geqq 0$, which satisfies (2.5), with $V_k + C_k > 0$ for $k \geqq M$. We will use a successive approximations argument to show that there is a sequence $v = \{v_k\}$, $k \geqq M$, which satisfies

$$v_k = \sum_{n=k}^{\infty} \frac{v_n^2}{v_n + c_n} + \sum_{n=k}^{\infty} a_{n+1}, \quad k \geqq M. \tag{2.9}$$

It will then follow by Lemma 2 that equation (2.2) is non-oscillatory.

We define a sequence of successive approximations

$v^j = \{v_k^j\}$, $k \geqq M$, $j \geqq 0$, as follows:

$$\begin{aligned} v_k^0 &= V_k, \quad k \geqq M, \\ v_k^j &= \sum_{n=k}^{\infty} \frac{(v_n^{j-1})^2}{v_n^{j-1} + c_n} + \sum_{n=k}^{\infty} a_{n+1}, \quad k \geqq M, \ j \geqq 1. \end{aligned} \tag{2.10ab}$$

We must first show that the sequences v^j, $j \geqq 1$, are well-defined by (2.10ab). Since $v_k^0 = V_k$, $k \geqq M$, we have

$$v_k^0 + c_k \geqq v_k^0 + C_k = V_k + C_k > 0, \quad k \geqq M.$$

Then

$$0 \leqq \frac{(v_k^0)^2}{v_k^0 + c_k} \leqq \frac{(v_k^0)^2}{v_k^0 + C_k} = \frac{V_k^2}{V_k + C_k}, \quad k \geqq M. \tag{2.11}$$

Since the sequence V satisfies (2.5), it follows from (2.11) that the series in (2.10ab) converges for $j = 1$, for all $k \geqq M$. Therefore v_k^1 is well-defined by (2.10ab) for each $k \geqq M$. Furthermore, by (2.4), (2.10ab), and (2.11), we have

$$0 \leqq v_k^1 \leqq \sum_{n=k}^{\infty} \frac{(V_n)^2}{V_n + C_n} + \sum_{n=k}^{\infty} A_{n+1} = V_k = v_k^0, \quad k \geqq M;$$

i.e., $0 \leqq v_k^1 \leqq v_k^0$, $k \geqq M$.

Proceeding inductively, we assume that v_k^j, $k \geq M$, has been defined by (2.10ab) for $j = 1,2,3,\ldots,i$ and that $0 \leq v_k^j \leq v_k^{j-1}$ for $k \geq M$, $j = 1,2,3,\ldots,i$. Then $v_k^i + c_k \geq c_k > 0$, $k \geq M$. Using Lemma 1, we then obtain

$$0 \leq \sum_{n=k}^{\infty} \frac{(v_n^i)^2}{v_n^i + c_n} \leq \sum_{n=k}^{\infty} \frac{(v_n^{i-1})^2}{v_n^{i-1} + c_n}, \ k \geq M. \tag{2.12}$$

It follows from (2.12) that v_k^{i+1} is well-defined by (2.10ab) for all $k \geq M$, and from (2.10ab) and (2.12) we have

$$0 \leq v_k^{i+1} \leq v_k^i, \ k \geq M.$$

Therefore, by induction, v_k^j is defined by (2.10ab) for all $j \geq 1$ and $k \geq M$, and

$$0 \leq v_k^j \leq v_k^{j-1}, \ j \geq 1, \ k \geq M. \tag{2.13}$$

Thus v_k^j is non-negative and non-increasing in j for each $k \geq M$ and we may define

$$v_k = \lim_{j\to\infty} v_k^j, \ k \geq M.$$

Note that $v_k \geq 0$, so that $v_k + c_k \geq c_k > 0$, $k \geq M$. Also, from (2.13) and Lemma 1, we obtain

$$0 \leq \frac{(v_k^j)^2}{v_k^j + c_k} \leq \frac{(v_k^{j-1})^2}{v_k^{j-1} + c_k}, \ j \geq 1, \ k \geq M. \tag{2.14}$$

Repeated application of (2.14) yields, with (2.13),

$$0 \leq \frac{(v_k^j)^2}{v_k^j + c_k} \leq \frac{(V_k)^2}{V_k + C_k} \qquad \text{for all } j \geq 0.$$

Thus the convergence of the first series in (2.10b) is uniform with respect to j. Consequently, we may take limits in (2.10ab) as $j \to \infty$ and obtain equation (2.9). It then follows from Lemma 2 that equation (2.2) is non-oscillatory, which completes the proof of the theorem.

Since several recent discussions of oscillation of solutions of difference equations have treated equation (1.2) above, we restate theorem 1 for equations of the form (1.2).

THEOREM 2. Given the difference equations

$$C_n x_{n+1} + C_{n-1} x_{n-1} = B_n x_n \tag{2.15}$$

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \tag{2.16}$$

assume that $0 < C_n \leqq c_n$ and $C_n \leqq K$, $n = 0,1,2,\ldots$, for some constant $K > 0$. If

$$0 \leqq \sum_{n=k}^{\infty} (c_n + c_{n-1} - b_n) \leqq \sum_{n=k}^{\infty} (C_n + C_{n-1} - B_n) < \infty$$

for all sufficiently large k, then, if (2.15) is non-oscillatory, (2.16) is non-oscillatory also.

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