MODIFIED GAUSS-LEGENDRE, LOBATTO AND RADAU CUBATURE FORMULAS FOR THE NUMERICAL EVALUATION OF 2-D SINGULAR INTEGRALS

P.S. THEOCARIS

Department of Theoretical and Applied Mechanics
The National Technical University of Athens
5, Heroes of Polytechnion Avenue,
Zographou, Athens 624, Greece

ABSTRACT   A numerical technique, first reported in 1979 in refs.[1] and [2], for the

numerical evaluation of two-dimensional *Cauchy-type* principal-value integrals, is

extended in this paper to include several cubature formulas of the *Radau* and *Lobatto*

types. For the construction of such a cubature formula the 2-D singular integral is

considered as an iterated one, and the second-order pole involved in this integral is

analyzed into a pair of complex poles. Based on this procedure, the methods of

numerical integration, valid for one-dimensional singular integrals, are extended to

the case of two-dimensional singular integrals. The cubature formulas of the *Lobatto*-

and *Radau*-type are now formulated to include the cases where some of the desired

abscissas may be chosen according to any appropriate criterion.

Moreover, the theory developed is enlarged to include the case of a 2-D

principal-value integral, containing a logarithmic singularity. The validity of the

results is illustrated by considering certain numerical examples. Furthermore, a

complete analysis of the convergence and the construction of error estimates is also

presented.

KEY WORDS AND PHRASES. *Numerical integration, Numerical evaluation of 2-D Singular*

Integrals, Cubature Formulas of the Radau and Lobatto types, Collocation points,

Error estimates for 2-D Principal Value Integrals.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 65D32, 65D30, 65G99, 41A55.

1. INTRODUCTION

Two-dimensional principal-value integrals are frequently encountered in engineering problems and especially in the three-diemensional theory of elasticity [3 to 5]. Yet, the methods for their numerical evaluation existing in the literature [3,6,7] are not general and efficient enough to be of a wide use. One such method, developed by *Cruse* [4], can be considered as the basis of the well-known *Boundary-Integral Equation* (BIE) *technique*. It should be noticed that, for the numerical evaluation of these integrals by using the BIE-method, the surface on which the integration should be performed must be discretized into infinitesimal flat pieces and the surface data of the density function should be assumed constant on each of them. On the other hand, a method based on the direct numerical integration [5,8] of the two-dimensional Cauchy principal-value integral was not generally accurate enough, thereby requiring an analytical attack on the integral.

Similar methods to the BIE-technique were proposed in refs.[7] and [8], where the numerical evaluation of two-dimensional Cauchy-type principal-value integrals on a circular region was considered, when the singularity was at the center of the circle. The results of [6] and [7] were generalized in [9] for the case of an arbitrary region.

This paper presents a general method for the numerical evaluation of two-dimensional principal-value integrals by constructing a cubature formula for their estimation. For the formation of such a cubature formula the two-dimensional integral is considered as an iterated one, and the involved singularity is analyzed by reduction into a pair of complex singularities [1,2]. Based on this fact, the methods of numerical integration, developed for one-dimensional singular integrals [9 to 10], are extended to the cases of two-dimensional singular integrals by introducing a properly estimated set of collocation points.

Moreover, a typical integral, which was previously evaluated by *Cruse* [4] in a different way, is again determined by using the proposed numerical technique. The coincidence of the results by the two methods is satisfactory.

On the other hand, the *Radau* and *Lobatto* integration rules are established for the numerical calculation of 2-D singular integrals. Several integrals, also

a logarithmic singularity, are calculated while special attention is given to the investigation of the convergence of the proposed method by obtaining estimate for the related error.

2. MODIFIED GAUSS-LEGENDRE CUBATURE FORMULAS.

Consider the two-dimensional singular integral $I(\xi_0, \eta_0)$, defined on a plane region S (Fig.1, [9]):

$$I(\xi_0, \eta_0) = \int_S w(\xi, \eta) \frac{f(\xi_0, \eta_0, \vartheta)}{r^2} u(\xi, \eta) d\xi d\eta \quad, \tag{2.1}$$

where:

$$(\xi - \xi_0) + i(\eta - \eta_0) = re^{i\vartheta} \quad, \tag{2.2}$$

and $w(\xi, \eta)$ is a weight function of the form

$$w(\xi, \eta) = w_1(\xi) w_2(\eta) \quad, \tag{2.3}$$

where the functions w_1 and w_2 are included in *Table I*.

Under the assumptions that:

i) the density function $u(\xi, \eta)$ of the singular integral $I(\xi_0, \eta_0)$ is a bounded and *Hölder* continuous function is S;

ii) if the surface has points at infinity, then $u(\xi, \eta) = \theta(r^{-k})(k>0)$ and;

iii) the characteristic function $f(\xi_0, \eta_0, \vartheta)$ is bounded and for a fixed pole $X(\xi_0, \eta_0)$, is continuous with respect to ϑ, *Tricomi* [11] has shown that the necessary and sufficient condition for the existence of the singular integral [2.1], in the principal value sense, is that is characteristic $f(\xi_0, \eta_0, \vartheta)$ satisfies the condition [12]

$$\int_0^{2\pi} f(\xi_0, \eta_0, \vartheta) d\vartheta = 0 \quad. \tag{2.4}$$

Let us now consider a square area T of sides equal to 2 units, belonging to S, which surrounds the second order pole $X(\xi_0, \eta_0)$. Then, it is easy to show that the singular integral I becomes

$$I(\xi_0, \eta_0) = \int_{S-T} w(\xi, \eta) \frac{f(\xi_0, \eta_0, \vartheta)}{r^2} u(\xi, \eta) d\xi d\eta + I_1(\xi_0, \eta_0) \quad, \tag{2.5}$$

where

$$I(\xi_0, \eta_0) = \oint_T w(\xi, \eta) \frac{f(\xi_0, \eta_0, \vartheta)}{r^2} u(\xi, \eta) d\xi d\eta .$$

(2.6)

Here, the first integral on the right-hand side of Eq.(2.5) is a regular one, while the second (that is the integral I_1) must be considered in a principal-value sense.

Integrals of the form (2.6) are frequently encountered in the 3-dim theory of elasticity [3], where the characteristic $f(\xi_0, \eta_0, \vartheta)$ satisfies the condition (2.4). Then, according to reference [12], the following expression is valid:

$$I_1(\xi_0, \eta_0) = \int_{-1}^{1} w_1(\xi) \frac{(I_A - I_B)}{\xi - \xi_0} d\xi ,$$

(2.7)

where:

$$I_A = \int_{-1}^{1} w_2(\eta) \frac{f(\xi_0, \eta_0, \vartheta)}{\eta - z} u(\xi, \eta) d\eta ,$$

(2.8)

and:

$$I_B = \int_{-1}^{1} w_2(\eta) \frac{f(\xi_0, \eta_0, \vartheta)}{\eta - \bar{z}} u(\xi, \eta) d\eta ,$$

(2.9)

with:

$$z = \eta_0 + i(\xi - \xi_0) .$$

(2.10)

First, we note that Eqs.(2.8) and (2.9) can be approximated by quadratures of the form [10]:

$$\ddot{I}_A \simeq \sum_{k=1}^{n} A_k \frac{f(\xi_0, \eta_0, \xi, \eta_k)}{\eta_k - \eta_0} u(\xi, \eta_k) - 2f(\xi_0, \eta_0, \xi) u(\xi, \eta_0) K_n(\eta_0) ,$$

(2.11)

(if $\eta_0 \neq \eta_k$ $(k = 1, 2, \ldots, n)$)

and:

$$I_B \simeq \sum_{\substack{k=1 \\ k \neq m}}^{n} A_k \frac{f(\xi_0, \eta_0, \xi, \eta_k)}{\eta_k - \eta_0} u(\xi, \eta_k) + A_m \frac{d[f(\xi_0, \eta_0, \xi, \eta) u(\xi, \eta)]}{d\eta} \bigg|_{\eta = \eta_0} - 2f(\xi_0, \eta_0, \xi) u(\xi, \eta_0) \Lambda_n(\eta_0),$$

(2.12)

(if $\eta_0 = \eta_m$ $(k = 1, 2, \ldots, n)$) ,

with:

$$K_n(t) = \frac{q_n(t)}{\sigma_n(t)} , \quad (\eta_0 \neq \eta_k) ,$$

(2.13)

$$\Lambda_n(t) = \frac{1}{\sigma'_t(t)}\left[q'_n(t) + \frac{1}{4}A_m\sigma''_n(t)\right] \; , \; (\eta_0 = \eta_m) \; . \tag{2.14}$$

The functions $K_n(t)$, $\Lambda_n(t)$ are systematically tabulated in ref.[12], according to the various numerical integration rules.

At the same time the principal value I_1 can be approximated by:

$$I_1 \simeq \sum_{\ell=1}^{m} A_m \frac{(\tilde{I}_A(\xi_0,\eta_0,\xi_m)-\tilde{I}_B(\xi_0,\eta_0,\xi_m))}{\xi_m-\xi_0} - 2(\tilde{I}_A(\xi_0,\eta_0)-\tilde{I}_B(\xi_0,\eta_0))K_n(\xi_0) \; , \tag{2.15}$$

where the functions $\tilde{I}_A$, $\tilde{I}_B$ are given by the expressions (2.11) and (2.12).

Let us consider now the special case where:

$$w(\xi,\eta) = 1 \; , \; (\textit{Gauss-Legendre} \text{ numerical integration rule}), \tag{2.16}$$

and:

$$f(\xi_0,\eta_0,\vartheta) = \frac{\partial r}{\partial \xi} = \cos\vartheta \; . \tag{2.17}$$

Then, the singular integral $I_1(\xi_0,\eta_0)$ becomes:

$$I_1 = \int_T \frac{\partial r/\partial \xi}{(\xi-\xi_0)^2+(\eta-\eta_0)^2}u(\xi,\eta)d\xi d\eta \; , \tag{2.18}$$

Moreover, the quadrature (2.15) may be rewritten as:

$$I_1 \simeq \int_{-1}^{1} \frac{\text{Im}(I_2(\xi_0,\eta_0,\eta))}{\eta-\eta_0}d\eta \; , \tag{2.19}$$

where $\text{Im}(I_2(\xi_0,\eta_0,\eta))$ denotes the imaginary part of the complex function:

$$I_2(\xi_0,\eta_0,\eta) = \int_{-1}^{1}u(\xi,\eta)d(r/\xi-z)+\int_{-1}^{1}\frac{r}{(\xi-z)^2}u(\xi,\eta)d\xi \; , \tag{2.20}$$

It is already well established [12] that, if η_0 is selected as a root of the *Legendre* function of the second kind and order m given by:

$$Q_m(\eta_0) = 0 \; , \tag{2.21}$$

a general method for the numerical evaluation of the integral (2.18) can be readily established. This can be achieved by making one-dimensional numerical integrations (that is by using product rules [11]). Finally, it was concluded that:

$$I_1(\xi_0,\eta_0) = \sum_{k=1}^{m} A_k \frac{\text{Im}(I_2(\xi_0,\eta_0,\eta_k))}{(\eta_k-\eta_0)} \; , \tag{2.22}$$

where η_k are the abscissas and A_k the corresponding weights of the numerical

integration rule. Applying this, it may be observed that the following improper

integral is needed to be calculated numerically:

$$I_3(\xi_0,\eta_0) = \int_{-1}^{1} \frac{u(z,\eta)Q_n(z)}{P_n(z)|\eta-\eta_0|}d\eta \ ,\tag{2.23}$$

where $Q_n(z)$ denotes the *Legendre* function of the second kind and order n, whereas

$P_n(z)$ expresses the *Legendre* polynomial of degree n. The existence of I_2 in an

ordinary sense requires that:

$$Q_n(\xi_0) = 0 \ ,\tag{2.24}$$

where n denotes the number of abscissas used for the numerical evaluation of the

integral I_2, and ξ_0 must be selected as a root of the *Legendre* function of the second

kind of order n.

Therefore, the pair of real numbers (ξ_0,η_0), which are considered as a root of

the *Legendre* function of the second kind of order (n,m) respectively, defines a

collocation point $X(\xi_0,\eta_0)$. The roots of the *Legendre* function of the second kind

are systematically calculated and tabulated in ref.[10].

Finally it should be mentioned that, if the cubature formula (2.22) is applied

to a set of points (ξ_0,η_0) different than the corresponding collocation points, the

resulting numerical values should be invalid, and usually diverge. The case where

$f(\xi_0,\eta_0,\vartheta)=\partial r/\partial\eta$ or $e^{in\vartheta}$ can be treated analogously.

3. APPLICATION OF THE CUBATURE FORMULAS

As an application of the above developments the following two-dimensional

Cauchy-type principal-value integral was evaluated:

$$I_4 = \oint_{T} \frac{\cos\vartheta}{r^2}ds \ ,\tag{3.1}$$

where the variables are defined again by Eq.(2.2) and T denotes the square with

$-1<\xi,\eta<1$.

On the basis of the results of Cruse [4], this integral was evaluated in a closed

form as:

$$I_4 = \ell n \frac{\left\{1-\eta_0+[(1+\xi_0)^2+(1-\eta_0)^2]^{1/2}\right\}\left\{-1-\eta_0+[(1-\xi_0)^2+(1+\eta_0)^2]^{1/2}\right\}}{\left\{-1-\eta_0+[(1+\xi_0)^2+(1+\eta_0)^2]^{1/2}\right\}\left\{1-\eta_0+[(1-\xi_0)^2+(1-\eta_0)^2]^{1/2}\right\}}\tag{3.2}$$

On the other hand, the integral I_4 was evaluated numerically by applying the above

suggested *Gauss-Legendre* method. The results are given in Table II for several collocation points $X_i(\xi_0, \eta_0)$.

It is obvious from this table that, by increasing the orders (m×n) of the *Legendre* polynomials, the values for I_4 converge fast and smoothly towards their corresponding theoretical values derived on the basis of Eq.(3.2)

More precisely, in Table II, the numerically estimated values for $I_4(\xi_0, \eta_0)$ are given for an arbitrarily selected set of collocation points, that is for the points:

$$X_1(0.36062\ 31751\ ,\ 0.36062\ 31751)\ ,$$
$$X_2(0.37149\ 22970\ ,\ 0.54039\ 34260)\ ,$$
$$X_3(0.54794\ 77112\ ,\ 0.95094\ 46082)\ ,$$

The coordinates of the collocation points X_1, X_2, X_3 are selected in both cases to be the roots of the *Legendre* polynomials of the second kind of orders m=n=8, 16 and 32 respectively [10].

Comparing the numerical values of the integral I_4 with their corresponding exact values given by relation (3.2), it may be implied that a rapid convergence exists. Indeed, the results presented in Table II seem to vconverge rapidly to their corresponding theoretical values, that is, for the case where 8×8 abscissae are used, the values of I_4 was found to be within 29 percent of the exact value, while for the 16×16 case the discrepancy between these two values decreases to within 6 percent and finally, for the 32×32 case it becomes only 0.3 percent.

Finally, it is worth mentioning that the rate of convergence of the suggested numerical integration rule was mainly affected by the oscillatory behavior of the characteristic of the two-dimensional Cauchy principal-value integral I_4. In the present case the characteristic function coincides with the function cosθ.

4. NUMERICAL EVALUATION OF 2-D PRINCIPAL VALUE INTEGRALS WITH A LOGARITHMIC

   SINGULARITY

The theory developed for the numerical evaluation of 2-D principal-value integrals, containing an algebraic singularity, is extended to include the case of Cauchy principal-value integrals involving a *logarithmic singularity*. Although such singular integrals are hardly found in physical problems, the derivation of a

corresponding cubature formula is motivated by the fact that these integrals are a natural extension of the 2-D Cauchy principal-value integrals involving an algebraic singularity, extensively examined in the previous paragraph. We generalize now the cubature formula (2.15), which is valid for two-dimensional singular integrals, to singular integrals of the following form, presenting a logarithmic singularity:

$$I_5(x_0,y_0) = \int_S w_0(x,y) \frac{f_0(x_0,y_0,\vartheta)}{(x-x_0)^2+(y-y_0)^2} u_0(x,y)dS \ ,$$

$$(x_0,y_0) \in S \ , \tag{4.1}$$

on a plane finite region S (Fig.1), where,

$$(x-x_0)+i(y-y_0) = re^{i\vartheta} \ , \tag{4.2}$$

while the logarithmic density function is expressed by

$$w_0(x,y) = \frac{\ln^m\left\{(x-x_0)^2+(y-y_0)^2\right\}}{\left\{(x-x_0)^2+(y-y_0)^2\right\}^{\lambda/2}} \ ,$$

$$(m = 1,2,3,\ldots) \ , \ \lambda \geq 1 \ , \tag{4.3}$$

Moreover, $I_5(x_0,y_0)$ can be written as:

$$I_5 = \int_S w(r,\vartheta)\frac{f(\vartheta)}{r^2}u(r,\vartheta)dS \ ,$$

with:

$$\tag{4.4}$$

$$f(\vartheta) = f_0(x_0,y_0,\vartheta) \ , \ u(r,\vartheta) = u_0(x,y) \ , \ w(r,\vartheta) = w_0(x,y) \ .$$

In the following it was assumed that the functions u and f satisfy the same conditions as previously stated (see Eq.(2.4)). On the other hand, the boundary of S (Fig.1) can be described by:

$$R = R(\vartheta) \ , \ \vartheta \in [0,2\pi] \ , \tag{4.5}$$

where R is considered as a single-valued, although the case of multivaluedness does not introduce any further considerable difficulties.

We shall now derive the cubature formula for the 2-D singular integral I_5 in accordance with relations (4.4) to (4.5) and reference [9]:

$$I_5 = \int_0^{2\pi} f(\vartheta)d\vartheta \int_0^{R(\vartheta)} w(r,\vartheta)\frac{u(r,\vartheta)}{r}dr \ . \tag{4.6}$$

Equation (4.6) suggests the possibility of considering I_5 as a product of two one-

dimensional integrals, by applying for the outer integral a product rule (i.e the trapezoidal rule with L abscissas), while for the inner integral the already known finite-part integration quadratures [13]:

$$I_6 = \frac{2\pi}{L}\sum_{i=0}^{L-1}\left(R\left(\frac{2\pi i}{L}\right)\right)^{1-\lambda}\left\{\sum_{j=0}^{s}(x_j)\ell n^{(s-j)}\left[R\left(\frac{2\pi i}{L}\right)\right]^{(s-j)} + \sum_{k=1}^{N}{}_jw_k u\left[R\left(\frac{2\pi i}{L}\right)\right]\right] + $$

$$+ \frac{\ell n^{(s+1)}\left(R\left(\frac{2\pi i}{L}\right)\right)}{(s+1)(\lambda-1)!}\ \frac{d^{(\lambda-1)}\left[u\left(0,\frac{2\pi i}{L}\right)\right]}{dx^{(\lambda-1)}}\right\} \ . \qquad (4.7)$$

We observe that, if j=0 in the double sum, the corresponding weights ${}_0w_k$ are those of the quadrature formula for finite-part integrals involving an algebraic singularity and which are tabulated in [13]. The weights ${}_jw_k$ for j≠0 are also included in [13]. Finally x_j denote the equally spaced nodes $x_j=(j-1)/N$, (j=1,2,...,N).

As an application of the above developments the following 2-D Cauchy principal-value integral was determined:

$$I_7 = \int_S w(\xi,\eta)\frac{\cos\theta}{r^2}u(\xi,\eta)d\xi d\eta \ , \qquad (4.8)$$

where:

$$w(\xi,\eta) = \ell n^m(r)/r^\lambda \ , \quad (\xi-\xi_0)+i\eta = re^{i\theta} \ , \qquad (4.9)$$

The numerical results are presented in *Table III* for various positions ξ_0 of the second order pole $X(\xi_0,\eta_0)$ inside the unit disk (Fig.2) with a weight function of the form (4.9), (m=1, λ=1/2) or (m=2, λ=1) and a density function $u(\xi,\eta)=\xi$ or $u(\xi,\eta)=\exp(\xi)$.

The total error of the cubature formula (4.7) consists of one term depending upon the trapezoidal rule which is applied around the angle θ (the number of abscissas of this rule is denoted by the integer L), while the other one term is that of the finite-part integration rule, which is applied along the radius r of the unit disk of Fig.2 (the number of abscissas of this rule is denoted by the integer N).

In order to check the rate of convergence of the proposed cubature (4.7), as L increases, we consider the special case of the density function $u(\xi,\eta)=\xi$ for which the error term belonging to the finite-part integration is zero (or equivalently independent of N) (Table III(a)). By observing this Table we can see that as L

increases from L=18 to L=36 or L=72 four additional digits of the numerical results are fixed (independently of the values of the abscissa N), ξ_0=0.2 or 0.8 and (m=1, λ=1/2).

On the other hand, in order to check the rate of convergence of the cubature formula (4.7) as a function of the integer N, we consider the special case for which the density function $u(\xi,\eta)$ has the exponential form $\exp(\xi)$, while L is taken to be large enough (i.e. L≥72) in order to minimize the trapezoidal error effects inherent to the numerical calculations (Table III(b)).

By observing this table we can see that as N increases (from N=3 to 4 or 6), at least one additional digit of the numerical result is fixed ξ_0=0.2 or 0.6 and (m=1, λ=1/2). The results given in table III are for the first time directly calculated for a two dimensional singular integral of the form (4.8), with a weight function different from unity, by applying a cubature formula of the form (4.7) for various positions of the second order pole $X(\xi_0,\eta_0)$ inside the unit disk S (Fig.2). It should be also mentioned that the corresponding results referenced in the bibliography are those of *Gabdulkhaev* [6,7] who succeeded in proposing a cubature formula for the singular integral I_7 in the special case where the weight function $w(\xi,\eta)$=1 and the second order pole $X(\xi_0,\eta_0)$ coincides with the center of the unit disk S, that is $X(\xi_0,\eta_0)$=X(0,0).

Furthermore, we can calculate a large variety of singular integrals of the form (4.8) by applying the cubature formula (4.7). As an example we can mention the cases *(m=2, λ=1)* (L=54, N=8) for which (ξ_0,I_7)=(0.2,61.442289) or (0.4,74866826) or (0.6,90.038705) or (0.8,101.075295), $u(\xi,\eta)$=$\exp(\xi)$. Another numerical example is: *(m=0, λ=0)*, $u(\xi,\eta)=a\xi^2+b\xi^2+\gamma\xi^2+\delta\xi+\epsilon$, (L=72, N=4), $(\xi_0,\alpha,\beta,\gamma,\delta,\epsilon,I_7)$=(0.2, 0.5, 0.5, 1.5,1.0,1.0,-0.807914) or (0.4,0.5,0.5,1.5,1.0,1.0,-4.080431) or (0.6,0.5,0.5,1.5, 1.0,1.0,-12.089886) or (0.2,0.5,1.5,1.0,0.5,0.5,0.792218) etc. Another numerical example is *(m=1,λ=1/2)* (L=72, N=8), $u(\xi,\eta)=a\xi+b$, (ξ_0,a,b,I_7)=(0.2,0.5,1.0, -6.2408940) or (0.4,0.5,1.0,-5.9932751) or (0.6,0.5,1.0,-5.0806756) etc.

As a conclusion, it can be claimed that a powerful numerical technique is developed for the numerical investigation of 2-D singular integrals of the form (4.8), defined over a planar region S and presenting a logarithmic singularity of

any type.

The proposed method seems to be new and can be considered as a direct generalization of several previous attempts to solve the simpler problem of the numerical calculation of the 2-D singular integral (4.8) for the special case $w(\xi,\eta)=1$, $\xi_0=0$, $\eta_0=0$, where S denotes the unit disk of Fig.2 (see: for example [3] to [8]).

5. MODIFIED GAUSS-LOBATTO AND GAUSS-RADAU CUBATURE FORMULAS

In problems of elasticity it is convenient to use cubature formulas where some of the abscissas are given in advance, whereas the others are free ·to be chosen according to some appropriate criterion. The case when a single station should be fixed in advance in the classical *Gauss* integration theory is generally called the *Gauss-Radau* integration, whereas the case when two preassigned stations are needed is called the *Gauss-Lobatto* rule.

In the following we shall develop appropriate integration rules for both modified types of integration. For this purpose we consider again the singular integral (2.1). By introducing a polar coordinate system with its origin coinciding with the second-order pole of the singular integral (2.1) $X(\xi_0,\eta_0)$, we obtain that:

$$f(\vartheta) = f(\xi_0,\eta_0,\vartheta) \ ,$$
$$u(r,\vartheta) = u(\xi,\eta) \ ,$$
$$dS = r \ dr \ d\vartheta \ .$$

(5.1)

Based on Eq.(5.1) the principal value of the integral (2.1) may be written as:

$$I_8 = \lim_{\varepsilon \to 0} \int_0^{2\pi} f(\vartheta) \int_\varepsilon^{R(\vartheta)} \frac{u(r,\vartheta)}{r} dr \ d\vartheta \ .$$

(5.2)

where the weight function $w(\xi,\eta)$ equals unity, and $R(\vartheta)$ is defined by Eq.(4.5).

Equation (5.2) suggests the possibility of considering I_8 as a product of two one-dimensional integrals expressed by:

$$I_8 = \int_0^{2\pi} \phi(\vartheta)d\vartheta = \frac{2\pi}{m} \sum_{i=0}^{m-1} \phi\left(\frac{2\pi i}{m}\right) \ ,$$

(5.3)

and:

$$\phi(\vartheta) = f(\vartheta) \int_0^{R(\vartheta)} \frac{u(r,\vartheta)}{r} dr \ ,$$

(5.4)

where the integral on the right-hand side of Eq.(5.4) may be defined in the finite-

part sense [13].

As regards the numerical evaluation of the integrals of the form (5.4), it is valid that:

$$\int_0^{R(\vartheta)} \frac{u(r,\vartheta)}{r} dr \simeq \sum_{k=1}^{n} A_k u(R(\vartheta)\rho_k, u) + u(0,\vartheta)\ell n R(\vartheta) , \qquad (5.5)$$

where ρ_k, A_k are the abscissas and the weights respectively formally tabulated in ref.[13].

The previously developed cubature formula may be extended as follows to the *Radau* and *Lobatto* integration rules;

i) In the case of *Radau*-type formula with the point x=0 as the one preassigned station, the following one-dimensional quadrature formula seems to be valid [13,9]:

$$\oint_0^1 \frac{u(x)}{x} dx = \sum_{i=1}^{n} \frac{w_i^*}{x_i^*+1}\left[u\left(\frac{x_i^*+1}{2}\right)-u(0)\right] , \qquad (5.6)$$

where w_i^* are the weights of the classical *Gauss-Legendre* quadrature formula and x_i^* denotes a zero of the *Legendre* polynomial or order n. By combining expressions (5.3) and (5.6) together with a properly selected trapezoidal rule (with m abscissas), we obtain the following cubature formula for the numerical evaluation of the singular integral (5.2).

$$I_s \simeq \frac{2\pi}{m} \sum_{i=0}^{m-1} f\left(\frac{2\pi i}{m}\right)\left\{\sum_{j=1}^{n} \frac{w_j^*}{x_j^*+1}\left[u\left(\frac{x_j^*+1}{2} , \frac{2\pi i}{m}\right) - \right.\right.$$

$$\left.\left. -u\left(0,\frac{2\pi i}{m}\right)\right] + u\left(0,\frac{2\pi i}{m}\right)\ell n\left[R\left(\frac{2\pi i}{m}\right)\right]\right\} , \qquad (5.7)$$

where x_j^* are identical with the zeros of the *Legendre* polynomials of order n, while w_j^* are the weights of the classical *Gauss-Legendre* quadrature formula.

ii) The second important case of the *Lobatto* type quadrature, with the points x=0, 1 as preassigned stations, may be treated similarly. For this case it is valid that [13]:

$$\oint_0^1 \frac{u(x)}{x} dx \simeq \sum_{i=1}^{n} \frac{w_i}{1-x_i^2}\left[\frac{x_i-1}{2}u(0) - \frac{x_i+1}{2}u(1)+u\left(\frac{x_i+1}{2}\right)\right] - u(0) + u(1) , \qquad (5.8)$$

where w_i, x_i are the weights and the stations, derived from the special *Jacobi*

polynomials $P^{(1,0)}(x)$. These coefficients are systematically tabulated in ref.[14].

By properly combining expressions (5.5) and (5.8), we may conclude that, for the case of the *Lobatto*-type integration rule, the following modified cubature formula is valid:

$$I_8 \simeq \frac{2\pi}{m} \sum_{i=0}^{m-1} f\left(\frac{2\pi i}{m}\right) \left\{ \sum_{j=1}^{n} \frac{w_j}{1-x_j^2} \left[\frac{x_j-1}{2} u\left(0,\frac{2\pi i}{m}\right) - \frac{x_j+1}{2} u\left(1,\frac{2\pi i}{m}\right) + \right. \right.$$

$$\left. \left. +u\left(\frac{x_j+1}{2},\frac{2\pi i}{m}\right) \right] + u\left(1,\frac{2\pi i}{m}\right) + u\left(0,\frac{2\pi i}{m}\right) \left[\ln\left[R\left(\frac{2\pi i}{m}\right)\right] - 1 \right] \right\} . \qquad (5.9)$$

6. ERROR ESTIMATES FOR 2-DIMENSIONAL PRINCIPAL VALUE INTEGRALS

An error estimate is derived for the cubature formula, which was proposed in the previous paragraph (Eqs.(4.6) or (5.2)), and the convergence of this formula is established.

Let us consider now the two-dimensional singular integral of the form:

$$H = \int_0^{2\pi} w(\vartheta)f(\vartheta)H_A(\vartheta)d\vartheta , \qquad (6.1)$$

where:

$$H_A(\vartheta) = \int_0^1 w(r)\frac{u(r,\vartheta)}{r}dr + p(n)\frac{u^{(n-1)}(0,\vartheta)}{(n-1)!} \times \ln|R(\vartheta)| , \qquad (6.2)$$

with $w(r)=r^{1-q}$ and:

$$p(n) = \begin{cases} 1, & \text{if } q \text{ is an integer} \\ \\ 0, & \text{if } q \text{ is a rational number.} \end{cases} \qquad (6.3)$$

Here the variables r and ϑ have been already introduced in relation (4.2), while the functions $u(r,\vartheta)$, $f(\vartheta)$ satisfy the conditions (2.4), and $R(\vartheta)$ is given again by Eq.(4.5).

Next, we can assume that the function $u(z,\vartheta)$ is an analytic function of z, regular in the disk $|z|<\varepsilon$, which contains the integration interval [0,1]; i.e.

$$u(z,\vartheta) = \sum_{k=0}^{\infty} b_k z^k , \qquad (6.4)$$

with:

$$b_k = \frac{u^{(k)}(0,\vartheta)}{k!} . \qquad (6.5)$$

Then, according to the developments of *Hardy* ([15] p.83), the integral:

$$H_B = \int_0^{\infty} e^{-t}F(zt)dt , \qquad (6.6)$$

where:

$$F(z) = \sum_{k=0}^{\infty} b_k \frac{z^k}{k\Gamma(1+k)} \quad , \tag{6.7}$$

represents a radial continuation of the function $u(r,\vartheta)$ on a definite region of summability. This practically means that the function $u(z,\vartheta)$ has the form:

$$u(z,\vartheta) = \int_0^{\infty} e^{-t} \left[\sum_{k=0}^{\infty} b_k \frac{(tz)^k}{k!} \right] dt \quad . \tag{6.8}$$

Then, the error term can be calculated as follows:

$$\left| R_n(u) \right| = \left| \int_0^1 w(r) \frac{u(r,\vartheta)}{r} dr - \sum_{k=1}^{n} w_k u(r_k,\vartheta) \right| = \left| \int_0^{\infty} e^{-t} \sum_{k=0}^{\infty} \frac{t^{n+k} b_{n+k}}{(n+k)!} c_{n,k} dt \right| \quad , \tag{6.9}$$

with:

$$c_{n,k} = \int_0^1 w(r) r^{n+k-1} dr - \sum_{k=1}^{n} w_k x_k^{n+k} \quad . \tag{6.10}$$

Furthermore, it follows from Eq.(6.9) that:

$$\left| R_n(u) \right| \leq \int_0^{\infty} e^{-t} \sum_{k=0}^{\infty} \left| \frac{t^{n+k} b_{n+k}}{(n+k)!} c_{n,k} \right| dt \quad , \tag{6.11}$$

or, after applying Hölder's inequality to Eq.(6.11), we obtain:

$$\left| R_n(u) \right| \leq \int_0^{\infty} e^{-t} \left[\sum_{k=0}^{\infty} \left| \frac{t^{n+k} b_{n+k}}{(n+k)!} \right|^q \right]^{1/q} \left[\sum_{k=0}^{\infty} |c_{n,k}|^p \right]^{1/q} dt \quad , \tag{6.12}$$

with:

$$1/p + 1/q = 1 \quad .$$

Thus, the error bound can be written as:

$$\left| R_n(u) \right| \leq \psi_1(n,p) \psi_2(u,q) \quad , \tag{6.13}$$

where:

$$\psi_1(n,p) = \left[\sum_{k=0}^{\infty} |c_{n,k}|^p \right]^{1/q} \tag{6.14}$$

and

$$\psi_2(u,q) = \int_0^{\infty} e^{-t} \left[\sum_{k=0}^{\infty} \left| \frac{t^{n+k} b_{n+k}}{(n+k)!} \right|^p \right]^{1/q} \quad . \tag{6.15}$$

We shall now consider the following important cases:

i) The case of $\psi_1(n,2)$, $\psi_2(u,2)$, that is the Euclidean norm of ψ, for which it is valid that:

$$\psi_1^2(n,2) = \sum_{k=0}^{\infty} c_{n,k}^2 \ , \tag{6.16}$$

ii) The case $\psi_1(n,\infty)$, $\psi_2(u,1)$ will be examined next, by which the supremum norm may be obtained, that is:

$$\psi_1(n,\infty) = \sup_{k=0,1,\ldots} |c_{n,k}| \ . \tag{6.17}$$

The main problem of the present analysis concerns the conditions of existence of the functions $\psi_1(n,z)$ and $\psi_1(n,\infty)$.

In order to prove the existence of the function $\psi_1(n,z)$, we investigate the series $\sum_k c_{n,k}^2$. We can rewrite Eq.(6.16) in the form:

$$\psi_1^2(n,2) = \sum_{k=0}^{\infty} L_{n+k} -2 \sum_{k=0}^{\infty} L_{n+k} S_k + \sum_{k=0}^{\infty} S_k^2 \ , \tag{6.18}$$

with:

$$L_{n+k} = \int_0^1 x^{n+k-\lambda} dx \ , \tag{6.19}$$

$$S_k = \sum_{i=1}^{n} w_i x_i^{n+k} \ . \tag{6.20}$$

Since it is valid that:

$$|S_k| < \sum_{i=1}^{n} |w_i| x_i^{n+k} < \left[\sum_{i=1}^{n} |w_i| \right] (\max x_i)^{n+k} \tag{6.21}$$

which means that the series $\left[\sum_{i=1}^{n} |w_i| (\max x_i)^{n+k} \right]$ is a convergent majorant series for $\sum_k S_k$, then it can be concluded that the second and third series in Eq.(6.21) are absolutely convergent. On the other hand, the first series on the right-hand side in Eq.(6.18) represents a general harmonic series, which converges absolutely. Therefore the first, second and third series in Eq.(6.18) are absolutely convergent and as consequence, $\psi_1(n,2)$ exists. On the other hand, the function ψ_1 can be calculated numerically by simply applying Eq.(6.18).

As an example we have that (n=4, $\psi_1(n,2)$=0,289), (n=6, ψ_1=0,223), (n=8, ψ_1=0,186), (n=10, ψ_1=0,162), (n=12, ψ_1=0,145), (n=14, ψ_1=0,132) are the calculated values of the factor ψ_1.

By properly combining expressions (6.1) to (6.3), we obtain that:

$$H = \int_0^{2\pi} \varphi_1(\vartheta) d\vartheta + \int_0^{2\pi} \varphi_2(\vartheta) d\vartheta - r \int_0^{2\pi} \varphi_3(\vartheta) d\vartheta \ , \tag{6.22}$$

where:

$$\varphi_1(\vartheta) = \sum_{i=1}^{n} w_i u(r_i, \vartheta) \ , \qquad\qquad (6.23)$$

$$\varphi_2(\vartheta) = R_n(u, \vartheta) \ , \qquad\qquad (6.24)$$

$$\varphi_3(\vartheta) = p(n)\frac{u^{(n-1)}(0,\vartheta)}{(n-1)!}\ell n|R(\vartheta)| \ . \qquad\qquad (6.25)$$

Here the quantities w_i, r_i, R_n, $p(n)$, $R(\vartheta)$ are defined by relations (4.2), (4.5) and (6.13). As a conclusion, the total error of the cubature formula (6.1) is given by:

$$R_t(u,f) \equiv R_n(u,\varphi_1)+R_n(u,\varphi_2)+R_n(u,\varphi_3) \ , \qquad\qquad (6.26)$$

where $R_n(u,\varphi_i)$ is the error expression of the trapezoidal rule for the finite integration interval $[0,2\pi]$.

In order to determine the error $R_n(u,\varphi_i)$, we consider the function $\psi(z)$ defined by:

$$\psi(z) = \begin{cases} \pi\exp\{+i\pi[\nu(z-a)-\lambda]\}, \ I_m z>0 \\ \\ \pi\exp\{-i\pi[\nu(z-a)-\lambda]\}, \ I_m z<0 \ , \end{cases} \qquad\qquad (6.27)$$

where a,ν and λ are real constants, λ being such that $0\leq\lambda<1$. Because of the fact that this function is analytic in each half-plane, we obtain, according to the developments of Donalson and Elliott [16], the relation:

$$R_n(u,\varphi_i) = \frac{1}{2\pi i}\int_C \frac{\psi(z)}{\mu(z)}\varphi_i(z)dz \ , \qquad\qquad (6.28)$$

where $\psi(z)$ is given by the relation (6.27) and $\mu(z)$ is defined by:

$$\mu(z) = -\sin\pi[\nu(z-a)-\lambda] \ , \qquad\qquad (6.29)$$

Moreover the functions $\varphi_i(z)$ are defined by Eqs.(6.23) to (6.25). The curve C, which surrounds the interval $[0,2\pi]$, is defined in ref.[16] (p.584).

The integrals of the form (6.28) can be calculated directly and, because of the analyticity of their integrands, they are finite and bounded. Therefore, by applying Eq.(6.26), the error of the cubature formula (4.7) follows as a definite and bounded quantity.

7. CONCLUSIONS

In the present paper the general method, already introduced in refs.[1] and [2],

is extended to include the cases of the *Lobatto-* and *Radau*-integration rules, for the

calculation of 2-D singular integrals. This construction of the cubature formulas of

the *Gaussian* type has the following advantages, in comparison with the approximate

techniques of Cruse et al.[3]:

i) The density function $u(r,\vartheta)$ is no longer approximated by an algebraic polynomial

$(ax+b$ or ax^2+bx+c $(a,b,c=constants))$, in order to calculate the integrals or to

solve the 2-D singular integral equations resulting from physical problems under

consideration.

ii) The range of integration S is no longer divided into small triangles etc in

order to calculate the C.P.V. of the 2-D integrals.

iii) It permits for the first time, the calculation of √2-D singular integrals with a

weight function $w(r,\vartheta)$ different from unity.

iv) The error quantities and the proof of convergence of the resulting Gaussian

cubature formulas is established.

v) The concept of *collocation points* is extended to three-dimensional problems by

using the techniques of complex analysis [7].

As a consequence, correct numerical values are found for certain 3-D problems

with a much smaller computer-time requirement.

vi) The numerical technique developed in this paper is simpler and less time-

consuming method than the BIE-method, since it does not need to discretize the

surface into small pieces, on which the integration must be done while the surface data

of the density function do not need to be considered as constant. This is an

advantage of the suggested method as compared with the BIE-method [4,6].

vii) The method developed can be easily extended to solve numerically systems of

two-dimensional singular integral equations [10]. From this point of view it seems

to be the two-dimensional counterpart of the already established theory for the

numerical solution of one-dimensional singular integral equations.

T A B L E I

$w_2 \backslash w_1$	1	$\dfrac{1}{\sqrt{1-\xi}}$	$\dfrac{1}{\sqrt{\xi}}$	$\dfrac{1}{\sqrt{1-\xi^2}}$	$e^{-\xi}$	$\dfrac{1}{\sqrt{\xi}}e^{-\xi}$
1	1					
$\dfrac{1}{\sqrt{1-\eta}}$	$\dfrac{1}{\sqrt{1-\eta}}$	$\dfrac{1}{\sqrt{(1-\eta)(1-\xi)}}$				
$\dfrac{1}{\sqrt{\eta}}$	$\dfrac{1}{\sqrt{\eta}}$	$\dfrac{1}{\sqrt{\eta(1-\xi)}}$	$\dfrac{1}{\sqrt{\xi\eta}}$			
$\dfrac{1}{\sqrt{1-\eta^2}}$	$\dfrac{1}{\sqrt{1-\eta^2}}$	$\dfrac{1}{\sqrt{(1-\eta)(1+\eta)(1-\xi)}}$	$\dfrac{1}{\sqrt{\xi(1-\eta)(1+\eta)}}$	$\dfrac{1}{\sqrt{(1-\xi^2)(1-\eta^2)}}$		
$e^{-\eta}$	$e^{-\eta}$	$\dfrac{e^{-\eta}}{\sqrt{1-\xi}}$	$\dfrac{e^{-\eta}}{\sqrt{\xi}}$	$\dfrac{e^{-\eta}}{\sqrt{1-\xi^2}}$	$e^{-(\xi+\eta)}$	
$\dfrac{1}{\sqrt{\eta}}e^{-\eta}$	$\dfrac{e^{-\eta}}{\sqrt{\eta}}$	$\dfrac{e^{-\eta}}{\sqrt{\eta(1-\xi)}}$	$\dfrac{e^{-\eta}}{\sqrt{\xi\eta}}$	$\dfrac{e^{-\eta}}{\sqrt{\eta(1-\xi^2)}}$	$\dfrac{e^{-(\xi+\eta)}}{\sqrt{\eta}}$	$\dfrac{e^{-(\xi+\eta)}}{\sqrt{\xi\eta}}$

Table I: The weight function $w(\xi,\eta)=w_1(\xi)w_2(\eta)$ for various quadrature rules (Gauss Legendre (G.L.), Modified Gauss Legendre (M.G.L.), Gauss Chebyshev (G.CH.), Gauss-Laguerre G.LAG., Gauss-Hermite G.H.).

T A B L E   II

(m×n)	Collocation Points $X_i(\xi_0, \eta_0) i = 1,2,3$	Numerical value of I_4 (Eq.(22))	Theoretical value of I_4 (Eq.(26))
(8×8)	X_1(0.36062 31751, 0.36062 31751)	-1.342879	-1.045333
(16×16)	X_2(0.37149 22970, 0.54039 34260)	-1.069729	-1.011953
(32×32)	X_3(0.54794 77112, 0.95094 46082)	-1.194753	-1.191160

Table II: Numerical and theoretical values of the integral $I_3 = \int_S \dfrac{\cos\vartheta}{r^2} dS$ for m×n=(8×8), (16×16) and (32×32).

T A B L E   III

(α)		
ξ_0 \ L	0.2	0.8
18	-12.5143188	-8.7161754
36	-12.5147725	-8.7161762
72	-12.5147725	-8.7161762

(β)		
ξ_0 \ L	0.2	0.6
3	-15.0669552	-20.1688455
4	-15.3306996	-20.5379592
6	-15.3382977	-20.5853835

Table III: (a) Numerical evaluation of the 2-D singular integral I_7 (eqn.34) for the case where (m=1, λ=1/2) and u(ξ,η)=ξ;

(b) Numerical evaluation of the 2-D singular integral I_7 (eqn.34) for the case where (m=1, λ=1/2, u(ξ,η)=expξ and L=72.
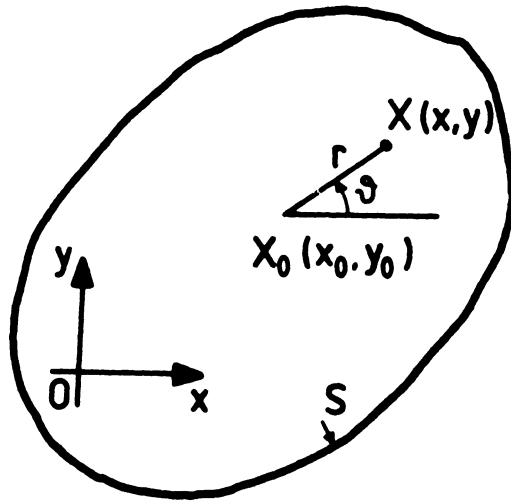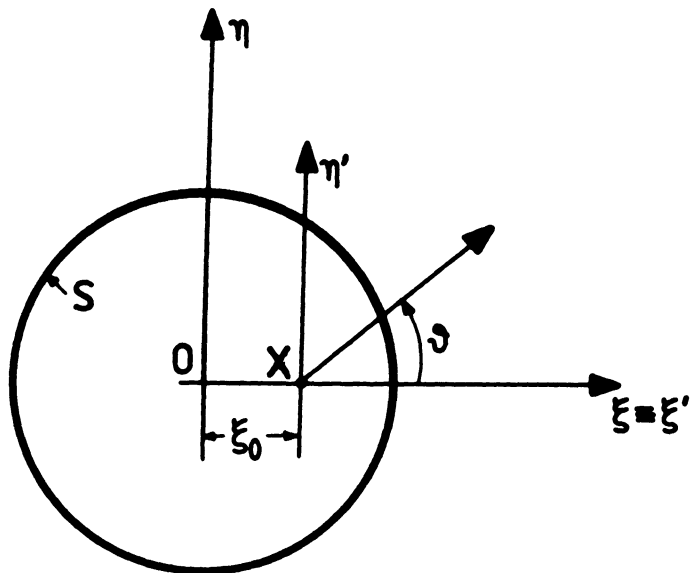
Fig. 1. Geometry of the two-dimensional region.

Fig. 2. Geometry of a unit disk S with a second-order pole   X (ξ_0, η_0) inside it.

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