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ASYMPTOTIC RELATIONSHIPS BETWEEN TWO HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

TAKASI KUSANO

Department of Mathematics
Faculty of Science
Hiroshima University
Hiroshima, Japan

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ABSTRACT. Some asymptotic relationships between the two ordinary differential equations

$$x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = 0, \tag{1}$$

$$y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_n(t)y = f(t,y), \tag{2}$$

are studied. Conditions are given that lead to an asymptotic equivalence between certain of the solutions of (1) and certain of the solutions of (2). The case where the perturbation f(t,y) depends on a functional argument is also discussed.

KEY WORDS AND PHRASES. Ordinary differential equations, asymptotic relations, functional differential equations.
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1. INTRODUCTION.

We are concerned with some asymptotic relationships between the following two differential equations

$$x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = 0, \tag{1.1}$$

$$y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_n(t)y = f(t,y), \tag{1.2}$$

where $p_i: [t_0,\infty) \to R$, $1 \le i \le n$, and $f: [t_0,\infty)\times R \to R$ are continuous functions. We obtain conditions that lead to an asymptotic equivalence between certain of the solutions of the linear equation (1.1) and certain of the solutions of the perturbed linear equation (1.2). No restriction is placed on the behavior of solutions of (1.1) which may be oscillatory or nonoscillatory. The example given at the end of this note deals with the case where (1.1) has both oscillatory and nonoscillatory solutions. Our results generalize those of Hallam [1] for second order equations.

2. MAIN RESULTS.

In what follows we denote by $W(\Phi_1,\dots,\Phi_m)(t)$ the Wronskian of $\Phi_1(t),\dots\Phi_m(t)$.

Let a fundamental system of solutions of (1.1), $\{x_1(t),\dots,x_n(t)\}$, be fixed and suppose that there exist positive continuous functions $x_i^*(t)$, $\xi_i^*(t)$, $1 \le i \le n$, which satisfy the inequalities

$$|x_i(t)| \le x_i^*(t) \quad \text{on} \quad [t_0,\infty), \quad 1 \le i \le n, \tag{2.1}$$

$$\left|\frac{W(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)(t)}{W(x_1,\dots,x_n)(t)}\right| \le \xi_i^*(t) \quad \text{on} \quad [t_0,\infty), \quad 1 \le i \le n. \tag{2.2}$$

Note that $W(x_1,\dots,x_n)(t)$ is a constant multiple of $\exp(-\int_{t_0}^{t} p_1(s)ds)$.

With regard to (1.2) we assume that $f(t,y)$ satisfies the inequality

$$|f(t,y)| \le \Phi(t,|y|) \quad \text{on} \quad [t_0,\infty)\times R, \tag{2.3}$$

where $\Phi : [t_0,\infty)\times R_+ \to R_+$, $R_+ = [0,\infty)$, is continuous and nondecreasing in the second variable for each fixed t.

THEOREM 1. *Suppose conditions* (2.1), (2.2), *and* (2.3) *are satisfied. Also suppose that, for some* k, $1 \le k \le n$, *and some constant* $c > 1$,

$$\int_{t_0}^{\infty} \xi_i^*(s)\Phi(s,cx_k^*(s))ds < \infty, \quad 1 \le i \le n, \tag{2.4}$$

and

$$\frac{x_i^*(t)}{x_k^*(t)} \int_t^{\infty} \xi_i^*(s)\Phi(s,cx_k^*(s))ds = o(1) \quad \text{as } t \to \infty \tag{2.5}$$

for $1 \le i \le n$ *with* $i \ne k$.

Then there exists a solution $y(t)$ *of equation* (1.2) *such that*

$$y(t) = x_k(t) + o(x_k^*(t)) \quad \text{as } t \to \infty. \tag{2.6}$$

In addition, for any solution $y_k(t)$ *of* (1.2) *satisfying the inequality* $|y_k(t)| \le cx_k^*(t)$ on $[t_0,\infty)$, *there exists a solution* $x(t)$ *of* (1.1) *such that*

$$x(t) = y_k(t) + o(x_k^*(t)) \quad \text{as } t \to \infty. \tag{2.7}$$

PROOF. In view of (2.4) and (2.5) we can choose $t_1 > t_0$ so large that

$$\int_{t_1}^{\infty} \xi_k^*(s)\Phi(s,cx_k^*(s))ds < \frac{c-1}{2} \tag{2.8}$$

and

$$\frac{x_i^*(t)}{x_k^*(t)} \int_t^{\infty} \xi_i^*(s)\Phi(s,cx_k^*(s))ds < \frac{c-1}{2(n-1)}, \quad t \geq t_1, \tag{2.9}$$

for $1 \leq i \leq n$ with $i \neq k$. Let $C[t_1,\infty)$ be the locally convex space of continuous functions on $[t,\infty)$ with the compact open topology and consider the closed convex subset F of $C[t_1,\infty)$ defined by

$$F = \{y \in C[t_1,\infty) : |y(t)| \leq cx_k^*(t), \quad t \geq t_1\}.$$

Define the operator $\mathcal{J}: F \to C[t_1,\infty)$ by the formula

$$y(t) = x_k(t) + \sum_{i=1}^{n} (-1)^{n-i-1} x_i(t) \int_t^{\infty} \mathfrak{w}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)(s)f(s,y(s))ds, \tag{2.10}$$

where

$$\mathfrak{w}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)(t) = \frac{W(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)(t)}{W(x_1,\ldots,x_n)(t)}. \tag{2.11}$$

We seek for a fixed point $y = y(t)$ of $\mathcal{J}$ in F. Using the identities

$$\sum_{i=0}^{n} (-1)^{n-i} x_i^{(j)}(t)W(x_1,\ldots,x_{i+1},\ldots,x_n)(t) = 0, \quad 0 \leq j \leq n-2, \tag{2.12}$$

$$\sum_{i=0}^{n} (-1)^{n-i} x_i^{(n-1)}(t)W(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)(t) = \mathfrak{w}(x_1,\ldots,x_n)(t), \tag{2.13}$$

we easily see that a fixed point of $\mathcal{J}$ is a solution of equation (1.2).

i) $\mathcal{J}$ <u>maps</u> F <u>into</u> F. This follows immediately from (2.1), (2.2), (2.3), (2.8), (2.9) and (2.10).

ii) $\mathcal{J}$ <u>is continuous</u>. Let $\{y_\nu\}$ be a sequence in F converging uniformly to $y \in F$

on every compact subinterval of $[t_1,\infty)$. Consider any compact subinterval of the form $[t_1,t_2]$. Let $\varepsilon > 0$ be given and choose $t_3 > t_2$ so that

$$\int_{t_3}^{\infty} \xi_i^*(s)\Phi(s,cx_k^*(s))ds < \frac{\varepsilon}{4nM}, \quad 1 \le i \le n, \tag{2.14}$$

where $M = \max_{1\le i\le n} \max_{t\varepsilon[t_1,t_2]} x_i^*(t)$. Take an integer N such that

$$\xi_i^*(s)|f(s,y_\nu(s)) - f(s,y(s))| < \frac{\varepsilon}{2nM(t_3-t_1)}, \quad s \in [t_1,t_3],\ 1 \le i \le n, \tag{2.15}$$

for all $\nu \ge N$; this is possible since $f(t,y)$ is continuous and $\{y_\nu\}$ converges uniformly to y on $[t_1,t_3]$.

Using (2.14) and (2.15), we have

$$|\mathcal{J}y_\nu(t) - \mathcal{J}y(t)| \le \sum_{i=1}^{n} x_i^*(t) \int_{t_1}^{t_3} \xi_i^*(s)|f(s,y_\nu(s)) - f(s,y(s))|ds$$

$$+ 2\sum_{i=1}^{n} x_i^*(t) \int_{t_3}^{\infty} \xi_i^*(s)\Phi(s,cx_k^*(s))ds < \varepsilon$$

for all $t \in [t_1,t_2]$ and all $\nu \ge N$. This shows that $\mathcal{J}$ is continuous on F.

iii) $\mathcal{J}$ F <u>is relatively compact</u>. From (2.10), (2.11) and (2.12) (with $j = 0$) we obtain

$$|(\mathcal{J}y)'(t)| \le |x_k'(t)| + \sum_{i=1}^{n} |x_i'(t)| \int_t^{\infty} \xi_i^*(s)\Phi(s,cx_k^*(s))ds.$$

On any compact subinterval of $[t_0,\infty)$ the right-hand side of the above inequality is bounded by a constant independent of $y \in F$. Therefore $\mathcal{J}$ F is equicontinuous on every compact subinterval of $[t_1,\infty)$ and its relative compactness follows from the Ascoli-Arzela theorem.

Applying now the Schauder-Tychonoff fixed-point theorem, we see that the operator $\mathcal{J}$ has a fixed point $y = y(t)$ in F. As remarked earlier, $y(t)$ is a solution of (1.2).

That $y(t)$ satisfies the order relation (2.6) follows from the inequality

$$|y(t) - x_k(t)| \leq \sum_{i=1}^{n} x_i^*(t) \int_t^\infty \xi_i^*(s)\Phi(s,cx_k^*(s))ds$$

with the help of hypotheses (2.4) and (2.5).

To prove the opposite relationship between the solutions of (1.1) and (1.2), let $y_k(t)$ be a solution of (2.2) satisfying the inequality $|y_k(t)| \leq cx_k^*(t)$ on $[t_0,\infty)$. Then it is easy to verify that the function $x(t)$ defined by

$$x(t) = y_k(t) + \sum_{i=1}^{n} (-1)^{n-i} x_i(t) \int_t^\infty W(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)(s)f(s,y_k(s))ds \tag{2.16}$$

is a solution of (1.1) which satisfies the order relations (2.7). Thus the proof of Theorem 1 is complete.

THEOREM 2. *Let conditions* (2.1), (2.2) *and* (2.3) *be satisfied. Suppose that, for some integer* k, $1 \leq k \leq n$, *and some constant* $c > 1$,

$$\int_{t_0}^\infty \xi_k^*(s)\Phi(s,cx_k^*(s))ds < \infty \tag{2.17}$$

and

$$\frac{x_i^*(t)}{x_k^*(t)} \int_{t_0}^t \xi_i^*(s)\Phi(s,cx_k^*(s))ds = o(1) \quad \text{as } t \to \infty \tag{2.18}$$

for $1 \leq i \leq n$ *with* $i \neq k$.

Then there exists a solution $y(t)$ *of* (1.2) *such that* (2.6) *is satisfied. In addition, if* $y_k(t)$ *is any solution of* (1.2) *satisfying the inequality* $|y_k(t)| \leq cx_k^*(t)$, *then there exists a solution* $x(t)$ *of* (1) *such that* (2.7) *is satisfied.*

PROOF. It suffices to proceed as in the proof of Theorem 1 by replacing (2.10) and (2.16), respectively, by

$$\mathcal{J}y(t) = x_k(t) + (-1)^{n-k-1} x_k(t) \int_t^\infty W(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)(s)f(s,y(s))ds$$
$$+ \sum_{\substack{i=1 \\ i\neq k}}^{n} (-1)^{n-i} x_i(t) \int_{t_1}^t W(s_1,\dots,x_{i-1},x_{i+1},\dots,x_n)(s)f(s,y(s))ds \tag{2.19}$$

and

$$x(t) = y_k(t) + (-1)^{n-k} x_k(t) \int_t^\infty W(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)(s) f(s,y_k(s))ds$$

$$+ \sum_{\substack{i=1 \\ i \neq k}}^{n} (-1)^{n-i-1} x_i(t) \int_{t_1}^{t} W(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)(s) f(s,y_k(s))ds. \quad (2.20)$$

The details will be left to the reader.

It is not hard to see that the above arguments can be applied to establish similar asymptotic relationship between (1.1) and the functional differential equation

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \dots + p_n(t)y(t) = f(t,y(g(t))), \quad (2.21)$$

where $p_i(t)$ and $f(t,y)$ are as before and $g : [t_0,\infty) \to R$ is a continuous function such that $\lim_{t\to\infty} g(t) = \infty$. For example, we have the following analogue of Theorem 1.

THEOREM 3. *Let conditions* (2.1), (2.2), and (2.3) *be satisfied. Suppose that, for some* k, $1 \le k \le n$, *and some constant* $c > 1$,

$$\int_{t_0}^{\infty} \xi_i^*(s)\Phi(s,cx_k^*(g(s)))ds < \infty, \qquad 1 \le i \le n, \quad (2.22)$$

$$\frac{x_i^*(t)}{x_k^*(t)} \int_t^\infty \xi_i^*(s)\Phi(s,cx_k^*(g(s)))ds = o(1) \quad \text{as } t \to \infty \quad (2.23)$$

for $1 \le i \le n$ *with* $i \neq k$.

Then (2.21) *has a solution* $y(t)$ *which satisfies* (2.6). *In addition, if* $y_k(t)$ *is a solution of* (2.21) *satisfying* $|y_k(t)| \le cx_k^*(t)$, *then* (1.1) *has a solution* $x(t)$ *such that* (2.7) *is satisfied*.

3. EXAMPLE.

Consider the third order differential equations

$$x''' + x = 0, \quad (3.1)$$

$$y''' + y = b(t)|y|^\gamma \operatorname{sgn} y, \quad (3.2)$$

where $\gamma \ge 0$ is a constant and $b : [0,\infty) \to R$ is a continuous function. The functions $x_1(t) = (2/3\sqrt{3})e^{-t}$, $x_2(t) = e^{t/2}\cos(\sqrt{3}/2)t$, $x_3(t) = e^{t/2}\sin(\sqrt{3}/2)t$ form a fundamental system of solutions of (3.1) such that $W(x_1,x_2,x_3)(t) = 1$. We can take

$x_1^*(t) = (2/3\sqrt{3})e^{-t}$, $x_2^*(t) = x_3^*(t) = e^{t/2}$. $\xi_1^*(t) = (\sqrt{3}/2)e^t$, $\xi_2^*(t) = \xi_3^*(t) = (2/3)e^{-t/2}$.

Suppose that

$$\int_0^\infty e^{(1-\gamma)t}|b(t)|dt < \infty. \tag{3.3}$$

Theorem 1 (with k = 1) then ensures that (3.2) has a solution $y_1(t)$ such that

$$y_1(t) = x_1(t) + o(e^{-t}) \quad \text{as } t \to \infty. \tag{3.4}$$

Suppose next that

$$\int_0^\infty e^{(1+(\gamma/2))t}|b(t)|dt < \infty. \tag{3.5}$$

Then, applying Theorem 1 (with k = 2 and k = 3), we see that (3.2) has solutions $y_2(t)$ and $y_3(t)$ such that

$$y_2(t) = x_2(t) + o(e^{t/2}) \quad \text{as } t \to \infty, \tag{3.6}$$

and

$$y_3(t) = x_3(t) + o(e^{t/2}) \quad \text{as } t \to \infty. \tag{3.7}$$

Obviously, $y_1(t)$ is nonoscillatory, whereas $y_2(t)$ and $y_3(t)$ are oscillatory. Since (3.5) is stronger than (3.3), (3.5) guarantees the existence of all the three solutions $y_1(t)$, $y_2(t)$, and $y_3(t)$ listed above.

From Theorem 1 it also follows that, in case (3.5) holds, if $y(t)$ is a solution of (3.2) satisfying

$$|y(t)| \leq c\, e^{t/2} \tag{3.8}$$

for some constant $c > 1$, then there exists a solution $x(t)$ of (3.1) such that $x(t) = y(t) + o(e^{t/2})$ as $t \to \infty$. Note that not all solutions of (3.2) are subject to this estimate. In fact, equation (3.2) with $\gamma = 3$ and $b(t) = 28e^{-6t}$ has a solution $y(t) = e^{3t}$, even though (3.5) is satisfied.

Finally, consider the functional differential equation

$$y'''(t) + y(t) = b(t)|y(t + \sin t)|^\gamma \operatorname{sgn} y(t + \sin t), \tag{3.9}$$

where γ and $b(t)$ are as above. Appealing to Theorem 1, we conclude that if (3.5) holds, (3.9) has three solutions $y_1(t)$, $y_2(t)$, and $y_3(t)$ with properties (3.4), (3.6), and (3.7), respectively.

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