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ON A CLASS OF POLYNOMIALS ASSOCIATED WITH THE PATHS IN A GRAPH AND ITS APPLICATION TO MINIMUM NODES DISJOINT PATH COVERINGS OF GRAPHS

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ABSTRACT. Let G be a graph. With every path α of G , let us associate a weitht w_α . With every spanning subgraph C of G consisting of paths α_1 , α_2 , ..., α_k , let us associate the weight

$$w(C) = \prod_{i=1}^{k} w_{\alpha_i} .$$

The path polynomial of G is

$$\Sigma\, w\, (C) ,$$

where the summation is taken over all the spanning subgraphs of G whose components are paths. Some basic properties of these polynomials are given. The polynomials are then used to obtain results about the ninimum number of node disjoint path coverings in graphs.

KEY WORDS AND PHRASES. Graph polynomial, generating function, path cover, spanning subgraphs, incorporated graph, path-to-point conering number, island decomposition.

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1. INTRODUCTION.

The graphs considered here will be finite, undirected, and without loops or multiple edges, unless otherwise stated. Let G be such a graph. A *path* in G is a subgraph of G which is a tree with nodes of valencies 1 and 2 only. A *path cover* of G is a spanning subgraph of G , whose components are all paths. (We regard an isolated node to be a path with no edges). With every path α in G , let us associate a wieght w_α . With every path cover (or simply *cover*) C of G , let us associate the weight

$$w(C) = \prod_{i=1}^{r} w_{\alpha_i} ,$$

where α_i (i=1, 2,...,r) are the components of C. Then the *path polynomial* of G is

$$\Sigma\, w\,(C),$$

where the summation is taken over all the path covers in G. This polynomial belongs to the family of F-polynomials defined in Farrell [1].

In this paper, we will assign the weight w_k to paths with k nodes. Therefore the path polynomial of G will be a polynomial in the indeterminates $w_1, w_2, w_3, \ldots$ etc. We will denote this polynomial by $P(G;\underline{w})$, where $\underline{w} = (w_1, w_2, w_3, \ldots)$ is a *general weight vector*. If we put $w_i = w$ for all i, then the resulting polynomial in w will be called the *simple path polynomial* of G. This polynomial will be denoted by P(G;w).

First of all, we will give a fundamental theorem on path polynomials and then use it to derive an algorithm for finding the polynomials. Basic properties of the polynomials will then be discussed. We will then derive formulae from which the path polynomials of trees could be obtained, and also expressions for path polynomials of multigraphs.

Finally, we will use the polynomials to extend some known results about minimum node disjoint path coverings of graphs. We refer the reader to Harary [2] for the basic definitions in graph theory.

2. THE FUNDAMENTAL THEOREM AND ALGORITHM.

Let e be an edge in the graph G. By *path incorporating* (or simply *incorporating*) e, we mean that e is distinguished in some way (for example, coloured) and required to belong to every path covering of G that we consider. Consider the set of all path covers of G. We can partition this set into two classes, (i) those containing a specified edge e, and (ii) those which do not. The covers which do not contain e will be covers of the graph G′ obtained from G by deleting e. The covers which contain e, will be covers of the graph G* obtained from G by incorporating e. Thus we have the following theorem.

THEOREM 1. (The Fundamental Theorem for Path Polynomials) Let G be a graph and e an (unincorporated) edge of G. Let G′ be the graph obtained from G by deleting e, and G* the graph obtained from G by incorporating e. Then

$$P(G;\underline{w}) = (P(G';\underline{w}) + P(G^*;\underline{w}).$$

By an *incorporated graph*, we will mean a graph whose edges are all incorporated. Since a circuit cannot be a subgraph of any path cover, we have the following result.

LEMMA 1. If G* contains an incorporated circuit, then

$$P(G^*;\underline{w}) = 0.$$

Since no path can have a node of valency greater than 2, we have

LEMMA 2. If G* contains more than two incorporated edges incident to a node, then

$$P)G^*;\underline{w}) = 0.$$

The Fundamental Algorithm for Path Polynomials (or simply the *reduction process*) consists of repeated applications of Theorem 1 to the graph G, until we obtain graphs G_i for which $P(G_i;\underline{w})$ could be immediately written down. Lemmas 1 and 2 imply some useful simplifications to the algorithm. If a node has two incorporated edges incident to it, we can immediately delete all the unincorporated edges which are adjacent to that node. Also, we can also immediately remove from the graph any edge that completes a circuit with a set of incorporated edges.

We have programmed the algorithm on a computer and have used it to generate path polynomials of several kinds of graphs. Hand computation of these polynomials could be very tedious if the graph is non-trivial. We note that a procedure for obtaining path covers of a tree is given in Slater [3].

3. SOME BASIC PROPERTIES OF PATH POLYNOMIALS.

It is clear that the terms of $P(G;\underline{w})$ are of the form $A\, w_1^{k_1} w_2^{k_2} \ldots w_r^{k_r} \ldots$, where A is the number of path covers of G containing k_1 isolated nodes, k_2 edges, k_3 paths length 3 etc. In the case of the simple path polynomial $P(G;w)$, the terms are of the form $A_r w^r$, where A_r is the number of covers of G with r components. The following results are immediate consequences of the definitions.

THEOREM 2. Let G be a graph with p nodes. Then the coefficient of w_p in $P(G;\underline{w})$ or the coefficient of w in $P(G;w)$, is the number of Hamiltonian paths in G.

THEOREM 3. If the coefficient of w in $P(G;w)$ is nonzero, then G is connected.

THEOREM 4. Let G be a graph with p nodes. Let r be the smallest exponent of w in $P(G;w)$. Then all terms with larger powers then r, up to w^p, must occur in $P(G;w)$ with nonzero coefficients (i.e., $P(G;w)$ has no gaps).

PROOF. The existence of a term in w^r implies the existence of a cover with r components. By deleting an appropriate number of edges, we can obtain a cover with k components, for all $r < k \le p$. The results therefore follows.

The simple cutnode theorem which holds for chromatic polynomials (See Read [4]) does not hold for path polynomials. However, we can obtain an analogous result when the graph G consists of two independent subgraphs B_1 and B_2 "chained together" by a path of length 2 as shown below in Figure 1.

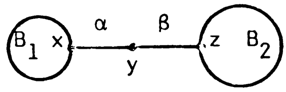

Figure 1.

By attaching a graph A to a graph B, we mean the identification of a node of A with a node of B so as to obtain a new graph in which A and B are subgraphs.

Let the edges of the path be α and β and the nodes x, y, and z as shown in Figure 1. Let us denote by G_1 and G_2 respectively, the graphs formed by attaching the edge α to B_1 and the edge β to B_2. It is clear that any path cover of G_1 can be "combined" with any path cover of G_2 (using node y) to yield a path cover of G. This leads to the following lemma.

LEMMA 3. Let G be a graph consisting of graphs B_1 and B_2 chained by a path of length 2. Let α and β be the edges of the path, which are adjacent to B_1 and B_2 respectively. Let H_1 be the graph B_1 with α attached to it and H_2 the graph B_2 with β attached to it. Then

$$P(G;w) = w^{-1} P(H_1;w)P(H_2;w).$$

This lemma can easily be extended to any finite chain of independent graphs chained by paths of length 2.

THEOREM 5. Let G be a graph consisting of k graphs B_1, $B_2, \ldots, B_k$ chained together by paths of length 2. Then

$$P(G;w) = w^{1-k} \prod_{i=1}^{k} P(H_i;w),$$

where H_1 is B_1 with an edge attached to it, and for $i = 2, 3, \ldots, k-1$, H_i is B_i with two edges attached to it. H_k is B_k with one edge attached to it.

4. PATH POLYNOMIALS OF GRAPHS WITH PATHS ATTACHED

In this section we will use the symbol G to denote the simple path polynomial of the graph G. The following lemma gives the simple path polynomial of a graph containing a *twig* (an edge incident to a node of valency 1).

LEMMA 4. Let G_1 be the graph formed by attaching a twig α to a graph G. Then

$$G_1 = wG + G^*,$$

where G^* is G_1 with α incorporated.

PROOF. Apply the reduction process to the graph G_1 by deleting α.

This result can be easily extended to the graph Q_n consisting of a graph G with n twigs <u>having no node in common</u>. We will use the notation $Q_n^*(r)$ for the graph Q_n with r of the n twigs incorporated, where $r \leq n$.

THEOREM 6.

$$Q_n = \sum_{r=0}^{n} w^r \, Q^*_{n-r}(1), \text{ with } Q^*_0(1) = G.$$

$$= w^n Q + \sum_{k=0}^{n-1} w^k \sum_{r=0}^{n-k} Q^*_{n-r-k}(1).$$

PROOF. Apply the reduction process to the graph Q_n by deleting a twig. This yields

$$Q_n = w\, Q_{n-1} + Q_n^*(1)$$

Now, apply the process to Q_{n-1}. This gives

$$Q_n = w[w\, Q_{n-2} + Q^*_{n-1}(1)] + Q_n^*(1).$$

By applying the reduction process to the graphs Q_{n-k} $(1<k\leq n-1)$ in a similar manner, the result is obtained.

LEMMA 5.

$$Q_n = w \sum_{r=0}^{n-1} Q_n^* -1\,(r) + Q_n^*(n) \quad (Q^*_{n-1}(0) \equiv Q_{n-1}.$$

PROOF. Apply the reduction process to the graph Q_n by deleting a twig. This yields

$$Q_n = w\, Q_{n-1} + Q_n^*(1).$$

$$= w\, Q_{n-1} + [w\, Q^*_{n-1}(1) + Q_n^*(2)],$$

by applying the reduction process to the graph $Q_n^*(1)$, by deleting a twig. The result follows by continuing the process on $Q_n^*(2)$, then $Q_n^*(3)$, etc.

THEOREM 7. Let G_n be the graph consisting of a graph G with n twigs attached to a single node, say node x. Then

$$G_n = w\ G_{n-1} + w^{n-1}\ G^* + (n-1)w^{n-1}\ G'',$$

where the graph G'' is the graph G-{x} and G* is G with an incorporated twig attached to x.

PROOF. We can use Lemma 5. Since $G^*_{n-1}(r) = 0$ for $r > 2$, we get

$$G_n = w\ G_{n-1} + w\ G^*_{n-1}(1) + G^*_n(2). \tag{4.1}$$

Apply the reduction process to the graph $G^*_{n-1}(1)$ be deleting a twig at node x. This yields

$$G^*_{n-1}(1) = w\ G^*_{n-2}(1) + G^*_{n-1}(2)$$

$$= w[w\ G^*_{n-3}(1) + G^*_{n-2}(2)] = G^*_{n-1}(2).$$

By substituting in (4.1), we get

$$G_n = w\ G_{n-1} + w^3 G^*_{n-3}(1) + w^2\ G^*_{n-2}(2) + w\ G^*_{n-1}(2) + G^*_n(2).$$

We can continue the reduction process on the graphs $G^*_i(1)$, until i reduces to 2. Now

$$G^*_2(1) = w\ G^*_1(1) + G^*_2(2)$$

Hence we get

$$G_n = w\ G_{n-1} + w^{n-2}\ (w\ G^*_1(1)) + \sum_{r=0}^{n-2} w^r\ G^*_{n-r}(2). \tag{4.2}$$

Since two edges are incorporated at node x in each of the graphs $G^*_{n-r}(2)$, none of the other edges adjacent to node x can be used in any path cover of $G^*_{n-r}(2)$. Hence,

$$G^*_{n-r}(2) = w^{n-r-1}\ G'',$$

since every cover of $G^*_{n-r}(2)$ must contain the (n-r-2) nodes adjacent to node x as isolated nodes (with weight w^{n-r-2}) and a path with 3 nodes (including node x) as a component (with weight w).

$$\Rightarrow G_n = w\ G_{n-1} + w^{n-1}\ G^* + \sum_{r=0}^{n-2} w^r . W^{n-r-1}\ G'' \quad (G^*_1(1) \equiv G^*)$$

$$= w\ G_{n-1} + w^{n-1}\ G^* + (n-1)\ w^{n-1}\ G'',$$

as required.

Notice that Theorem 7 could also be proved combinatorially by specifying one of the n twigs and considering when it does not appear in a path cover ($w\ G_{n-1}$), when it is the only twig that appears ($w^{n-1}G^*$), and when it appears with one of the other n-1 twigs ($(n-1)w^{n-1}G''$). The analytical proof has been given to empahsize the use of Theorem 3.

Theorems 6 and 7 are useful for finding path polynomials of trees. For example, let T be the tree shown below in Figure 2 (a)

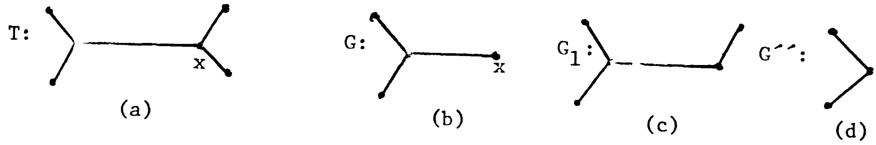

Figure 2

By using Theorem 7, with G as the graph shown in Figure 2 (b), we get

$$T = G_2 = w\ G_1 + w\ G^* = w\ G''$$

$$= w\ (w^5+4w^4+6w^3=3w^2) + w(w^4+3w^3+3w^2) + w(w^3+2w^2+w)$$

$$= w^6 + 5w^5 + 10w^4 + 8w^3 + w^2.$$

Let G(n) be the graph obtained by attaching a path with n (>0) nodes to a graph G. We can apply the reduction process to the graph G(n) by deleting the terminal edge of the path. This yields

$$G(n) = w\ G(n-1) + G^*(n), \tag{4.3}$$

where the graph G*(n) is the graph G(n) with the terminal edge of the path incorporated.

There is no difference between the simple path polynomials of the graphs G*(n) and G(n-1) since the incorporated edge and the node of valency 2 to which it is attached can be regarded as the terminal "node" of G(n-1). Hence

$$G(n) = (1 + w)\ G(n - 1).$$

$$= (1 + w)^{n-1}\ G(1).$$

By using Lemma 4, for the graph G(1) ($\equiv G_1$), we get the following results.

THEOREM 8

$$G(n) = (1+w)^{n-1}\ (wG + G^*)$$

where the graph G* is the graph G(n) with the entire path incorporated.

By taking the graph G as an isolated node, we obtain the following corollary.

COROLLARY 8.1. Let P_n be the path with n nodes. Then

$$P(P_n;w) = w(1 + w)^{n-1}$$

5. PATH POLYNOMIALS OF MULTIGRAPHS.

Let G^n be a multigraph containing two nodes x and y joined by n edges. In any cover of G^n, either nodes x and y are adjacent, or they are not. If they are, then there are n ways of choosing an edge xy. If they are not, then the covers will be covers of the graph G' obtained from G^n by deleting the n edges joining nodes x and y. The covers in which x and y are adjacent are covers of the graph G*, obtained from G^n by incorporating an edge xy and deleting the n-1 remaining edges xy. This discussion leads to the following theorem.

THEOREM 9.

$$P(G^n;\underline{w}) = P(G';\underline{w}) + nP(G^*;\underline{w})$$

Let G^1 be the graph obtained from $G^{n'}$ by deleting n-1 of the edges which join x to y. We can apply the reduction process to G^1 by removing the edge xy. This yields

$$P(G^1;\underline{w}) = P(G';\underline{w}) + P(G^*;\underline{w}).$$

Hence, $$P)G^*;\underline{w}) = P(G^1;\underline{w}) - P(G';\underline{w}).$$

By substituting for $P(G^*;\underline{w})$ in Theorem 9, we obtain the following results:

THEOREM 10.

$$P(G^n;\underline{w}) = nP(G^1;\underline{w}) - (w-1)\ P(G';\underline{w}).$$

The expression for $P(G^n;\underline{w})$, given in the above theorem, is useful when G^1 is a graph whose path polynomial is known or could be easily found.

6. APPLICATION TO MINIMUM PATH COVERS IN GRAPHS.

The path polynomial of a graph contains much information about the paths in the graph. It might therefore be useful in an investigation involving the paths in a given graph. In this section, we illustrate the use of the polynomials by deriving and extending some known results.

Let us denote, by $\zeta(G)$, the minimun number of elements in any path cover of G. $\zeta(G)$ has been called the *path-to-point covering number* of G in Boesch, Chen and McHugh.

In the case of edge disjoint paths, the analogous parameter is call the *path number* (see Haray [6]). The path cover of G has been called an *island decomposition of G* by Goodman and Hedetniemi [7].

In [5], an investigation was made into the relation of ζ to other well-known graphical invariants and an algorithm was developed to determine ζ for trees. In Barnette [8] and Klee [9], ζ arose in the study of Hamiltonian graphs. In [7], an efficient algorithm was derived for finding the minimum number of edges needed to be added to a graph in order to make it Hamiltonian, i.e. the *Hamiltonian completion number* of the graph, denoted by hc(G). Clearly, when G is non-hamiltonian,

$$\zeta(G) = hc(G),$$

since a path cover with cardinalty k can be changed into a Hamiltonian path by adding k new edges.

It is clear that $\zeta(G)$ is the smallest power of w in $P(G;w)$. Let G_n be the graph described in Theorem 7. Then

$$P(G_n;w) = wP(G_{n-1};w) + w^{n-1}P(G^*;w) + (n-1)w^{n-1}P(G'';w).$$

Now G'' is a subgraph of the graph G_{n-1} with n less nodes. Any minimum cover C of G'' can be extended to a minimum cover of G_{n-1} by adding at least n-2 components, since either x can extend a path in C, - causing at least n-2 components to be added to C, or x cannot - again adding at least n-2 components to the cover. Hence

$$\zeta(G_{n-1}) \geq n-2 + \zeta(G'').$$

$$\Rightarrow \quad \zeta(wG_{n-1}) \geq n-1 + \zeta(G'').$$

It is clear that $\zeta(w^{n-1}G^*) = n + \zeta(G'')$.

Also, $\zeta((n-1)w^{n-1}G'') + n-1 + \zeta(G'')$.

Therefore, we obtain the following result.

COROLLARY 7.1.

$$\zeta(G_n) = n-1 + \zeta(G'').$$

This result was obtained in [5] (Theorem 3(i)).

Notice that, if we put n=2, we get

$$\zeta(G_2) = 1 + \zeta(G'').$$

$$\Rightarrow \zeta(G'') = \zeta(G_2) - 1.$$

The result for the special case in which the graph G is a tree is given in [7] (Lemma 3).

From Theorem 8, we have

$$G(n) = (1+w)^{n-1}(wG+G^*).$$

$$\Rightarrow \zeta(G(n)) = \min (1+\zeta(G), \zeta(G^*)).$$

Any minimum cover of G can be extended to a minimum cover of G* by adding at most one component, the attached path. Therefore,

$$1 + \zeta(G) \geq \zeta(G^*).$$

Hence we have the following result.

COROLLARY 8.2.

$$\zeta(G(n)) = \zeta(G^*).$$

This corollary generalizes Theorem 3(ii) of [5].

If we put n=3 in Equation (4.3), we get

$$G(3) = wG(2) + G^*(3).$$

Since G*(3) has the terminal edge incorporated,

$$G^*(3) = G(2).$$

$$\Rightarrow G(3) + (1+w)\ G(2).$$

$$\Rightarrow \zeta(G(3)) = \zeta(G(2)). \qquad (6.1)$$

This result was also obtained in [5] (Theorem 3(ii)). In the special case where G is a tree and v is the terminal node of the attached path, we get

$$\zeta(T) = \zeta(T-\{v\}), \qquad (6.2)$$

where T is the tree consisting of G with the attached path and T-{v} is the tree obtained from T by removing node v.

By replacing ζ with hc in (6.2), we get

$$hc(T) = hc(T-\{v\}).$$

This result was obtained in [7] by a different technique.

Let us now consider the graph G of Theorem 5. It follows immediately from this theorem, that

$$\zeta(G) = I-k + \sum_{i=1}^{k} \zeta(H_i). \qquad (6.3)$$

Let G_1 be the graph of Lemma 4. Then G_1 and H_1 are identical. Therefore,

$$\zeta(H_1) = \zeta(G_1) = \zeta(G^*)$$

from Corollary 8.2.

Let x be the node of H_1 to which the twig is incident, and let B_i'' be the graph obtained from B_i by removing the nodes to which the twigs are incident. Any minimum cover of B_1'' can be extended to a minimum cover of G* by adding at most one component (the twig, in the case where all the nodes adjacent to x have degree 2) to the minimum cover. Hence

$$\zeta(G^*) = \zeta(H_1) \le 1 + \zeta(B_1'').$$

Similarly, for

$$r=2,\ 3,\ldots,\ k-1,$$

$$\zeta(H_r) \le 2 + \zeta(B_r''),$$

$$\zeta(H_k) \le 1 + \zeta(B_k'').$$

Hence, we have the following result, obtained by substituting for $\zeta(H_i)$ in (6.3).

THEOREM 11. Let G be a graph consisting of k graphs $B_1, B_2,\ldots, B_k$ chained together by paths of length 2. Let B_i'' be the graph obtained from B_i by removing the nodes to which the paths are chained. Then

$$\zeta(G) \le \sum_{i=1}^{k} \zeta(B_i'') + k-1.$$

Notice that, if the nodes which are common to the paths and the graph B_i all have valency 2, then we obtain the following corollary.

COROLLARY 11.1.

$$\zeta(G) = \sum_{i=1}^{k} \zeta(B_i) - k+1.$$

PROOF. In this case,

$$\zeta(H_i) = \zeta(B_i) \quad (i = 1, 2,\ldots, k),$$

since the attachment of a twig to a node of valency 1 cannot increase the minimum number of elements in a cover, as shown above in (6.1). The result therefore follows from (6.3).

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