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A SCALAR GEODESIC DEVIATION EQUATION AND A PHASE THEOREM

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ABSTRACT. A scalar equation is derived for η, the distance between two structureless test particles falling freely in a gravitational field:

$$\ddot{\eta} + (K - \Omega^2)\eta = 0.$$

An amplitude, frequency and a phase are defined for the relative motion. The phases are classed as elliptic, hyperbolic and parabolic according as

$$K - \Omega^2 > 0,$$
$$< 0,$$
$$= 0.$$

In elliptic phases we deduce a positive definite relative energy E and a phase-shift theorem. The relevance of the phase-shift theorem to gravitational plane waves is discussed.

KEY WORDS AND PHRASES. Geodesic deviation, gravitational radiation.

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1. INTRODUCTION.

It is well known that the geodesic deviation equation

$$\ddot{\eta}^i + R^i{}_{jk\ell}\, u^j\, \eta^k u^\ell = 0$$

in general relativity is a physical equation, because it relates the relative acceleration between neighbouring test particles to certain physical components of the Riemann-curvative tensor [7]. However, instead of working with the standard tensor form of the geodesic deviation equation, we use the 'scalar form'

$$\ddot{\eta} + (K - \Omega^2) = 0$$

which is derived in Sec. 2. The reason for adopting this approach is because of the difficulty in formally solving the tensor form of the equation except in the most trivial cases. The 'scalar form', though containing less information regarding the source field can be solved formally relatively easily. Further the analogy with the equation of a time-dependent simple harmonic oscillator can be exploited to give insight into the phenomenon governed by the equation.

For example, the formal solution of the scalar equation shows the motion of the test particles to be 'wave-like', where the amplitude of the wave is a function of the phase. Strictly this means that the motion may not be periodic at all, but as we shall see that the analogy is useful. Also this analogy enables us to define the concepts of phase, amplitude and relative energy associated with the motion. From the solution the existence of the bimodal character of the vibrations can be seen. The difference in the two modes, i.e., one mode tends to pull together while the other repels simultaneously, can also be seen. Thus the well known fact that the effect of any gravitational source on two test particles is tidal in character, can be seen. The scalar equation also has a constant of the motion associated with it.

In Sec. 4 the phase shift, due to the effect of a perturbation in the source, on the motion of the test particles is deduced.

2. THE SCALAR DEVIATION EQUATION.

The geodesic deviation equation is the equation of motion of the space-like part of the deviation vector η^i between two neighbouring test particles in a gravitational field, namely

$$\ddot{\eta}^i + R^i_{\ jk\ell}\, u^j \eta^k u^\ell = 0, \tag{2.1}$$

where

$$\eta^i u_i = 0,\ u^i u_i = 1,$$
$$i,j,k,\ell = 0,1,2,3,$$

u^i being the unit time-like tangent vector to a geodesic, and covariant differentiation along u^i is indicated by a '.' [9].

Let $\eta^i = \eta\mu^i$, where $\mu^i\mu_i = -1$. (2.2)

Then $\mu^i \overset{*}{=} (0,\mu^\alpha)$ are the direction cosines of the deviation vector in the local rest space.

On substituting (2.2) in (2.1) we obtain,

$$\ddot{\eta}\,\mu^i + 2\,\dot{\eta}\,\dot{\mu}^i + \eta\,\ddot{\mu}^i + \eta\,R^i{}_{jk\ell}\,u^j\mu^k u^\ell = 0. \tag{2.3}$$

Clearly $\mu_i\dot{\mu}^i = 0$, $\mu_i\ddot{u}_i + \dot{\mu}_i\dot{\mu}^i = 0$ and as $\dot{\mu}^i$ is space-like, i.e., $\dot{\mu}_i\dot{\mu}^i \equiv -\Omega^2 \leq 0$ then $\mu_i\ddot{\mu}^i = \Omega^2 \geq 0$.

On transvecting (2.3) with μ_i we get

$$\ddot{\eta} + (K - \Omega^2)\eta = 0, \tag{2.4}$$

where $K = -R_{ijk\ell}\,u^i\mu^j u^k\mu^\ell$ and the form of the geodesic deviation equation used in this article is obtained [10]. The resemblance of (2.4) to the equation of a time-dependent simple harmonic oscillator should be noted. All the details of the derivation are given in reference [2].

3. AMPLITUDE, PHASE AND RELATIVE ENERGY.

Let ρ be a particular solution of the differential equation

$$\ddot{\rho} + (K - \Omega^2)\rho = \frac{1}{\rho^3} \tag{3.1}$$

which never vanishes [5,8] then,

$$\frac{d}{ds}\left[\frac{\eta^2}{\rho^2} + (\rho\dot{\eta} - \eta\dot{\rho})^2\right] = 0,$$

i.e.

$$\frac{\eta^2}{\rho^2} + (\rho\dot{\eta} - \eta\dot{\rho})^2 = a^2 \tag{3.2}$$

is a conserved quantity for the equation of motion (2.4). It is usual to call $J = \frac{1}{2}a^2$, the Lewis invariant of (2.4). It has the character of an action [2,5]. On introducing the phase variable

$$\Phi(s) = \int_{s_c}^{s} \frac{ds'}{\rho^2(s')}$$

we can solve (3.2) for $\frac{\eta}{\rho}$ to get

$$\eta = a\rho \cos(\Phi) . \tag{3.3}$$

The amplitude $A(s) = a\rho$ (3.4)

is related to the phase by relation

$$A^2 \dot{\Phi} = a^2 . \tag{3.5}$$

When $K - \Omega^2 > 0$ we say that the relative motion is in an elliptic phase as Φ is real. Otherwise the phase is called hyperbolic or parabolic according as $K - \Omega^2 < 0$ or $= 0$. [1]. We will restrict our attention to the recurrence of elliptic phases and so we can invert (3.3)

$$\rho(s) = \frac{\eta(s)}{a} \{1 + (\int_{s_o}^{s} \frac{ds'}{\eta^2(s')})^2\}^{\frac{1}{2}} .$$

Thus we know the phase $\Phi(s)$ in terms of ρ and so in terms of η, i.e. in principle, the observable quantity η determines the phase $\Phi(s)$ and the amplitude $A(s)$ [11].

A further reason for the restriction to elliptic phases is that when a hyperbolic phase arises it always terminates in the onset of a new elliptic phase.

Since J is an exact invariant and has, as noted earlier, the dimensions of "energy/frequency" and as

$$\omega = \frac{1}{\rho^2} \tag{3.6}$$

is a frequency we can define an energy for the relative motion to be

$$E = J\omega = \tfrac{1}{2}[\frac{\eta^2}{\rho^4} + (\dot{\eta} - \frac{\dot{\rho}}{\rho}\eta)^2] . \tag{3.7}$$

If we make the adiabatic assumption, namely that $\frac{\dot{\omega}}{\omega}$ is small, then $\frac{\dot{\rho}}{\rho}$ and $\frac{\ddot{\rho}}{\rho}$ can be neglected and equation (3.7) gives

$$J = \frac{E}{\omega} = \tfrac{1}{2}[\frac{\dot{\eta}^2 + \omega^2\eta^2}{\omega}] ,$$

where $\omega^2 \simeq K - \Omega^2$ will hold approximately [5].

4. THE "PHASE-SHIFT" THEOREM.

The question of phase stability will now be discussed. This will lead to a 'phase shift' criterion.

Using (3.6) in (3.1) we get

$$I(s) = K - \Omega^2 = \omega^2 + \frac{\ddot{\omega}}{2\omega} - \frac{3}{4}(\frac{\dot{\omega}}{\omega})^2 . \tag{4.1}$$

For stability the variational derivative of (4.1) is taken which gives

$$2\omega\ \delta(I(s)) = \{\frac{\ddot{\omega}}{\omega} - \frac{3}{4}\left(\frac{\dot{\omega}}{\omega}\right)^2 - 4\omega^2\}\ \delta\omega + 3\ \frac{\dot{\omega}}{\omega}\ \delta\dot{\omega} - \delta\ddot{\omega}$$

whose solution [3,4] is

$$\delta\omega(s) = -\omega(s) \int_{s_0}^{s} \frac{\delta I(s'')}{\omega(s'')}\ \sin\{2 \int_{s''}^{s} \omega(s')ds'\}ds''. \tag{4.2}$$

If the relative motion of the test particles is to be stable, they must have smooth motions, or motions which can be varied continuously from one configuration to another. In terms of variations, it is possible to define a continuous parameter α which distinguishes one particular configuration from another. So (4.2) may be rewritten as,

$$\delta\omega(s,\alpha) = -\ \omega(s,\alpha) \int_{s_0}^{s} \frac{\delta I(s'',\alpha)}{\omega(s'',\alpha)} \sin\{2 \int_{s''}^{s}\omega(s',\alpha)ds'\}ds'', \tag{4.3}$$

where $\delta\omega(s,\alpha) = \frac{\partial\omega(s,\alpha)}{\partial\alpha}\, d\alpha \equiv \omega_\alpha(s,\alpha)d\alpha$ and we give α the range $0 \leq \alpha \leq 1$. Then on integrating (4.3) with respect to α we get

$$\Delta\omega(s) \equiv \omega(s,0) - \omega(s,1) = \int_0^1 d\alpha \int_{s_0}^{s} \frac{I_\alpha(s'',\alpha)}{\omega(s'',\alpha)} \sin\{2\int_{s''}^{s}\omega(s',\alpha)ds'\}ds''. \tag{4.4}$$

A further integrating of (4.4) with respect to proper time gives for the new left hand side

$$\Phi_0(s) - \Phi_1(s) = \int_{s_0}^{s} \Delta\omega(s''')\ ds''' \tag{4.5}$$

which is a phase shift seen in the relative motion of the two test bodies due to a perturbation in the source. For the new right hand side we get after changing the order of the integration variables, [3].

$$\int_0^1 d\alpha \int_{s_0}^{s} \frac{I_\alpha(s'',\alpha)}{\omega(s'',\alpha)} \sin^2\{ \int_{s''}^{s}\omega(s',\alpha)ds'\}ds''. \tag{4.6}$$

This expression is complicated, but it can be understood in the following way. Since the system has a time-dependent phase, so the first integral from the right hand side accounts for the phase. The multiplicative factor $I_\alpha(s'',\alpha)/\omega(s'',\alpha)$ is expected if one notes that from (4.1) it is the frequency shift of the system, which is also time-dependent. The second integral from the right hand side indicates the times for the observation of the phases. The final integral gives the average over all virtual configurations within that time period. For the time independent simple harmonic oscillator, since the motion is periodic this situation does not arise at all.

In the next section we apply the phase shift theorem to the plane wave metric.

5. THE GRAVITATIONAL PLANE WAVE.

The space-time metric

$$ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2 - G(x,y,u)\, du^2 \ ,$$

where $u = ct - z$, $G(x,y,u) = (x^2 - y^2)E(u)$

and $E(u)$ an arbitrary differentiable function of the retarded distance u, describes one mode of a plane gravitational wave [7]

$$\ddot{\eta}^1 = - E(u)\left(\frac{du}{ds}\right)^2 \eta^1 \ ,$$

$$\ddot{\eta}^2 = E(u)\left(\frac{du}{ds}\right)^2 \eta^2 \ ,$$

$$\eta^0 = \eta^3 = 0,$$

where du/ds is a Doppler shift factor and

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z.$$

Equation (2.4) becomes

$$\ddot{\eta} + \eta[E(u)\left(\frac{du}{ds}\right)^2(\mu^1\mu_1 - \mu^2\mu_2) - \Omega^2] = 0$$

or

$$\ddot{\eta} + \eta[E(u)\left(\frac{du}{ds}\right)^2(1-2\cos^2\theta) - \Omega^2] = 0,$$

where $\mu^2\mu_2 = \cos^2\theta$. From the effect of a gravitational shock wave on equation (2.4), which is given in the appendix, it is reasonable to assume that only the term $E_\alpha(u,\alpha)$ will be significant when the variation over different configurations are taken. Thus the phase-shift theorem reduces to

$$\Delta\Phi = \int_0^1 d\alpha \int_{s_0}^{s} \frac{E_\alpha(u,\alpha)}{(s'',\alpha)} \left(\frac{du}{ds''}\right)^2 (1-2\cos^2\theta_\alpha)\, \sin^2\{\int_{s''}^{s}\omega(s',\alpha)ds'\}ds''$$

$$= \int_0^1 d\alpha \int_{s_0}^{s} E_\alpha(u,\alpha)\left(\frac{du}{ds''}\right)^2(1-2\cos^2\theta_\alpha)\, \frac{\eta^2(s'',\alpha)}{a^2(\alpha)}\, \{1+(\int_{s_0}^{s''} \frac{ds'''}{\eta^2(s''',\alpha)})^2\}\times$$

$$\sin^2\{\int_{s''}^{s}\omega(s',\alpha)ds'\}ds''.$$

On examination we see that the phase-shift is expressed completely in terms of the observable quantity η.

APPENDIX.

As a gravitational shock wave passes over a pair of freely falling structureless test particles the discontinuity $[R_{ijk\ell}]$ in the curvature tensor causes a discontinuity in the relative acceleration given by [6].

$$[\ddot{\eta}^i] + [R^i{}_{jk\ell}]\, u^j \eta^k u^\ell = 0$$

or using (3) we get

$$[\ddot{\eta}]\mu^i + \eta[\ddot{\mu}^i] + \eta[R^i{}_{jk\ell}]\, u^j \mu^k u^\ell = 0 \tag{A1}$$

As $$[\mu_i\ \ddot{\mu}^i] = -\ [\dot{\mu}_i\ \dot{\mu}^i] = 0$$

we can transvect (A1) with μ_i to get

$$[\ddot{\eta}] + [K]\eta = 0. \tag{A2}$$

In the limit as a plane sandwich wave approaches such a shock front, all physical quantities in the coefficient of η other than the curvature tensor component K have negligible variation, which is the comment made in section 5.

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REFERENCES.

1. BORUVKA, O., Linear Differential Transformations of the Second Order, VEB Berlin 1967, English Translation by F.M. Arscott 1971, English Universities Press.

2. DOLAN, P., CHOUDHURY, P. and SAFKO, J.L., A "constant of the motion" for the geodesic deviation equation, Journ. Austral. Math. Soc. 22 Series B (1980), 28-33.

3. EL'SIN, M.I., The phase method and the classical method of Comparison, Doklady Akad. Nauk. SSSR (N.S.) 68 (1949), 813-816.

4. EL'SIN, M.I., On the method of evaluating the phase of a linear differential equation of the second order, Mathematika Ucenye Zapishi Moskov. Gosud. Univ. 45 (1940), 97-108.

5. LEWIS, H.R. JNR., Class of exact invariants for classical and quantum time-dependent harmonic oscilators, Journ. Math. Phys. 9 (1968), 1976-1988.

6. LICHNEROWICZ, A., Electromagnetic and Gravitational waves and Radiation in General Relativity, Annali di Mat. 50 (1960), 1-95.

7. PERES, A., Some Gravitational Waves, Phys. Rev. Lett. 3 (1959), 571-572.

8. PINNEY, E., The non-linear differential equation $y''+p(x)y+\frac{c}{y^3}=0$. Proc. Amer. Math. Soc. 1 (1950), 681.

9. PIRANI, F.A.E., On the physical significance of the Riemann tensor, Acta Physica Polon. 13 (1956), 389.

10. SYNGE, J.L., On the geometry of dynamics, Phil. Trans. Roy. Soc. 226A (1926), 31-106.

11. TAMOR, S., On extended adiabatic invariants, Jour. Nuclear Energy 1C (1960), 199-205.

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