Internat. J. Math. & Math. Sci.
Vol.6 No.4 (1983) 671-703

ADVANCED DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT DEVIATIONS

S.M. SHAH

Department of Mathematics
University of Kentucky
Lexington, Kentucky 40502

and

JOSEPH WIENER

Department of Mathematics
Pan American University
Edinburg, Texas 78539

(Received November 15, 1983)

ABSTRACT. Functional differential equations of advanced type with piecewise constant argument deviations are studied. They are closely related to impulse, loaded and, especially, to difference equations, and have the structure of continuous dynamical systems within intervals of unit length.

KEY WORDS AND PHRASES. Functional Differential Equation, Advanced Equation, Difference Equation, Piecewise Constant Deviation, Initial-Value Problem, Solution, Existence, Uniqueness, Backward Continuation, Growth, Stability.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 34K05, 34K10, 34K20, 39A10.

1. INTRODUCTION.

In [1] and [2] analytic solutions to differential equations with linear transformations of the argument are studied. The initial values are given at the fixed point of the argument deviation. Integral transformations establish close connections between entire and distributional solutions of such equations. Profound links exist also between functional and functional differential equations. Thus, the study of the first often enables one to predict properties of differential equations of neutral type. On the other hand, some methods for the latter in the special case when the deviation of the argument vanishes at individual points has been used to investigate functional equations [3]. Functional equations are directly related to difference equations of a discrete (for example, integer-valued) argument, the theory of which has been very intensively developed in the book [4] and in numerous subsequent papers. Bordering on difference equations are also impulse functional differential equations with impacts and switching, loaded equations (that is, those including values of the unknown solution for given constant values of the argument), equations

$$x'(t) = f(t, x(t), x(h(t)))$$

with arguments of the form $[t]$, that is, having intervals of constancy, etc. A substantial theory of such equations is virtually undeveloped [5].

In this article we study differential equations with arguments $h(t) = [t]$ and $h(t) = [t+n]$, where $[t]$ denotes the greatest-integer function. Connections are established between differential equations with piecewise constant deviations and difference equations of an integer-valued argument. Impulse and loaded equations may be included in our scheme too. Indeed, consider the equation

$$x'(t) = ax(t) + a_0x([t]) + a_1x([t+1]) \tag{1.1}$$

and write it as

$$x'(t) = ax(t) + \sum_{i=-\infty}^{\infty} (a_0x(i) + a_1x(i+1))(H(t-i) - H(t-i-1)),$$

where $H(t) = 1$ for $t > 0$ and $H(t) = 0$ for $t < 0$. If we admit distributional derivatives, then differentiating the latter relation gives

$$x''(t) = ax'(t) + \sum_{i=-\infty}^{\infty} (a_0x(i) + a_1x(i+1))(\delta(t-i) - \delta(t-i-1)),$$

where δ is the delta functional. This impulse equation contains the values of the unknown solution for the integral values of t. In the second section Eq. (1.1) is considered. The initial-value problem is posed at $t = 0$, and the solution is sought for $t > 0$. The existence and uniqueness of solution and of its backward continuation on $(-\infty, 0]$ is proved. Furthermore, an important fact is established that the initial condition may be posed at any point, not necessarily integral. Necessary and sufficient conditions of stability and asymptotic stability of the trivial solution are determined explicitly via coefficients of the given equation, and oscillatory properties of solutions to (1.1) are studied.

In the third part the foregoing results are generalized for equations with many deviations and systems of equations. We show that these equations are intrinsically closer to difference rather than to differential equations. In fact, the equations considered in this paper have the structure of continuous dynamical systems within intervals of unit length. Continuity of a solution at a point joining any two consecutive intervals then implies recursion relations for the solution at such points. The equations are thus similar in structure to those found in certain "sequential-continuous" models of disease dynamics as treated in [6]. We also investigate a class of systems that depending on their coefficients combine either equations of retarded, neutral or advanced type.

In the last section linear equations with variable coefficients are studied. First, the existence and uniqueness of solution on $[0, \infty)$ is proved for systems with continuous coefficients. A simple algorithm of computing the solution by means of continued fractions is indicated for a class of scalar equations. Then, a general estimate of the solutions growth as $t \to +\infty$ is found. Special consideration is given to the problem of stability, and for this purpose we employ a method developed earlier in the theory of distributional and entire solutions to functional differential equations. An existence criterion of periodic solutions to linear equations with periodic coefficients is established. Some nonlinear equations are also tackled.

Consider the nth order differential equation with N argument delays

$$x^{(m_0)}(t) = f(t, x(t), \ldots, x^{(m_0-1)}(t), x(t-T_1(t)), \ldots, x^{(m_1)}(t-T_1(t)), \ldots$$
$$\ldots, x(t-T_N(t)), \ldots, x^{(m_N)}(t-T_N(t))), \tag{1.2}$$

where all $T_i(t) \geq 0$ and $n = \max m_i$, $0 \leq i \leq N$. Here $x^{(k)}(t-T_i(t))$ is the kth derivative of the function $x(z)$ taken at the point $z = t-T_i(t)$. Often equations that can be reduced to the form (1.2) by a change of the independent variable are also discussed. A natural classification of functional differential equations has been suggested in [7]. In Eq. (1.2) let $\mu = \max m_i$, $1 \leq i \leq N$, and $\lambda = m_0 - \mu$. If $\lambda > 0$, then (1.2) is called an equation with retarded (lagging, delayed) argument. In the case $\lambda = 0$, Eq. (1.2) is of neutral type. For $\lambda < 0$, (1.2) is an equation of advanced type. Retarded differential equations with piecewise constant delays (EPCD) have been studied in [8, 9, 10]. Some results on neutral EPCD were announced in [9] and [10]. Together with the present paper, these works enable us to conclude that all three types of EPCD share similar characteristics. First of all, it is natural to pose the initial-value problem for such equations not on an interval but at a number of individual points. Secondly, in ordinary differential equations with a continuous vector field the solution exists to the right and left of the initial t-value. For retarded functional differential equations, this is not necessarily the case [11]. Furthermore, it appears that advanced equations, in general, lose their margin of smoothness, and the method of successive integration shows that after several steps to the right from the initial interval the solution may even not exist. However, two-sided solutions do exist for all types of EPCD. Finally, the striking dissimilarities between the solutions growth of algebraic differential and difference equations are well-known [12, 13]. Since EPCD combine the features of both differential and difference equations, their asymptotic behavior as $t \to \infty$ resembles in some cases the solutions growth of differential equations, while in others it inherits the properties of difference equations. In conclusion, we note that the separate study of Eq. (1.1) and more general equations with many argument deviations is motivated not only by instructive purposes and the possibility of obtaining some deeper results in this special case but, mainly, by the fact that (1.1) possesses properties of both advanced and neutral equations.

2. EQUATIONS WITH CONSTANT COEFFICIENTS.

Consider the scalar initial-value problem

$$x'(t) = ax(t) + a_0x([t]) + a_1x([t+1]),\ x(0) = c_0 \tag{2.1}$$

with constant coefficients. Here $[t]$ designates the greatest-integer function. We introduce the following

DEFINITION 2.1. A solution of Eq. (2.1) on $[0, \infty)$ is a function $x(t)$ that satisfies the conditions:

(i) $x(t)$ is continuous on $[0,\infty)$.

(ii) The derivative $x'(t)$ exists at each point $t \in [0, \infty)$, with the possible exception of the points $[t] \in [0, \infty)$ where one-sided derivatives exist.

(iii) Eq. (2.1) is satisfied on each interval $[n, n+1) \subset [0, \infty)$ with integral ends.

Denote

$$b_0(t) = e^{at} + a^{-1}a_0(e^{at} - 1),\ b_1(t) = a^{-1}a_1(e^{at} - 1), \tag{2.2}$$

$$\lambda = b_0(1)/(1 - b_1(1)).$$

THEOREM 2.1. Problem (2.1) has on $[0, \infty)$ a unique solution

$$x(t) = (b_0(\{t\}) + \lambda b_1(\{t\}))\lambda^{[t]} c_0, \tag{2.3}$$

where $\{t\}$ is the fractional part of t, if

$$b_1(1) \neq 1. \tag{2.4}$$

PROOF. Assuming that $x_n(t)$ and $x_{n+1}(t)$ are solutions of Eq. (2.1) on the intervals $[n, n+1)$ and $[n+1, n+2)$, respectively, satisfying the conditions $x_n(n) = c_n$ and $x_{n+1}(n+1) = c_{n+1}$, we have

$$x_n'(t) = ax_n(t) + a_0 c_n + a_1 c_{n+1}, \tag{2.5}$$

since $x_n(n+1) = x_{n+1}(n+1)$. The general solution of this equation on the given interval is

$$x_n(t) = e^{a(t-n)}c - a^{-1}(a_0 c_n + a_1 c_{n+1}),$$

with an arbitrary constant c. Putting here $t = n$ gives

$$c = (1 + a^{-1}a_0)c_n + a^{-1}a_1 c_{n+1}$$

and

$$x_n(t) = b_0(t-n)c_n + b_1(t-n)c_{n+1} \tag{2.6}$$

For $t = n+1$ we have

$$c_{n+1} = b_0(1)c_n + b_1(1)c_{n+1}, \quad n \geq 0$$

and inequality (2.4) implies

$$c_{n+1} = \frac{b_0(1)}{1 - b_1(1)} c_n .$$

With the notations (2.2), this is written as

$$c_{n+1} = \lambda c_n, \quad n \geq 0.$$

Hence,

$$c_n = \lambda^n c_0.$$

This result together with (2.6) yields (2.3). Formula (2.3) was obtained with the implicit assumption $a \neq 0$. If $a = 0$, then

$$x(t) = \left(1 + \frac{a_0 + a_1}{1 - a_1}\{t\}\right)\left(\frac{1 + a_0}{1 - a_1}\right)^{[t]} c_0,$$

which is the limiting case of (2.3) as $a \to 0$. The uniqueness of solution (2.3) on $[0, \infty)$ follows from its continuity and from the uniqueness of the problem $x_n(n) = c_n$ for (2.5) on each interval $[n, n+1]$. It remains to observe that hypothesis (2.4) is equivalent to

$$a_1 \neq a/(e^a - 1).$$

In the particular case $b_0(1) = 0$ we have $\lambda = 0$, and formula (2.3) holds true assuming $0^0 = 1$.

REMARK. If $b_1(1) = 1$, two possibilities may occur: the case $b_0(1) \neq 0$ implies $x(t) = 0$, and for $b_0(1) = 0$ problem (2.1) has infinitely many solutions.

COROLLARY. The solution of (2.1) cannot grow to infinity faster than exponentially as $t \to +\infty$.

PROOF. The values $b_0(\{t\})$ and $b_1(\{t\})$ in (2.3) are bounded. Therefore, $|x(t)| \leq m|\lambda|^{[t]}$, with some constant m.

THEOREM 2.2. The solution of (2.1) has a unique backward continuation on $(-\infty, 0]$ given by formula (2.3) if

$$b_0(1) \neq 0. \tag{2.7}$$

PROOF. If $x_{-n}(t)$ and $x_{-n+1}(t)$ denote the solutions of Eq. (2.1) on the intervals $[-n, -n+1)$ and $[-n+1, -n+2)$, respectively, satisfying $x_{-n}(-n) = c_{-n}$ and $x_{-n+1}(-n+1) = c_{-n+1}$, then by virtue of the condition $x_{-n}(-n+1) = x_{-n+1}(-n+1)$, it follows from the equation

$$x'_{-n}(t) = ax_{-n}(t) + a_0c_{-n} + a_1c_{-n+1}$$

that

$$x_{-n}(t) = e^{a(t+n)}c - a^{-1}(a_0c_{-n} + a_1c_{-n+1}),$$

where

$$c = (1 + a^{-1}a_0)c_{-n} + a^{-1}a_1c_{-n+1}.$$

Therefore,

$$x_{-n}(t) = (e^{a(t+n)} + (e^{a(t+n)} - 1)a^{-1}a_0)c_{-n} + a^{-1}a_1(e^{a(t+n)} - 1)c_{-n+1}.$$

With the notations (2.2), we have

$$x_{-n}(t) = b_0(t+n)c_{-n} + b_1(t+n)c_{-n+1} \tag{2.8}$$

Putting $t = -n+1$ gives

$$c_{-n+1} = b_0(1)c_{-n} + b_1(1)c_{-n+1}$$

and

$$c_{-n} = \frac{1 - b_1(1)}{b_0(1)} c_{-n+1}.$$

Finally, we write

$$c_{-n} = \lambda^{-1}c_{-n+1}, \quad n \geq 1$$

and

$$c_{-n} = \lambda^{-n} c_0$$

which together with (2.8) proves (2.3) for $t < 0$.

The initial-value problem for Eq. (2.1) may be posed at any point, not necessarily integral.

THEOREM 2.3. If conditions (2.4) and (2.7) hold, and

$$b_0(\{t_0\}) + \lambda b_1(\{t_0\}) \neq 0,$$

then the problem $x(t_0) = x_0$ for Eq. (2.1) has a unique solution on $(-\infty, \infty)$.

PROOF. By virtue of (2.4) and (2.7), the problem $x(0) = c_0$ for (2.1) has a unique solution on $(-\infty, \infty)$ given by (2.3). Hence,

$$x(t_0) = (b_0(\{t_0\}) + \lambda b_1(\{t_0\}))\lambda^{[t_0]} c_0$$

and

$$c_0 = (b_0(\{t_0\}) + \lambda b_1(\{t_0\}))^{-1} \lambda^{-[t_0]} x_0.$$

It remains to substitute this result in (2.3), to obtain

$$x(t) = (b_0(\{t\}) + \lambda b_1(\{t\}))(b_0\{t_0\} + \lambda b_1(\{t_0\}))^{-1} \lambda^{[t]-[t_0]} x_0.$$

THEOREM 2.4. The solution $x = 0$ of Eq. (2.1) is stable (respectively, asymptotically stable) as $t \to +\infty$, if and only if $|\lambda| \leq 1$ (respectively, $|\lambda| < 1$).

Proof follows directly from (2.3).

THEOREM 2.5. The solution $x = 0$ of Eq. (2.1) is stable (respectively, asymptotically stable) as $t \to +\infty$, if and only if

$$(a + a_0 + a_1)(a_1 - a_0 - \frac{a(e^a + 1)}{e^a - 1}) \geq 0 \tag{2.9}$$

(respectively, > 0).

PROOF. The inequality $|\lambda| \leq 1$ can be written as

$$-1 \leq \frac{b_0(1)}{1 - b_1(1)} \leq 1.$$

If $1 - b_1(1) > 0$, then

$$b_1(1) - 1 \leq b_0(1) \leq 1 - b_1(1)$$

and

$$b_1(1) + b_0(1) \leq 1, \quad b_1(1) - b_0(1) \leq 1. \tag{2.10}$$

Since (2.10) implies $b_1(1) \leq 1$, we analyze only (2.10). Taking into account (2.2) gives

$$a^{-1}a_1(e^a - 1) + e^a + a^{-1}a_0(e^a - 1) \leq 1,$$

$$a^{-1}a_1(e^a - 1) - e^a - a^{-1}a_0(e^a - 1) \leq 1.$$

From here, we have

$$a^{-1}(a_0 + a_1)(e^a - 1) \leq 1 - e^a,$$

that is,

$$a + a_0 + a_1 \leq 0, \tag{2.11}$$

and

$$a^{-1}(a_1 - a_0)(e^a - 1) \leq e^a + 1$$

which is equivalent to

$$a_1 - a_0 - \frac{a(e^a + 1)}{e^a - 1} \leq 0. \tag{2.12}$$

If $1 - b_1(1) < 0$, then

$$b_1(1) + b_0(1) \geq 1, \quad b_1(1) - b_0(1) \geq 1.$$

These inequalities imply $b_1(1) \geq 1$. Therefore, we consider only

$$a^{-1}a_1(e^a - 1) + e^a + a^{-1}a_0(e^a - 1) \geq 1,$$

$$a^{-1}a_1(e^a - 1) - e^a - a^{-1}a_0(e^a - 1) \geq 1.$$

These relations yield inequalities opposite to (2.11) and (2.12) and prove the theorem

COROLLARY. The solution $x = 0$ of the equation

$$x'(t) = ax(t) + a_0x([t]) \tag{2.13}$$

is stable (asymptotically stable) iff the inequalities

$$-a(e^a + 1)/(e^a - 1) \leq a_0 \leq - a$$

(strict inequalities) take place.

THEOREM 2.6. In each interval $(n, n+1)$ with integral ends the solution of Eq. (2.1) with the condition $x(0) = c_0 \neq 0$ has precisely one zero

$$t_n = n + \frac{1}{a} \ln \frac{a_0 + a_1e^a}{a + a_0 + a_1}$$

if

$$\left(a_0 + \frac{ae^a}{e^a - 1}\right)\left(a_1 - \frac{a}{e^a - 1}\right) > 0. \tag{2.14}$$

If (2.14) is not satisfied and $a_0 \neq -ae^a/(e^a - 1)$, $c_0 \neq 0$, then solution (2.3) has no zeros in $[0, \infty)$.

PROOF. For $\lambda \neq 0$, $c_0 \neq 0$, the equation $x(t) = 0$ is equivalent to

$$b_0(\{t\}) + \lambda b_1(\{t\}) = 0.$$

Hence,

$$b_0(\{t\})/b_1(\{t\}) = b_0(1)/(b_1(1) - 1)$$

and

$$\frac{e^{a\{t\}} + a^{-1}a_0(e^{a\{t\}} - 1)}{a^{-1}a_1(e^{a\{t\}} - 1)} = \frac{e^a + a^{-1}a_0(e^a - 1)}{a^{-1}a_1(e^a - 1) - 1}.$$

It follows from here that

$$e^{a\{t\}} = (a_0 + a_1e^a)/(a + a_0 + a_1).$$

If $a > 0$, then

$$1 \leq (a_0 + a_1e^a)/(a + a_0 + a_1) < e^a$$

and, by virtue of (2.4), the equality sign on the left must be omitted. The case

$$a + a_0 + a_1 > 0 \tag{2.15}$$

leads to

$$a_0 > - ae^a/(e^a - 1), \quad a_1 > a/(e^a - 1). \tag{2.16}$$

By adding inequalities (2.16) it is easy to see that (2.15) is a consequence of (2.16). If

$$a > 0, \; a + a_0 + a_1 < 0,$$

then

$$(a + a_0 + a_1)e^a < a_0 + a_1e^a < a + a_0 + a_1$$

and

$$a_0 < - ae^a/(e^a - 1), \; a_1 < a/(e^a - 1). \tag{2.17}$$

Again, the inequality $a + a_0 + a_1 < 0$ follows from (2.17). The case $a < 0$, together with (2.4), gives

$$e^a < (a_0 + a_1e^a)/(a + a_0 + a_1) < 1,$$

and assumption (2.15) yields (2.16). And if $a + a_0 + a_1 < 0$, then we obtain (2.17). Hypothesis (2.14) can also be written as $b_0(1)(b_1(1) - 1) > 0$ which is equivalent to $\lambda < 0$.

COROLLARY. In each interval $(n, n+1)$ the solution

$$x(t) = (e^{a\{t\}}(1 + \frac{a_0}{a}) - \frac{a_0}{a})(e^a(1 + \frac{a_0}{a}) - \frac{a_0}{a})^{[t]}c_0 \tag{2.18}$$

of (2.13) satisfying the condition $x(0) = c_0 \neq 0$ has precisely one zero

$$t_n = n + \frac{1}{a} \ln \frac{a_0}{a + a_0}$$

if

$$a_0 < - ae^a/(e^a - 1). \tag{2.19}$$

If (2.19) is not satisfied, solution (2.18) has no zeros in $[0, \infty)$.

From the last two theorems we obtain the following decomposition of the space (a, a_0, a_1).

1. If

$$(a_0 + \frac{ae^a}{e^a - 1})(a_1 - a_0 - \frac{a(e^a + 1)}{e^a - 1}) > 0, \tag{2.20}$$

solution (2.3) with $c_0 \neq 0$ has precisely one zero in each interval $(n, n+1)$ and $\lim x(t) = 0$ as $t \to + \infty$.

In fact, the solution possesses the required properties iff $-1 < \lambda < 0$, that is $\lambda(\lambda + 1) < 0$. With the notations (2.2), we have $b_0(1)(b_0(1) - b_1(1) + 1) < 0$ which yields (2.20).

2. If

$$(a + a_0 + a_1)(a_0 + \frac{ae^a}{e^a - 1}) < 0, \tag{2.21}$$

solution (2.3) with $c_0 \neq 0$ has no zeros in $(0, \infty)$ and $\lim x(t) = 0$ as $t \to + \infty$.

The properties take place iff $0 < \lambda < 1$. Hence, we consider the inequality $\lambda(\lambda - 1) < 0$ leading to (2.21).

3. If

$$(a_1 - \frac{a}{e^a - 1})(a_1 - a_0 - \frac{a(e^a + 1)}{e^a - 1}) < 0, \tag{2.22}$$

solution (2.3) with $c_0 \neq 0$ has precisely one zero in each interval $(n, n+1)$ and is unbounded on $[0, \infty)$.

The proof follows from the inequality $\lambda < - 1$ which is equivalent to $(b_1(1) - 1) \cdot(b_1(1) - b_0(1) - 1) < 0$ and gives (2.22).

4. If

$$(a + a_0 + a_1)(a_1 - \frac{a}{e^a - 1}) < 0, \tag{2.23}$$

solution (2.3) with $c_0 \neq 0$ has no zeros in $(0, \infty)$ and is unbounded.

Relation (2.23) results from $\lambda > 1$ which can be written as $(b_1(1) - 1)(b_0(1) + b_1(1) - 1) < 0$.

5. If $a_1 - a_0 - \dfrac{a(e^a + 1)}{e^a - 1} = 0$, $a_0 + \dfrac{ae^a}{e^a - 1} \neq 0$, solution (2.3) with $c_0 \neq 0$ has precisely one zero in each interval $(n, n+1)$ and is bounded on $[0, \infty)$ (but does not vanish as $t \to \infty$) since, in this case, $\lambda = -1$.

6. If $a_1 - a_0 - \dfrac{a(e^a + 1)}{e^a - 1} = 0$, $a_0 + \dfrac{ae^a}{e^a - 1} = 0$, then $a_1 = a/(e^a - 1)$, and problem (2.1) has infinitely many solutions.

7. If $a + a_0 + a_1 = 0$, $a_1 \neq a/(e^a - 1)$, then $x(t) = c_0$. Indeed, here we have $\lambda = 1$ and

$$x(t) = (b_0(\{t\}) + b_1(\{t\}))c_0 = a^{-1}(e^{a\{t\}}(a + a_0 + a_1) - a_0 - a_1)c_0.$$

8. Finally, if $a_0 + \dfrac{ae^a}{e^a - 1} = 0$, $a_1 - \dfrac{a}{e^a - 1} \neq 0$, then $x(t) = b_0(t)c_0$, for $0 \leq t < 1$, and $x(t) = 0$, for $t \geq 1$.

For Eq. (2.13) we have the following decomposition of the plane (a, a_0).

1. If $-a(e^a + 1)/(e^a - 1) < a_0 < -ae^a/(e^a - 1)$, solution (2.18) with $c_0 \neq 0$ has precisely one zero in each interval $(n, n+1)$ and $\lim x(t) = 0$ as $t \to \infty$.

2. If $-ae^a/(e^a - 1) < a_0 < -a$, then $x(t)$ has no zeros in $(0, \infty)$ and $\lim x(t) = 0$ as $t \to \infty$.

3. For $a_0 < -a(e^a + 1)/(e^a - 1)$ the solution has precisely one zero in each interval $(n, n+1)$ and is unbounded on $[0, \infty)$.

4. For $a_0 > -a$ the solution has no zeros in $(0, \infty)$ and is unbounded.

The case $a_0 = -a$ yields $x(t) = c_0$. If $a_0 = -ae^a/(e^a - 1)$, then $x(t) = b_0(t)c_0$ on $[0, 1)$, and $x(t) = 0$ on $[1, \infty)$. For $a_0 = -a(e^a + 1)/(e^a - 1)$ the solution $x(t)$ if bounded and oscillatory.

3. SOME GENERALIZATIONS.

For the problem

$$x'(t) = Ax(t) + A_0x([t]) + A_1x([t+1]), \quad x(0) = c_0 \tag{3.1}$$

in which A, A_0, A_1 are $r \times r$ - matrices and x is an r-vector, let

$$B_0(t) = e^{At} + (e^{At} - I)A^{-1}A_0, \quad B_1(t) = (e^{At} - I)A^{-1}A_1.$$

THEOREM 3.1. If the matrices A and $I - B_1(1)$ are nonsingular, then problem (3.1) has on $[0, \infty)$ a unique solution

$$x(t) = (B_0(\{t\}) + B_1(\{t\})(I - B_1(1))^{-1}B_0(1))(I - B_1(1))^{-[t]}B_0{}^{[t]}(1)c_0$$

and this solution cannot grow to infinity faster than exponentially.

Consider the equation

$$x'(t) = ax(t) + a_0x([t]) + a_1x([t+1]) + a_2x([t+2]),\ a_2 \neq 0 \tag{3.2}$$

with the initial conditions

$$x(0) = c_0,\ x(1) = c_1. \tag{3.3}$$

Let λ_1 and λ_2 be the roots of the equation

$$b_2(1)\lambda^2 + (b_1(1) - 1)\lambda + b_0(1) = 0, \tag{3.4}$$

where $b_0(t)$ and $b_1(t)$ are given in (2.2) and

$$b_2(t) = a^{-1}a_2(e^{at} - 1).$$

THEOREM 3.2. Problem (3.2) - (3.3) has on $[0, \infty)$ a unique solution

$$x(t) = b_0(\{t\})c_{[t]} + b_1(\{t\})c_{[t+1]} + b_2(\{t\})c_{[t+2]}, \tag{3.5}$$

where

$$c_{[t]} = (\lambda_1{}^{[t]}(\lambda_2c_0 - c_1) + (c_1 - \lambda_1c_0)\lambda_2^{[t]})/(\lambda_2 - \lambda_1), \tag{3.6}$$

and this solution cannot grow to infinity faster than exponentially.

PROOF. For $n \leq t < n+1$, Eq. (3.2) takes the form

$$x'(t) = ax(t) + a_0x(n) + a_1x(n+1) + a_2x(n+2)$$

with the general solution

$$x(t) = e^{a(t-n)}c - a^{-1}(a_0x(n) + a_1x(n+1) + a_2x(n+2)),$$

where c is an arbitrary constant. Hence, a solution $x_n(t)$ of Eq. (3.2) on the given interval satisfying the conditions

$$x(n) = c_n,\quad x(n+1) = c_{n+1},\quad x(n+2) = c_{n+2}$$

is

$$x_n(t) = e^{a(t-n)}c - a^{-1}(a_0c_n + a_1c_{n+1} + a_2c_{n+2}).$$

To determine the value of c, put $t = n$; then

$$c = (1 + a^{-1}a_0)c_n + a^{-1}a_1c_{n+1} + a^{-1}a_2c_{n+2},$$

and

$$x_n(t) = b_0(t-n)c_n + b_1(t-n)c_{n+1} + b_2(t-n)c_{n+2}. \tag{3.7}$$

By virtue of the relation

$$x_n(n+1) = x_{n+1}(n+1) = c_{n+1},$$

we have

$$c_{n+1} = b_0(1)c_n + b_1(1)c_{n+1} + b_2(1)c_{n+2}$$

whence

$$b_2(1)c_{n+2} + (b_1(1) - 1)c_{n+1} + b_0(1)c_n = 0, \quad n \geq 0. \tag{3.8}$$

We look for a nontrivial solution of this difference equation in the form $c_n = \lambda^n$. Then

$$b_2(1)\lambda^{n+2} + (b_1(1) - 1)\lambda^{n+1} + b_0(1)\lambda^n = 0,$$

and λ satisfies (3.4). If the roots λ_1 and λ_2 of (3.4) are different, the general solution of (3.8) is

$$c_n = k_1\lambda_1^n + k_2\lambda_2^n,$$

with arbitrary constants k_1 and k_2. In fact, it satisfies (3.8) for all integral n. In particular, for n = 0 and n = 1 this formula gives

$$k_1 + k_2 = c_0, \quad \lambda_1 k_1 + \lambda_2 k_2 = c_1,$$

and

$$k_1 = (\lambda_2 c_0 - c_1)/(\lambda_2 - \lambda_1), \quad k_2 = (c_1 - \lambda_1 c_0)/(\lambda_2 - \lambda_1).$$

These results, together with (3.7), establish (3.5) and (3.6).

If $\lambda_1 = \lambda_2 = \lambda$, then

$$c_{[t]} = \lambda^{[t-1]}(c_1[t] - \lambda c_0[t-1]),$$

which is the limiting case of (3.6) as $\lambda_2 \to \lambda_1$. Formula (3.5) was obtained with the implicit assumption $a \neq 0$. If $a = 0$, then

$$x(t) = c_{[t]} + (a_0c_{[t]} + a_1c_{[t+1]} + a_2c_{[t+2]})\{t\},$$

which is the limiting case of (3.5) as $a \to 0$. The uniqueness of solution (3.5) on $[0, \infty)$ follows from its continuity and from the uniqueness of the coefficients c_n for each $n \geq 0$. The conclusion about the solution growth is an implication of the estimates for expressions (3.6).

THEOREM 3.3. If $b_0(1) \neq 0$, the solution of (3.2) - (3.3) has a unique backward continuation on $(-\infty, 0]$ given by (3.5) - (3.6).

Theorem 2.3 establishes an important fact that the initial-value problem for Eq. (2.1) may be posed at any point, not necessarily integral. A similar proposition is true also for Eq. (3.2).

THEOREM 3.4. If $b_0(1) \neq 0$ and

$$\beta_0^2 + (1 - b_1(1))\beta_0\beta_1 + ((1 - b_1(1))^2 - 2b_0(1))\beta_0\beta_2 +$$

$$+ b_0(1)\beta_1^2 + b_0(1)(1 - b_1(1))\beta_1\beta_2 + b_0^2(1)\beta_2^2 \neq 0, \tag{3.9}$$

where

$$\beta_0 = b_0(\{t_0\}),\ \beta_1 = b_1(\{t_0\}),\ \beta_2 = b_2(\{t_0\}),$$

then the initial-value problem $x(t_0) = x_0$, $x(t_0 + 1) = x_1$ for (3.2) has a unique solution on $(-\infty, \infty)$.

PROOF. With the notation

$$\mu(t) = (\lambda_2^{[t]} - \lambda_1^{[t]})/(\lambda_2 - \lambda_1),$$

we obtain from (3.6) the equations

$$c_{[t_0+i]} = -b_0(1)\mu(t_0-1+i)c_0 + \mu(t_0+i)c_1,\ i = 0, 1, 2, 3.$$

Substituting these values in (3.5) for $t = t_0$ and $t = t_0 + 1$ and taking into account $\{t_0 + 1\} = \{t_0\}$ gives the system

$$\begin{aligned}
&(-b_0(1)(\mu(t_0 - 1)\beta_0 + \mu(t_0)\beta_1 + \mu(t_0 + 1)\beta_2)c_0 + \\
&\quad + (\mu(t_0)\beta_0 + \mu(t_0 + 1)\beta_1 + \mu(t_0 + 2)\beta_2)c_1 = x_0, \\
&(-b_0(1)(\mu(t_0)\beta_0 + \mu(t_0 + 1)\beta_1 + \mu(t_0 + 2)\beta_2)c_0 + \\
&\quad + (\mu(t_0 + 1)\beta_0 + \mu(t_0 + 2)\beta_1 + \mu(t_0 + 3)\beta_2)c_1 = x_1
\end{aligned} \tag{3.10}$$

for the unknowns c_0 and c_1. Computations show that

$$\begin{vmatrix} \mu(t_0 + m) & \mu(t_0 + n) \\ \mu(t_0 + p) & \mu(t_0 + n + p - m) \end{vmatrix} = -(b_0(1))^{[t_0]+n+p-m}(\lambda_2^{m-p} - \lambda_1^{m-p})(\lambda_2^{m-n} - \lambda_1^{m-n})/(\lambda_2 - \lambda_1)^2,$$

for any integers m, n, p. This result leads to the conclusion that the determinant of (3.10) coincides with the left-hand side of (3.9). Hence, we can express the coefficients c_0 and c_1 via the initial values x_0 and x_1 and substitute in (3.5) and (3.6), to find the solution $x(t)$.

THEOREM 3.5. The solution $x = 0$ of Eq. (3.2) is stable (respectively, asymptotically stable) as $t \to +\infty$, if and only if $|\lambda| \leq 1$ (respectively, $|\lambda| < 1$), $i = 1,2$.

Proof follows from (3.6) and from the boundedness of the values $b_j(\{t\})$.

THEOREM 3.6. The solution $x = 0$ of Eq. (3.2) is asymptotically stable if

$$\left(a_0 + \frac{ae^a}{e^a - 1}\right)a_2 < 0 \tag{3.11}$$

and either of the following hypotheses is satisfied:

(i) $\quad (a + a_0 + a_1 + a_2)\left(a_1 - \frac{a}{e^a - 1}\right) < 0,$

(ii) $\left(a_1 - \dfrac{a}{e^a - 1}\right)\left(- a_0 + a_1 - a_2 - \dfrac{a(e^a + 1)}{e^a - 1}\right) < 0,$

where the first factors in (i) and (ii) retain the sign of a_2.

PROOF. With the notations $b_j = b_j(1)$, it follows from (3.11) that

$$D^2 = (1 - b_1)^2 - 4b_0b_2 > 0,$$

hence the roots

$$\lambda_{1,\,2} = (1 - b_1 \pm D)/2b_2$$

of (3.4) are real. If $a_2 > 0$, the inequalities $|\lambda_i| < 1$ are equivalent to

$$1 - b_1 + D < 2b_2 \tag{3.12}$$

and

$$1 - b_1 - D > - 2b_2. \tag{3.13}$$

From (3.12) we obtain

$$b_1 + 2b_2 > 1, \quad b_0 + b_1 + b_2 > 1,$$

and (3.13) yields

$$b_1 - 2b_2 < 1, \quad - b_0 + b_1 - b_2 < 1.$$

The inequalities $b_1 + 2b_2 < b_0 + b_1 + b_2$ and $- b_0 + b_1 - b_2 < b_1 - 2b_2$ contradict to $b_0 < 0$ and $b_2 > 0$ which are consequences of (3.11) and $a_2 > 0$. Therefore, it remains to consider only

$$b_0 + b_1 + b_2 > 1, \quad - b_0 + b_1 - b_2 < 1,$$

that is,

$$b_2 > 1 - b_0 - b_1, \quad b_2 > - 1 - b_0 + b_1.$$

If $1 - b_0 - b_1 > - 1 - b_0 + b_1$, then $b_1 < 1$. In this case, a sufficient condition of asymptotic stability is

$$b_0 + b_1 + b_2 > 1, \quad b_1 < 1$$

which, in terms of the coefficients a_j, coincides with hypothesis (i). If $1 - b_0 - b_1 < - 1 - b_0 + b_1$, then $b_1 > 1$, and (ii) follows from the inequalities

$$b_1 > 1, \quad - b_0 + b_1 - b_2 < 1.$$

The case $b_0 > 0$, $b_2 < 0$ is treated similarly.

Let $x_n(t)$ be a solution of the equation

$$x'(t) = ax(t) + \sum_{i=0}^{N} a_i x([t+i]), \quad a_N \neq 0, \quad N \geq 2 \tag{3.14}$$

with constant coefficients on the interval $[n, n+1)$. If the initial conditions for (3.14) are

$$x(n+i) = c_{n+i}, \quad 0 \le i \le N$$

then we have the equation

$$x_n'(t) = ax_n(t) + \sum_{i=0}^{N} a_i c_{n+i}$$

the general solution of which is

$$x_n(t) = e^{a(t-n)}c - \sum_{i=0}^{N} a^{-1} a_i c_{n+i}, \quad a \neq 0$$

For t=n this gives

$$c_n = c - \sum_{i=0}^{N} a^{-1} a_i c_{n+i}$$

and

$$x_n(t) = e^{a(t-n)}c_n + (e^{a(t-n)} - 1) \sum_{i=0}^{N} a^{-1} a_i c_{n+i} \tag{3.15}$$

Taking into account that $x_n(n+1) = x_{n+1}(n+1) = c_{n+1}$ we obtain

$$c_{n+1} = e^a c_n + \sum_{i=0}^{N} (e^a - 1)a^{-1} a_i c_{n+i}.$$

With the notations

$$b_0 = e^a + a^{-1}a_0(e^a - 1), \quad b_i = a^{-1}a_i(e^a - 1), \quad i \ge 1$$

this equation takes the form

$$b_N c_{n+N} + b_{N-1}c_{n+N-1} + \dots + b_2 c_{n+2} + (b_1 - 1)c_{n+1} + b_0 c_n = 0. \tag{3.16}$$

Its particular solution is sought as $c_n = \lambda^n$; then

$$b_N \lambda^N + b_{N-1}\lambda^{N-1} + \dots + b_2 \lambda^2 + (b_1 - 1)\lambda + b_0 = 0. \tag{3.17}$$

If all roots $\lambda_1, \lambda_2, \dots, \lambda_N$ of (3.17) are simple, the general solution of (3.16) is given by

$$c_n = k_1\lambda_1^n + k_2\lambda_2^n + \dots + k_N\lambda_N^n, \tag{3.18}$$

with arbitrary constant coefficients. The initial-value problem for (3.14) may be posed at any N consecutive points. Thus we consider the existence and uniqueness of the solution to (3.14) for $t \ge 0$ satisfying the conditions

$$x(i) = c_i, \quad i = 0, 1, \dots, N - 1. \tag{3.19}$$

Then letting $n = 0, 1, \ldots, N - 1$ and $c_n = x(n)$ in (3.18) we get a system of equations with Vandermond's determinant $\det(\lambda_j^i)$ which is different from zero. Hence, the unknowns k_j are uniquely determined by (3.19). If some roots of (3.17) are multiple, the general solution of (3.16) contains products of exponential functions by polynomials of certain degree. The limiting case of (3.15) as $a \to 0$ gives the solution of (3.14) when $a = 0$. We proved

THEOREM 3.7. Problem (3.14) - (3.19) has a unique solution on $[0, \infty)$. This solution cannot grow to infinity faster than exponentially.

REMARK. If (3.17) has a zero root of multiplicity j, then (3.16) contains only the terms with $c_{n+j}, \ldots, c_{n+N}$. The substitution of n for n + j reduces the order of the difference equation (3.16) to N - j. In this case, the solution x(t) for $t \geq j$ depends only on the initial values $x(j), x(j+1), \ldots, x(N-1)$.

Consider the initial-value problem

$$x'(t) = Ax(t) + \sum_{i=0}^{N} A_i x^{(i)}([t]), \quad x(0) = c_0 \tag{3.20}$$

with constant $r \times r$ - matrices A and A_i and r-vector x. If $A_i = 0$, for all $i \geq 1$, then (3.20) is a retarded equation. For $A \neq 0$ and $A_i = 0 (i \geq 2)$, it is of neutral type. And if $A_i \neq 0$, for some $i \geq 2$, (3.20) is an advanced equation.

THEOREM 3.8. If the matrices A and $B = I - \sum_{i=1}^{N} A_i A^{i-1}$ are nonsingular, then problem (3.20) on $[0, \infty)$ has a unique solution

$$x(t) = E(\{t\})E^{[t]}(1)c_0, \tag{3.21}$$

where

$$E(t) = I + (e^{At} - I)S$$

and

$$S = A^{-1}B^{-1}(A + A_0).$$

PROOF. Assuming that $x_n(t)$ is a solution of Eq. (3.20) on the interval $n \leq t < n + 1$ with the initial conditions

$$x^{(i)}(n+) = c_{ni}, \quad i = 0, \ldots, N$$

we have

$$x_n'(t) = Ax_n(t) + \sum_{i=0}^{N} A_i c_{ni}. \tag{3.22}$$

Since the general solution $e^{A(t-n)}c$ of the homogeneous equation corresponding to (3.22) is analytic, we may differentiate (3.22) and put $t = n$. This gives

$$c_{n1} = Ac_{n0} + \sum_{i=0}^{N} A_i c_{ni},$$

$$c_{ni} = Ac_{n,\ i-1}\ , \quad i = 2, \ldots, N.$$

From the latter relation, $c_{ni} = A^{i-1}c_{n1}$, and from the first it follows that

$$c_{n1} = (A + A_0)c_{n0} + \sum_{i=1}^{N} A_i A^{i-1} c_{n1}.$$

Hence,

$$c_{ni} = A^{i-1}B^{-1}(A + A_0)c_{n0},\ i \geq 1.$$

The general solution of (3.22) is

$$x_n(t) = e^{A(t-n)}c - \sum_{i=0}^{N} A^{-1}A_i c_{ni},$$

with an arbitrary vector c. For $t = n$ we get $c = Sc_{n0}$ and

$$x_n(t) = E(t - n)c_{n0}. \tag{3.23}$$

Thus, on $[n, n+1)$ there exists a unique solution of Eq. (3.20) with the initial condition $x(n) = c_{n0}$ given by (3.23). For the solution of (3.20) on $[n-1, n)$ satisfying $x(n-1) = c_{n-1,\ 0}$ we have

$$x_{n-1}(t) = E(t - n + 1)c_{n-1,\ 0}.$$

The requirement $x_{n-1}(n) = x_n(n)$ implies $c_{n0} = E(1)c_{n-1,\ 0}$ from which

$$c_{n0} = E^n(1)c_0.$$

Together with (3.23) this result proves the theorem. The uniqueness of solution (3.21) on $[0, \infty)$ follows from its continuity and from the uniqueness of the problem $x(n) = c_{n0}$ for (3.20) on each interval $[n, n+1)$. It is easy to see that the restriction $\det A \neq 0$ is required not in the proof of existence and uniqueness but only in deriving formula (3.21).

THEOREM 3.9. If the matrix B is nonsingular, problem (3.20) on $[0, \infty)$ has a unique solution

$$x(t) = V_0(t)c_0, \quad 0 \leq t < 1$$

$$x(t) = V_{[t]}(t) \prod_{k=[t]}^{1} V_{k-1}(k)c_0, \quad t \geq 1$$

where

$$V_k(t) = e^{A(t-k)} + \int_k^t e^{A(t-s)}Cds$$

and

$$C = B^{-1}(A + A_0) - A.$$

COROLLARY. The solution of (3.20) cannot grow to infinity faster than exponentially.

PROOF. Since $||V_{k-1}(k)|| \leq m$ and $||V_{[t]}(t)|| \leq m$, where $m = (1 + ||C||)e^{||A||}$, we have

$$||x(t)|| \leq m^{t+1} \; ||c_0||.$$

REMARK. It is well known that if the spectrum of the matrix A lies in the open left halfplane, then for any function f(t) bounded on $[0, \infty)$ all solutions of the nonhomogeneous equation

$$x'(t) = Ax(t) + f(t)$$

are bounded on $[0, \infty)$. In general, this is not true for (3.20) which is a nonhomogeneous equation with a constant free term on each interval $[n, n+1)$. For instance, the solution

$$x(t) = (1 + (1 - e^{-\{t\}})\delta)(1 + (1 - e^{-1})\delta)^{[t]}$$

of the scalar problem

$$x'(t) = -x(t) + (1 + \delta)x([t]), \quad x(0) = 1, \quad \delta > 0$$

is unbounded on $[0, \infty)$ since

$$|x(t)| \geq (1 + (1 - e^{-1})\delta)^{[t]}.$$

The reason for this is the change of the free term as t passes through integral values.

THEOREM 3.10. If, in addition to the hypotheses of Theorem 3.8, the matrix E(1) is nonsingular, then the solution of (3.20) has a unique backward continuation on $(-\infty, 0]$ given by formula (3.21).

PROOF. If $x_{-n}(t)$ denotes the solution of (3.20) on $[-n, -n+1)$ with the initial condition $x_{-n}(-n) = c_{-n0}$, then

$$x_{-n}(t) = E(t + n)c_{-n0},$$

and changing n to n-1 gives

$$x_{-n+1}(t) = E(t + n - 1)c_{-n+1,\, 0}.$$

Since

$$x_{-n}(- n + 1) = x_{-n+1}(- n + 1) = c_{-n+1,\, 0}$$

we get the relation

$$c_{-n0} = E^{-1}(1)c_{-n+1,\ 0}$$

which proves the theorem.

THEOREM 3.11. If the matrices A, B, E(1) and $E(\{t_0\})$ are nonsingular, then Eq. (3.20) with the condition $x(t_0) = x_0$ has on $(-\infty, \infty)$ a unique solution

$$x(t) = E(\{t\})E^{[t]-[t_0]}(1)E^{-1}(\{t_0\})x_0.$$

PROOF. From (3.21) we have $c_0 = E^{-[t_0]}(1)x([t_0])$ and

$$x(t) = E(\{t\})E^{[t]-[t_0]}(1)x([t_0]).$$

Furthermore, from (3.23) it follows that

$$x([t_0]) = E^{-1}(\{t_0\})x_0.$$

It remains to substitute this result in the previous relation.

4. EQUATIONS WITH VARIABLE COEFFICIENTS

Along with the equation

$$x'(t) = A(t)x(t) + \sum_{i=0}^{N} A_i(t)x([t+i]),\ N \geq 2,\ (0 \leq t < \infty) \tag{4.1}$$

$$x(i) = c_i,\quad i = 0, \ldots, N-1$$

the coefficients of which are r x r - matrices and x is an r-vector, we also consider

$$x'(t) = A(t)x(t). \tag{4.2}$$

If A(t) is continuous, the problem $x(0) = c_0$ for (4.2) has a unique solution $x(t) = U(t)c_0$, where U(t) is the solution of the matrix equation

$$U'(t) = A(t)U(t),\ U(0) = I. \tag{4.3}$$

The solution of the problem $x(s) = c_0$, $s \in [0, \infty)$ for (4.2) is represented in the form

$$x(t) = U(t)U^{-1}(s)c_0.$$

Let

$$B_{0n}(t) = U(t)(U^{-1}(n) + \int_n^t U^{-1}(s)A_0(s)ds),$$

$$B_{in}(t) = U(t)\int_n^t U^{-1}(s)A_i(s)ds,\ i = 1, \ldots, N. \tag{4.4}$$

THEOREM 4.1. Problem (4.1) has a unique solution on $0 \leq t < \infty$ if A(t) and $A_i(t) \in C[0, \infty)$, and the matrices $B_{Nn}(n+1)$ are nonsingular for $n = 0, 1, \ldots$.

PROOF. On the interval $n \leq t < n + 1$ we have the equation

$$x'(t) = A(t)x(t) + \sum_{i=0}^{N} A_i(t)c_{n+i}, \quad c_{n+i} = x(n+i)$$

Its solution $x_n(t)$ satisfying the condition $x(n) = c_n$ is given by the expression

$$x_n(t) = U(t)(U^{-1}(n)c_n + \int_n^t U^{-1}(s) \sum_{i=0}^{N} A_i(s)c_{n+i}ds). \tag{4.5}$$

Hence, the relation $x_n(n+1) = x_{n+1}(n+1) = c_{n+1}$ implies

$$c_{n+1} = U(n+1)(U^{-1}(n)c_n + \int_n^{n+1} U^{-1}(s) \sum_{i=0}^{N} A_i(s)c_{n+i}ds).$$

With the notations (4.4), this difference equation takes the form

$$B_{Nn}(n+1)c_{n+N} + \dots + B_{2n}(n+1)c_{n+2} + (B_{1n}(n+1)-I)c_{n+1} + B_{0n}(n+1)c_n = 0.$$

Since the matrices $B_{Nn}(n+1)$ are nonsingular for all $n \geq 0$, there exists a unique solution $c_n (n \geq N)$ provided that the values $c_0, \dots, c_{N-1}$ are prescribed. Substituting the vectors c_n in (4.5) yields the solution of (4.1).

For the scalar problem

$$x'(t) = a(t)x(t) + a_0(t)x([t]) + a_1(t)x([t+1]) + a_2(t)x([t+2]),$$

$$x(0) = c_0, \quad x(1) = c_1$$

with coefficients continuous on $[0, \infty)$ we can indicate a simple algorithm of computing the solution. According to (4.4) and (4.5), we have

$$x_n(t) = B_{0n}(t)c_n + B_{1n}(t)c_{n+1} + B_{2n}(t)c_{n+2}, \tag{4.6}$$

with

$$U(t) = \exp(\int_0^t a(s)ds).$$

The coefficients c_n satisfy the equation

$$B_{2n}(n+1)c_{n+2} + (B_{1n}(n+1) - 1)c_{n+1} + B_{0n}(n+1)c_n = 0, \; n \geq 0.$$

Denote

$$d_0(n+1) = -B_{0n}(n+1)/B_{2n}(n+1), \; d_1(n+1) = (1 - B_{1n}(n+1))/B_{2n}(n+1),$$

$$r_n = c_{n+1}/c_n.$$

Then from the relation

$$c_{n+2} = d_1(n+1)c_{n+1} + d_0(n+1)c_n$$

it follows that

$$r_{n+1} = d_1(n+1) + \frac{d_0(n+1)}{r_n},$$

which yields

$$r_1 = d_1(1) + d_0(1)/r_0,$$

$$r_2 = d_1(2) + d_0(2)/r_1 = d_1(2) + \cfrac{d_0(2)}{d_1(1) + \cfrac{d_0(1)}{r_0}} ,$$

and continuing this procedure leads to the continued-fraction expansion

$$r_n = d_1(n) + \frac{d_0(n)}{d_1(n-1)+} \; \frac{d_0(n-1)}{d_1(n-2)+} \; \cdots \; \frac{d_0(2)}{d_1(1)+} \; \frac{d_0(1)}{c_1/c_0}$$

and to the formula

$$c_n = r_1 r_2 \cdots r_{n-1} c_0$$

for the coefficients of solution (4.6).

THEOREM 4.2. Assume that the matrices $A(t)$, $A_0(t)$, $A_1(t)$ are continuous on $0 \le t < \infty$, and let

$$\omega(t) = \max||A(s)||, \quad \omega_i(t) = \max||A_i(s)||, \; 0 \le s \le t, \quad i = 0, 1.$$

If for all $t \in [0, \infty)$,

$$\omega_1(t)e^{\omega(t)} \le q < 1, \tag{4.7}$$

then the solution of the problem

$$x'(t) = A(t)x(t) + A_0(t)x([t]) + A_1(t)x([t+1]), \; x(0) = c_0 \tag{4.8}$$

satisfies the estimate

$$||x(t)|| \le (1-q)^{-t-1}(\omega_0(t+1) + 1)^{t+1}e^{(t+1)\omega(t+1)}||c_0||, \; 0 \le t < \infty. \tag{4.9}$$

PROOF. According to (4.5), the solution $x_n(t)$ of Eq. (4.8) on $n \le t < n + 1$ with the condition $x(n) = c_n$ is given by

$$x_n(t) = B_{0n}(t)c_n + B_{1n}(t)c_{n+1}. \tag{4.10}$$

Since $x_n(n+1) = x_{n+1}(n+1) = c_{n+1}$, we have

$$c_{n+1} = B_{0n}(n+1)c_n + B_{1n}(n+1)c_{n+1},$$

and

$$(I - B_{1n}(n+1))c_{n+1} = B_{0n}(n+1)c_n. \tag{4.11}$$

The matrix $U(t, s) = U(t)U^{-1}(s)$ is the solution of Eq. (4.3) satisfying $U(s, s) = I$. Therefore, it is the sum of the absolutely convergent series

$$U(t, s) = I + \int_s^t A(s_1)ds_1 + \int_s^t A(s_1)ds_1 \int_s^{s_1} A(s_2)ds_2 + \dots$$

whence

$$||U(t, s)|| \le e^{(t-s)\omega(t)}$$

and turning to (4.4) we find that for $n \le t \le n + 1$,

$$||B_{0n}(t)|| \le (\omega_0(t) + 1)e^{\omega(t)}, \quad ||B_{1n}(t)|| \le \omega_1(t)e^{\omega(t)} \tag{4.12}$$

By virtue of (4.7) and the latter inequality, the matrices $I - B_{1n}(n+1)$ have inverses for all $n \ge 0$ and

$$||(I - B_{1n}(n+1))^{-1}|| \le 1 + ||B_{1n}(n+1)|| + ||B_{1n}(n+1)||^2 + \dots \le (1 - q)^{-1}.$$

From the relation

$$c_{n+1} = (I - B_{1n}(n+1))^{-1}B_{0n}(n+1)c_n$$

it follows that

$$||c_{n+1}|| \le (1 - q)^{-1}(\omega_0(n+1) + 1)e^{\omega(n+1)}||c_n||,$$

and

$$||c_n|| \le (1 - q)^{-n}(\omega_0(n) + 1)^n e^{n\omega(n)}||c_0||.$$

The application of this result together with (4.12) to (4.10) yields

$$||x_n(t)|| \le (e^{\omega(t)}(\omega_0(t) + 1)(1 - q)^{-n}(\omega_0(n) + 1)^n e^{n\omega(n)} +$$

$$+ q(1 - q)^{-n-1}e^{\omega(t)}(\omega_0(n+1) + 1)^{n+1}e^{(n+1)\omega(n+1)})||c_0||.$$

This inequality implies (4.9) since the functions $\omega(t)$, $\omega_0(t)$ and $(1 - q)^{-t}$ are increasing.

THEOREM 4.3. The solution of

$$x'(t) = ax(t) + a_0(t)x([t]) + a_1(t)x([t+1]), \quad x(0) = c_0 \tag{4.13}$$

tends to zero as $t \to +\infty$ if the following hypotheses are satisfied: (i) a is constant, $a_0(t)$, $a_1(t) \in C[0, \infty)$; (ii) $\sup|a_1(t)| = q_1 < 1$ on $[0, \infty)$ and, starting with some $t_0 \ge 0$,

$$-\frac{a(e^a + 1 - q_1)}{e^a - 1} < m_0 \le M_0 < -a - \frac{aq_1}{e^a - 1}, \tag{4.14}$$

where $m_0 = \inf a_0(t)$, $M_0 = \sup a_0(t)$, $t \geq t_0$.

Proof. The solution of Eq. (4.13) is given by (4.10), with

$$B_{0n}(t) = e^{a(t-n)} + \int_n^t e^{a(t-s)} a_0(s)ds, \quad B_{1n}(t) = \int_n^t e^{a(t-s)} a_1(s)ds$$

and $n \leq t < n + 1$. Since the functions $B_{0n}(t)$ and $B_{1n}(t)$ are bounded on $[0, \infty)$, it remains to show that $\lim c_n = 0$ as $n \to \infty$. The condition $x_n(n+1) = c_{n+1}$ implies (4.11), from which $c_n/c_{n-1} = B_{0,\ n-1}(n)/(1 - B_{1,\ n-1}(n))$ and

$$c_n = r_1 r_2 \ \cdots \ r_{n-1} c_0,$$

where

$$r_i = (e^a + \int_{i-1}^{i} e^{a(i-s)} a_0(s)ds)/(1 - \int_{i-1}^{i} e^{a(i-s)} a_1(s)ds).$$

By virtue of (i) and the first part of (ii),

$$|1 - \int_{i-1}^{i} e^{a(i-s)} a_1(s)ds| \geq 1 - \int_{i-1}^{i} |a_1(s)|ds \geq 1 - q_1.$$

Therefore, $1 - B_{1,\ i-1}(i) \neq 0$ for all $i \geq 1$. Let j be a natural number such that $j \geq t_0$, then due to (4.14) we have

$$B_{0,\ i-1}(i) = e^a + \int_{i-1}^{i} e^{a(i-s)} a_0(s)ds \leq e^a + M_0(e^a - 1)/a,$$

$$B_{0,\ i-1}(i) \geq e^a + m_0(e^a - 1)/a, \quad i \geq j$$

and

$$B_{0,\ i-1}(i) < e^a - a(e^a - 1)/a - aq_1(e^a - 1)/a(e^a - 1) = 1 - q_1$$

$$B_{0,\ i-1}(i) > e^a - a(e^a + 1 - q_1)(e^a - 1)/a(e^a - 1) = -(1 - q_1).$$

Hence,

$$|B_{0,\ i-1}(i)| \leq q(1 - q_1), \quad |1 - B_{1,\ i-1}(i)| \geq 1 - q_1, \quad 0 \leq q < 1$$

and

$$|r_i| \leq q, \quad i \geq j.$$

On the other hand, for $i < j$ we use the inequalities

$$|B_{0,\ i-1}(i)| \leq e^a + M(e^a - 1)/a = p_1, \quad M = \sup|a_0(t)|, \ t \in [0, \infty)$$

and

$$|r_i| \leq p_1/(1 - q_1) = p.$$

The boundedness of the function $|a_0(t)|$ on $[0, \infty)$ is a result of its continuity and of (4.14). To finish the proof, we write

$$|c_n| = |c_0| \prod_{i=1}^{j-1} |r_i| \prod_{i=j}^{n-1} |r_i| \leq p^{j-1} q^{n-j} |c_0|, \; n \geq j + 1$$

and here the second factor vanishes as $n \to \infty$. The method of proof suggests also the following

THEOREM 4.4. The solution of (4.13) tends to zero as $t \to +\infty$ if the coefficient a is constant, the functions $a_0(t)$, $a_1(t) \in C[0, \infty)$, $a_1(t) \leq 0$ for all $t \in [0, \infty)$ and, starting with some $t_0 \geq 0$,

$$-a(e^a + 1)/(e^a - 1) < m_0 \leq M_0 < -a.$$

THEOREM 4.5. The solution of the scalar problem

$$x'(t) = ax(t) + \sum_{i=0}^{N} a_i(t)x([t+i]), \; N \geq 2 \tag{4.15}$$

$$x^{(i)}(0) = c_i, \quad i = 0, \ldots, N - 1$$

tends to zero as $t \to +\infty$ if the following hypotheses are satisfied.

(i) The coefficient a is constant, the functions $a_i(t) \in C[0, \infty)$.

(ii) The functions $a_i(t) (i = 0, 1, \ldots, N - 1)$ are bounded on $[0, \infty)$.

(iii) The function $a_N(t)$ is positive, monotonically increasing, and $\lim a_N(t) = \infty$ as $t \to \infty$.

Proof. The solution of Eq. (4.15) on the interval $n \leq t < n+1$ is given by the formula

$$x_n(t) = \sum_{i=0}^{N} B_{in}(t)c_{n+i}, \; x_n(n) = c_n$$

where

$$B_{0n}(t) = e^{a(t-n)} + \int_n^t e^{a(t-s)} a_0(s)ds, \; B_{in}(t) = \int_n^t e^{a(t-s)} a_i(s)ds,$$

$$(i = 1, \ldots, N).$$

Since the functions $B_{i,[t]}(t)$ are bounded on $[0, \infty)$, it remains to prove that $c_n \to 0$ as $n \to \infty$. The continuity of the solution to (4.15) implies $x_n(n+1) = c_{n+1}$ and leads to the equation

$$c_{n+N} = \sum_{i=0}^{N-1} d_i(n+1)c_{n+i}, \tag{4.16}$$

with

$$d_1(n+1) = (1 - B_{1n}(n+1))/B_{Nn}(n+1), \quad d_i(n+1) = -B_{in}(n+1)/B_{Nn}(n+1)$$

$$(i = 0, 2, \ldots, N-1).$$

We have

$$B_{Nn}(n+1) = \int_n^{n+1} e^{a(n+1-s)} a_N(s)ds = \frac{e^a - 1}{a} a_N(\xi_n), \quad n < \xi_n < n+1,$$

hence, by virtue of (iii), $B_{Nn}(n+1) > 0$ for all $n \geq 0$ and $\lim B_{Nn}(n+1) = \infty$ as $n \to \infty$. On the other hand, due to (ii), the values

$$B_{in}(n+1) = \int_n^{n+1} e^{a(n+1-s)} a_i(s)ds, \quad i = 0, \ldots, N-1$$

are uniformly bounded for all $n \geq 0$ since

$$|B_{in}(n+1)| \leq e^{|a|} \sup|a_i(t)|, \quad t \in [0, \infty).$$

Therefore,

$$|d_i(n+1)| \leq M/a_N(\xi_n),$$

with some constant $M > 0$. The positive variable

$$\delta_n = M/a_N(\xi_n)$$

is monotonically decreasing to zero as $n \to \infty$. From (4.16) it follows that

$$|c_{n+N}| \leq \delta_n \sum_{i=0}^{N-1} |c_{n+i}|. \tag{4.17}$$

To evaluate the coefficients c_n, we employ the method developed in [14] for the study of distributional solutions to functional differential equations. Denote

$$M_n = \max |c_j|, \quad 0 \leq j \leq n.$$

Then

$$|c_{n+N}| \leq \delta_n NM_{n+N-1}.$$

Starting with some natural k,

$$\delta_n N < 1, \quad |c_{n+N}| < M_{n+N-1},$$

and

$$M_{n+N} = M_{n+N-1}, \quad n > k.$$

Hence,

$$M_{n+N} = M_{k+N}, \quad n \geq k. \tag{4.18}$$

The application of (4.18) to (4.17) successively yields

$$|c_{k+N+1}| \le \delta_{k+1} N M_{k+N}, \ |c_{k+N+2}| \le \delta_{k+2} N M_{k+N}, \ \ldots$$

$$\ldots, \ |c_{k+2N}| \le \delta_{k+N} N M_{k+N}.$$

Since the sequence $\{\delta_n\}$ is decreasing it follows that

$$|c_{k+N+i+1}| \le \delta_{k+1} N M_{k+N}, \quad i = 0, \ldots, N-1.$$

Now we put $n = k+N+1, k+N+2, \ldots, k+2N$ in (4.17) and use the latter inequalities to obtain

$$|c_{k+2N+1}| \le (\delta_{k+1}N)^2 M_{k+N}, \ |c_{k+2N+2}| \le (\delta_{k+1}N)^2 M_{k+N}, \ \ldots$$

$$\ldots, \ |c_{k+3N}| \le (\delta_{k+1}N)^2 M_{k+N},$$

that is,

$$|c_{k+2N+1+i}| \le (\delta_{k+1}N)^2 M_{k+N}, \quad i = 0, \ldots, N-1.$$

Continuation of the iteration process shows that

$$|c_{k+jN+1+i}| \le (\delta_{k+1}N)^j M_{k+N},$$

for all natural j and $i = 0, \ldots, N-1$, and the inequality $\delta_{k+1}N < 1$ implies $c_{k+jN+1+i} \to 0$ as $j \to \infty$.

THEOREM 4.6. Assume that the coefficients in (4.1) are continuous periodic matrices of period 1. If there exists a periodic solution $x(t)$ of period 1 to Eq. (4.1), then $\lambda = 1$ is an eigenvalue of the matrix

$$S = \sum_{i=0}^{N} B_{i0}(1),$$

where $B_{i0}(t)$ are given in (4.4), and $x(0)$ is a corresponding eigenvector. Conversely, if $\lambda = 1$ is an eigenvalue of S and c_0 is a corresponding eigenvector and if the matrices $B_{N0}(1)$ and $B_{00}(1)$ are nonsingular, then Eq. (4.1) with the conditions

$$x(i) = c_0, \ i = 0, \ldots, N-1 \tag{4.19}$$

has a unique solution on $-\infty < t < \infty$, and it is 1-periodic.

PROOF. Suppose that $x(t)$ is a solution of period 1 to Eq. (4.1) and let $x(0) = c_0 \neq 0$. Since $x(n) = x(0)$, for all integral n, we have $c_n = c_0$ and, according to (4.5), the assumption $c_0 = 0$ would mean $x(t) = 0$. Putting $t = n+1$ in (4.5) gives

$$c_0 = U(n+1)(U^{-1}(n) + \sum_{i=0}^{N} \int_n^{n+1} U^{-1}(s)A_i(s)ds)c_0.$$

From the relation $A(t+n) = A(t)$ it follows that $U(t+n)$ is a solution of Eq. (4.3). Hence, $U(t+n) = U(t)C_n$, and to determine the matrix C_n we put here $t = 0$ which yields $U(n) = U(0)C_n$, that is $C_n = U(n)$. Now we can write

$$U(t+n) = U(t)U(n),\ U(n+1)U^{-1}(n) = U(1),\ U^{-1}(t+n) = U^{-1}(n)U^{-1}(t)$$

and since $A_i(t+n) = A_i(t)$ we observe that

$$\int_n^{n+1} U^{-1}(s)A_i(s)ds = \int_0^1 U^{-1}(s+n)A_i(s)ds = U^{-1}(n)\int_0^1 U^{-1}(s)A_i(s)ds,$$

Therefore,

$$(U(1) + \sum_{i=0}^{N} U(1)\int_0^1 U^{-1}(s)A_i(s)ds - I)c_0 = 0$$

and, taking into account (4.4), we conclude that

$$(\sum_{i=0}^{N} B_{i0}(1) - I)c_0 = 0. \tag{4.20}$$

This proves the first part of the theorem.

According to (4.5), the solution of problem (4.1) for $t \geq 0$ is

$$x(t) = U(t)(U^{-1}([t])c_{[t]} + \sum_{i=0}^{N} (\int_{[t]}^{t} U^{-1}(s)A_i(s)ds)c_{[t+i]}), \tag{4.21}$$

and if the coefficients of (4.1) are 1-periodic, it has a unique backward continuation on $(-\infty, 0]$ given by the same formula (4.21). Indeed, on the interval $[-n, -n+1)$ Eq. (4.1) becomes

$$x'_{-n}(t) = A(t)x_{-n}(t) + \sum_{i=0}^{N} A_i(t)c_{-n+i},$$

where $c_{-n+i} = x_{-n}(-n+i)$, and its solution

$$x_{-n}(t) = U(t)(U^{-1}(-n)c_{-n} + \sum_{i=0}^{N} (\int_{-n}^{t} U^{-1}(s)A_i(s)ds)c_{-n+i}) \tag{4.22}$$

can be written in the form (4.21). It remains to show that the coefficients $c_{[t]}$ satisfy the same difference equation for both positive and negative values of t. From the derivation of (4.20) we note that

$$c_{n+1} = \sum_{i=0}^{N} B_{i0}(1)c_{n+i}, \quad n \geq 0. \tag{4.23}$$

For $t = -n+1$ formula (4.22) gives

$$c_{-n+1} = U(-n+1)(U^{-1}(-n)c_{-n} + \sum_{i=0}^{N} (\int_{-n}^{-n+1} U^{-1}(s)A_i(s)ds)c_{-n+i})$$

Since $A(t-n) = A(t)$ and $A_i(t-n) = A_i(t)$, we have

$$U(t-n) = U(t)U(-n), \; U(-n+1)U^{-1}(-n) = U(1), \; U^{-1}(t-n) = U^{-1}(-n)U^{-1}(t)$$

and

$$\int_{-n}^{-n+1} U^{-1}(s)A_i(s)ds = \int_0^1 U^{-1}(s-n)A_i(s)ds) = U^{-1}(-n) \int_0^1 U^{-1}(s)A_i(s)ds$$

The equation

$$c_{-n+1} = U(1)(c_{-n} + \sum_{i=0}^{N} (\int_0^1 U^{-1}(s)A_i(s)ds)c_{-n+i} \tag{4.24}$$

for the coefficients c_{-n} coincides with Eq. (4.23) for c_n. The representation of (4.24) as

$$B_{00}(1)c_{-n} = c_{-n+1} - \sum_{i=1}^{N} B_{i0}(1)c_{-n+i} \tag{4.25}$$

shows that since the matrix $B_{00}(1)$ is nonsingular, conditions (4.19) uniquely determine the values c_{-n} for all $n \geq 1$. This proves the existence and uniqueness of solution to Eq. (4.1) satisfying (4.19). To establish its periodicity, we put $n = 0$ in (4.23) and invoking (4.19) obtain

$$(\sum_{i=0}^{N-1} B_{i0}(1) - I)c_0 + B_{N0}(1)c_N = 0.$$

Subtracting this relation from (4.20) gives $B_{N0}(1)(c_0 - c_N) = 0$, and $c_N = c_0$. For $n = 1$, it follows from (4.23) that

$$(\sum_{i=0}^{N-1} B_{i0}(1) - I)c_0 + B_{N0}(1)c_{N-1} = 0,$$

and combining this result with (4.20) yields $c_{N-1} = c_0$. Continuing this procedure confirms that $c_n = c_0$ for all $n \geq 0$. To prove that $c_{-n} = c_0$ for $n \geq -1$, we consider (4.25) together with (4.20). Finally, on each interval $[n, n+1)$ problem (4.1) - (4.19) is reduced to the same equation

$$x'(t) = A(t)x(t) + (\sum_{i=0}^{N} A_i(t))c_0, \; x(n) = c_0$$

which concludes the proof.

The initial-value problem

$$x'(t) = A(t)x(t) + \sum_{i=0}^{N} A_i(t)x^{(i)}([t]), \; x(0) = c_0 \tag{4.26}$$

with $r \times r$ - matrices $A(t)$ and $A_i(t) \in C^{N-1}[0, \infty)$ and r-vector $x(t)$, can be treated similarly to (3.20). Assuming that $x_n(t)$ is a solution of Eq. (4.26) for $n \leq t < n + 1$ satisfying the conditions

$$x^{(i)}(n) = c_{ni}, \quad i = 0, \ldots, N$$

we have

$$x_n'(t) = A(t)x_n(t) + \sum_{i=0}^{N} A_i(t)c_{ni} \tag{4.27}$$

Since the general solution of the homogeneous equation corresponding to (4.27) is N times differentiable, we may differentiate (4.27) $i - 1$ times and put $t = n$. This gives

$$c_{ni} = \sum_{j=0}^{i-1} \binom{i-1}{j} A^{(i-1-j)}(0)c_{nj} + \sum_{k=0}^{N} A_k^{(i-1)}(0)c_{nk}, \; i = 1, \ldots, N. \tag{4.28}$$

The formula for $x_n(t)$ is

$$x_n(t) = U(t)(U^{-1}(n)c_{n0} + \sum_{i=0}^{N} \int_n^t U^{-1}(s)A_i(s)c_{ni}ds),$$

where $U(t)$ is the solution of (4.3), and $x_n(n+1) = c_{n+1,\,0}$ implies

$$c_{n+1,\,0} = U(n+1)(U^{-1}(n)c_{n0} + \sum_{i=0}^{N} \int_n^{n+1} U^{-1}(s)A_i(s)c_{ni}ds). \tag{4.29}$$

Now we solve the system (4.28) for the unknowns c_{ni} $(i = 1, \ldots, N)$ and substitute these values in (4.29), to obtain a recursion relation between $c_{n+1,\,0}$ and c_{n0}. Since $c_{00} = c_0$ is given we can find c_{n0} (and c_{ni}) for all $n \geq 1$. Thus, we proved

THEOREM 4.6. Problem (4.26) has a unique solution on $[0, \infty)$ if the matrices $A(t)$ and $A_i(t) \in C^{N-1}[0, \infty)$ and system (4.28) has a unique solution with respect to the vectors c_{ni}.

THEOREM 4.7. Suppose the equation

$$x'(t) = A(t)x(t) + B(t)x([t+N]) + f(t, x([t]), x([t+1]), \ldots, x([t+N-1])), \quad (N \geq 2) \tag{4.30}$$

satisfies the following hypotheses.

(i) The matrices $A(t)$, $B(t) \in C[0, \infty)$, and f is a continuous function in the space of its variables.

(ii) The matrices $\int_n^{n+1} U^{-1}(t)B(t)dt$ are nonsingular for all $n \geq 0$, where $U(t)$ is the solution of (4.3).

Then (4.30) with the initial conditions (4.1) has a unique solution on $[0, \infty)$.

PROOF. On $n \leq t < n + 1$ Eq. (4.30) takes the form

$$x_n'(t) = A(t)x_n(t) + B(t)c_{n+N} + f(t, c_n, c_{n+1}, \ldots, c_{n+N-1}),$$

$$c_j = x(j), \quad j = n, n+1, \ldots, n+N.$$

Hence,

$$x_n(t) = U(t)(U^{-1}(n)c_n + \int_n^t U^{-1}(s)B(s)c_{n+N}ds +$$

$$+ \int_n^t U^{-1}(s)f(s, c_n, c_{n+1}, \ldots, c_{n+N-1})ds).$$

For $t = n + 1$ we have

$$c_{n+1} = U(n+1)(U^{-1}(n)c_n + (\int_n^{n+1} U^{-1}(s)B(s)ds)c_{n+N} + \tag{4.31}$$

$$+ \int_n^{n+1} U^{-1}(s)f(s, c_n, c_{n+1}, \ldots, c_{n+N-1})ds).$$

Hypothesis (ii) allows to solve (4.31) for c_{n+N} and to find all coefficients c_n successively by employing conditions (4.1).

In conclusion, we consider the scalar equation

$$x'(t) = f(x(t), x([t]), x([t+1])), \quad x(0) = c_0, \quad (0 \leq t < \infty). \tag{4.32}$$

If in the ordinary differential equation with parameters

$$x' = f(x, \lambda, \mu), \quad f(x, \lambda, \mu) \in C(R^3) \tag{4.33}$$

$f(x, \lambda, \mu) \neq 0$ everywhere, then there exists a general integral

$$F(x, \lambda, \mu) = t + g(\lambda, \mu),$$

with an arbitrary function $g(\lambda, \mu)$. If $x(t)$ can be extended over all $t \geq 0$, then for the solution $x_n(t)$ of (4.32) on $n \leq t < n + 1$ we have the equation

$$x_n' = f(x_n, c_n, c_{n+1}), \quad x_n(n) = c_n. \tag{4.34}$$

Hence, if $\lambda = c_n$, $\mu = c_{n+1}$, then

$$F(x_n, c_n, c_{n+1}) = t + g(c_n, c_{n+1}).$$

For $t = n$ this gives

$$F(c_n, c_n, c_{n+1}) = n + g(c_n, c_{n+1}),$$

and

$$F(x_n, c_n, c_{n+1}) = t + F(c_n, c_n, c_{n+1}) - n. \tag{4.35}$$

Now we take the solution of (4.32) on $[n+1, n+2)$ that satisfies $x(n+1) = c_{n+1}$. Since $x_n(n+1) = x_{n+1}(n+1)$ we put $t = n+1$ in (4.35) and get

$$F(c_{n+1}, c_n, c_{n+1}) - F(c_n, c_n, c_{n+1}) = 1.$$

This equation can be written as

$$\int_{c_n}^{c_{n+1}} \frac{dx}{f(x, c_n, c_{n+1})} = 1, \tag{4.36}$$

which also follows directly by integrating (4.34) from $t = n$ to $t = n+1$. After this, we relax the condition that $f(x, \lambda, \mu) \neq 0$ everywhere. It is easy to see that we may always suppose $f(x, c_n, c_{n+1}) \neq 0$, $x \in (c_n, c_{n+1})$ since the assumption $f(c, c_n, c_{n+1}) = 0$ for some $c \in (c_n, c_{n+1})$ leads to the conclusion that the constant function $x_n(t) = c$ is a solution of Eq. (4.34) on the interval $n \leq t < n + 1$. However, this solution does not satisfy the condition $x_n(n) = c_n$ because $c \neq c_n$. On the other hand, if $f(c_n, c_n, c_{n+1}) = 0$, then $x_n(t) = c_n$ is a solution of problem (4.34) on $[n, n+1)$. By virtue of the condition $x_n(n+1) = x_{n+1}(n+1) = c_{n+1}$, we have $c_n = c_{n+1}$ and $f(c_n, c_n, c_n) = 0$. Thus, we proved

THEOREM 4.8. Assume the following hypotheses:

(i) $f(x, \lambda, \mu) \in C(R^3)$ and $f(x, x, x) \neq 0$ for all real x.

(ii) The solutions of Eq. (4.33) can be extended over $[0, \infty)$.

(iii) Eq. (4.36) has a unique solution with respect to c_{n+1}.

Then problem (4.32) has a unique solution on $[0, \infty)$ given by formula (4.35) for

$t \in [n, n+1]$, where

$$F(x, \lambda, \mu) = \int \frac{dx}{f(x, \lambda, \mu)} .$$

If $f(c_n, c_n, c_n) = 0$, then $x(t) = c_n$ is a solution on $[n, \infty)$ of Eq. (4.32) with the condition $x(n) = c_n$.

REFERENCES

1. Shah, S.M. and Wiener, J. Distributional and entire solutions of ordinary differential and functional differential equations, Inter. J. Math. & Math Sci. 6(2), (1983), 243-270.

2. Cooke, K.L. and Wiener, J. Distributional and analytic solutions of functional differential equations, J. Math. Anal. and Appl. (to appear).

3. Pelyukh, G.P. and Sharkovskii, A.N. Introduction to the Theory of Functional Equations (Russian), Naukova Dumka, Kiev, 1974.

4. Halanay, A. and Wexler, D. Teoria Calitativă a Sistemelor cu Impulsuri, Acad. RSR, Bucuresti, 1968.

5. Myshkis, A.D. On certain problems in the theory of differential equations with deviating argument, Uspekhi Mat. Nauk 32(2), (1977), 173-202.

6. Busenberg, S. and Cooke, K.L. Models of vertically transmitted diseases with sequential-continuous dynamics, in Nonlinear Phenomena in Mathematical Sciences, Lakshmikantham, V. (editor), Academic Press, New York, 1982, 179-187.

7. Kamenskii, G.A. Towards the general theory of equations with deviating argument, DAN SSSR 120(4), (1958), 697-700.

8. Cooke, K.L. and Wiener, J. Retarded differential equations with piecewise constant delays, J. Math. Anal. and Appl. (to appear).

9. Wiener, J. Differential equations with piecewise constant delays, in Trends in the Theory and Practice of Nonlinear Differential Equations, Lakshmikantham, V. (editor), Marcel Dekker, New York, 1983, 547-552.

10. Wiener, J. Pointwise initial-value problems for functional differential equations, in Proceedings of the International Conference on Differential Equations at The University of Alabama in Birmingham, March 21-26, 1983, Knowles, Ian and Lewis, Rogers (editors), to appear.

11. Hale, J.K. Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.

12. Shah, S.M. On real continuous solutions of algebraic difference equations, Bull. Amer. Math. Soc. 53(1947), 548-558.

13. Shah, S.M. On real continuous solutions of algebraic difference equations, II, Proc. Nat. Inst. Sci. India 16(1950), 11-17.

14. Wiener, J. Generalized-function solutions of linear systems, J. Differ. Equat. 38(2), (1980), 301-315.