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LIE TRANSFORMS AND MOTION OF A CHARGED PARTICLE IN A PERIODIC MAGNETIC FIELD

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ABSTRACT. We use the Lie series averaging method to obtain a complete second order solution for motion of a charged particle in a spatially periodic magnetic field. A comparison is made with the first order solution obtained previously by Coffey.

KEY WORDS AND PHRASES. Lie Transforms, magnetic moment, relative fluctuation, second order solution.

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1. INTRODUCTION.

Sometime ago Coffey [1] applied his formulation of degenerate perturbation theory to the interaction between a charged particle and a spatially periodic magnetic field. Coffey's results refute the argument of Dragt [2] and Wentzel [3] who tried to show that a resonant interaction between a charged particle and a periodic magnetic field would cause a break down of the adiabatic invariance of the particle in orbital magnetic moments and thus would be responsible for the removal of protons from the inner Van Allen Belt. Coffey obtained a first order solution to this problem by his perturbation method. But, as has been pointed out by Coffey, the period of secular motion varies as $\varepsilon^{1/2}$ and not as ε. Hence, a second order calculation is of interest in this problem especially if it shows significant effect on the relative fluctuation of the magnetic moment. But, for second order calculations, Coffey's method becomes very much involved. In this work we show that a well established averaging method, using the Lie series, gives complete solution of the problem discussed above up to second order without involved calculations.

Lie transforms and Lie series were found to be very useful in analysing the oscillation of a weakly nonlinear system. For further details see Kamel [4], Nayfeh [5].

Interested readers may see the details of application of Lie transforms in case of Van der Pol's equation, in ref [5] (page 209). Our Technique was a straight forward extension of the method discussed in [5]. Our first order results completely agree with those obtained by Coffey, while second order calculations show that a relative fluctuation of the magnetic moment of the order of unity cannot be ruled out even for periodic disturbances with wave lengths smaller than those anticipated in first order calculations. In some cases, the term proportional to ε^2 dominates over other terms.

2. LIE SERIES AND LIE TRANSFORM.

We present here some basic steps which will be required in our calculations. Consider a system of differential equations given by

$$\frac{\partial X}{\partial t} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} f_n(X) \tag{2.1}$$

where X and f_n are column matrices and ε is a small parameter. The basic idea is to introduce a transformation from X to Y so that (2.1) becomes

$$\frac{\partial Y}{\partial t} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} g_n(Y) \tag{2.2}$$

where g contains only long period terms.

We do this by a near identity transform (see ref. [5] p. 209)

$$X = X(Y,t) = Y + \varepsilon X_1(Y) + \varepsilon^2 X_2(Y) + \dots\dots\dots \tag{2.3}$$

which can be written as

$$\frac{\partial X}{\partial t} = W(X,\varepsilon),\ X(\varepsilon=0) = Y \tag{2.4}$$

In other words, in order to obtain solution of (2.1) for large t, it will be necessary to find the solution of (2.4) for small ε. The algorithm is as follows. Under the transformation defined by (2.3), a vector

$$f(X,\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} f_n(X) \tag{2.5}$$

is transformed to

$$f(X,\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} f^n(Y) \tag{2.6}$$

The relation between g_n and f_n in (2.1) and (2.2) can be found from step by step calculations:

$$g_0(Y) = f_0(Y) \tag{2.7}$$

$$g_1(Y) = f_1(Y) + L_1 f_0 \tag{2.8}$$

$$\text{where } L_1 g = \left(\frac{\partial (g)_i}{\partial x_j}\right)(W)_j - \left(\frac{\partial (W)_i}{\partial x_j}\right)(g)_j \tag{2.9}$$

We choose W_1 in such a way that g_1 contains only long period terms. Thus W_1 is known. Then a second order calculation gives

$$g_2 = f_2 + L_1 f_1 + L_1 g_1 + L_2 f_0 \tag{2.10}$$

where

$$L_2 f_0 = \frac{\partial f_0}{\partial X} W_2 - \frac{\partial W_2}{\partial X} f_0 \tag{2.11}$$

Again we choose W_2 to remove the short period terms in the r.h.s. of (2.5) and get g_2 and so on. Thus, step by step calculations yield a solution to the desired order.

3. EQUATION OF MOTION OF A CHARGED PARTICLE IN A PERIODIC MAGNETIC FIELD.

The nonrelativistic Hamiltonian H which describes the system is

$$H = \left(\frac{1}{2m}\right)\left\{\underline{p} - (e/c)\underline{A}\right\}^2 \tag{3.1}$$

where $\underline{A}$ is the vector potential

$$\underline{A} = \left(-B_0 y, \frac{(B_1)}{k} \text{Cos } kz, 0\right) \tag{3.2}$$

which gives the magnetic field $\underline{B}$ as

$$\underline{B} = (B_1, \text{Sin } kz, 0, B_0) \tag{3.3}$$

H can be written as

$$H = \left(\frac{1}{2m}\right)\left[(p_x + m\omega_0 y)^2 + ((p_y - \frac{(m\omega_1)}{k}\text{Cos } kz)^2 + p_z^{\,2})\right] \tag{3.4}$$

$$\varepsilon = \frac{\omega 1}{\omega 0} = \frac{B1}{B0} \tag{3.5}$$

and ε is a small parameter.

Since we aim to find a complete second order solution of the equations of motion and to compare, wherever possible, our results with those obtained by Coffey, we use the same variables and notation as Coffey to derive the equations of motion.

H can be transformed to the hamiltonian h given by

$$h = J + \left(\frac{1}{2m\omega_0}\right)pz^2 - \left(\frac{\varepsilon}{k}\right)(2m\omega_0 J)^{1/2}\operatorname{Cos}\Psi \operatorname{Cos} kz + \left(\frac{\varepsilon^2 \omega_0}{2k^2}\right)\operatorname{Cos}^2 kz \tag{3.6}$$

Then the equations of motion are

$$J' = \left(-\frac{\varepsilon}{2k}\right)(2m\omega_0 J)^{1/2}[\operatorname{Sin}(\Psi + kz) + \operatorname{Sin}(\Psi - kz)] \tag{3.7}$$

$$p'_z = -\left(\frac{\varepsilon}{2}\right)(2m\omega_0 J)^{1/2}[\operatorname{Sin}(\Psi + kz) - \operatorname{Sin}(\Psi - kz)] + \frac{\varepsilon^2 m\omega_0}{k}\operatorname{Cos} kz \operatorname{Sin} kz \ldots. \tag{3.8}$$

$$kz' = \frac{kpz}{m\omega_0} \tag{3.9}$$

(For relations between $pz \cdot \frac{k}{k'}$ I and $m\omega_0$ see ref [1])

$$\Psi' = 1 - \left(\frac{\varepsilon}{2k}\right)\left(\frac{m\omega_0}{2J}\right)^{1/2}[\operatorname{Cos}(\Psi + kz) + \operatorname{Cos}(\Psi - kz)] \tag{3.10}$$

where $J' = \frac{\partial J}{\partial t}$ etc.

$$c = \omega_0 t \tag{3.11}$$

and J, P_z, and Z, y are related to the cartesion co-ordinates of the guiding centre and their conjugate momenta p_x, p_z, p_z in the following way:

$$x = r - \left(\frac{2J}{m\omega_0}\right)^{1/2}\cos\Psi \tag{3.12}$$

$$y = -\left(\frac{1}{m\omega_0}\right)p_r + \left(\frac{2J}{m\omega_0}\right)^{1/2}\operatorname{Sin}\Psi. \tag{3.13}$$

$$p_x = p_r,\ p_y = (2m\omega_0 J)^{1/2}\operatorname{Cos}\Psi \tag{3.14}$$

$$p_z = P_z \tag{3.15}$$

We apply Lie transform theory to the system of equations (3.7) to (3.10) i.e., we write (3.7) to (3.10) in the form

$$\frac{\partial X}{\partial t} = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}f_n(X) \tag{3.16}$$

where

$$X = \begin{bmatrix} J \\ P_z \\ \phi_1 \\ \phi_2 \end{bmatrix} \tag{3.17}$$

$$f_0 = \begin{bmatrix} 0 \\ 0 \\ 1 + \frac{kp_z}{m\omega_0} \\ 1 - \frac{kp_z}{m\omega_0} \end{bmatrix} \tag{3.18}$$

$$f_1 = \begin{bmatrix} -\frac{A}{k} aJ^{1/2} \operatorname{Sin}\phi_1 - \frac{1}{k} aJ^{1/2} \operatorname{Sin}\phi_2 \\ -aJ^{1/2} \operatorname{Sin}\phi_2 + aJ^{1/2} \operatorname{Sin}\phi_2 \\ -\frac{a}{2k} J^{-1/2} \operatorname{Cos}\phi_1 - \frac{a}{2k} J^{-1/2} \operatorname{Cos}\phi_2 \\ -\frac{a}{2k} J^{-1/2} \operatorname{Cos}\phi_1 - \frac{a}{2k} J^{-1/2} \operatorname{Cos}\phi_2 \end{bmatrix} \tag{3.19}$$

$$f_2 = \begin{bmatrix} 0 \\ \frac{m\omega_0}{2k} \operatorname{Sin} 2(\phi_1 - \phi_2) \\ 0 \\ 0 \end{bmatrix} \tag{3.20}$$

$$a = \frac{1}{2} (2m\omega_0)^{1/2} \tag{3.21}$$

4. FIRST ORDER SOLUTIONS.

We write

$$g_1(Y) = f_1(Y) + L_1 f_0 \tag{4.1}$$

where

$$L_1 f_0 = \frac{\partial (f_0)i}{\partial x_j} W_{1j} - \frac{\partial (W_1)i}{\partial n_j} (f_0)j \tag{4.2}$$

where W_1 is a column matrix given by

$$W_1 = \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{bmatrix} \tag{4.3}$$

We choose W_1 in such a way that g_1 contains only long period terms. For that purpose, we assume that ϕ_1 contributes to only small amplitude rapid fluctuations and ϕ_2 gives rise to secular motion. From (4.1) we have,

$$g_1 = \begin{bmatrix} -\left(1+\frac{kp_z}{m\omega_0}\right)\frac{\partial\Psi_1}{\partial\phi_1} - \left(1-\frac{kp_z}{m\omega_0}\right)\frac{\partial\Psi_1}{\partial\phi_2} \\ -\left(1+\frac{kp_z}{m\omega_0}\right)\frac{\partial\Psi_2}{\partial\phi_1} - \left(1-\frac{kp_z}{m\omega_0}\right)\frac{\partial\Psi_2}{\partial\phi_2} \\ -\left(1+\frac{kp_z}{m\omega_0}\right)\frac{\partial\Psi_3}{\partial\phi_1} - \left(1-\frac{kp_z}{m\omega_0}\right)\frac{\partial\Psi_3}{\partial\phi_2} + \frac{k\Psi_2}{m\omega_0} \\ -\frac{k\Psi_2}{m\omega_0} - \left(1+\frac{kp_z}{m\omega_0}\right)\frac{\partial\Psi_4}{\partial\phi_1} - \left(1-\frac{kp_z}{\omega_0}\right)\frac{\partial\Psi_4}{\partial\phi_2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{k}aJ^{\frac{1}{2}}\operatorname{Sin}\phi_1 - \frac{1}{k}aJ^{\frac{1}{2}}\operatorname{Sin}\phi_2 \\ -aJ^{\frac{1}{2}}\operatorname{Sin}\phi_1 + aJ^{\frac{1}{2}}\operatorname{Sin}\phi_2 \\ -\frac{a}{2k}J^{-\frac{1}{2}}\operatorname{Cos}\phi_1 - \frac{a}{2k}J^{-\frac{1}{2}}\operatorname{Cos}\phi_2 \\ -\frac{a}{2k}J^{-\frac{1}{2}}\operatorname{Cos}\phi_1 - \frac{a}{2k}J^{-\frac{1}{2}}\operatorname{Cos}\phi_2 \end{bmatrix} \tag{4.4}$$

g_1 contains long period term only. Therefore, we choose Ψ in such a way that

$$\frac{\partial\Psi}{\partial\phi_2} = 0 \tag{4.5}$$

and

$$\left(1+\frac{kp_z}{m\omega}\right)\frac{\partial\Psi_1}{\partial\phi_1} = -\frac{1}{k}aJ^{\frac{1}{2}}\operatorname{Sin}\phi_1 \text{ etc.} \tag{4.6}$$

So that

$$W_1 = \begin{bmatrix} \frac{1}{k\omega_2}a\operatorname{Cos}\phi_1 J^{\frac{1}{2}} + A_1 \\ \frac{1}{k\omega_2}a\operatorname{Cos}\phi_1 J^{\frac{1}{2}} + A_2 \\ -\frac{a}{2k\omega_2}J^{-\frac{1}{2}}\operatorname{Sin}\phi_1 + \frac{ak}{m\omega_0\omega_2{}^2}\operatorname{Sin}\phi_1 J^{\frac{1}{2}} + A_3 \\ -\frac{a}{2k\omega_2}J^{-\frac{1}{2}}\operatorname{Sin}\phi_1 - \frac{ak}{m\omega_0\omega_2{}^2}\operatorname{Sin}\phi_1 J^{\frac{1}{2}} + A_4 \end{bmatrix} \tag{4.7}$$

where A_1 is a function of J and p_z.

After standard calculations, we get

$$g_1 = \begin{bmatrix} -\frac{a}{k} J^{*\frac{1}{2}} \mathrm{Sin}\phi_2 \\ aJ^{*\frac{1}{2}} \mathrm{Sin}\phi_2 \\ -\frac{a}{2k} J^{*-\frac{1}{2}} \mathrm{Cos}\phi_2 \\ -\frac{a}{2k} J^{*-\frac{1}{2}} \mathrm{Cos}\phi_2 \end{bmatrix} \tag{4.8}$$

writing $\phi_2 = \sigma$ we have

$$J^{*\prime} = -\frac{\varepsilon a}{k} J^{*\frac{1}{2}} \mathrm{Sin}\sigma \tag{4.9}$$

$$p_z^{*\prime} = \varepsilon a\ J^{*\frac{1}{2}} \mathrm{Sin}\sigma \tag{4.10}$$

$$\sigma^{*\prime} = 1 - \frac{kp_z^{*\prime}}{m\omega_0} - \frac{\varepsilon}{2k} aJ^{*-\frac{1}{2}} \mathrm{Cos}\sigma \tag{4.11}$$

where * denotes the secular part.

Remembering that $a = \left(\frac{m\omega_0}{2}\right)^{\frac{1}{2}}$, these equations are identical with those obtained by Coffey. As the first order solutions have been discussed in an extensive manner in reference [1], we will not go into the details of the solutions of (3.7) to (3.10) except to point out the fact that the secular part exhibits periodic behaviour and gives stable solutions.

5. SECOND ORDER CALCULATIONS; For the second order, we have

$$g_2 = f_2 + L_1 f_1 + L_1 g_1 + L_2 f_0 \tag{5.1}$$

Since g_1 consists of long period terms only and W_1 short period terms, we have

$$L_1 g = 0 \tag{5.2}$$

Also,

$$L_2 f_0 = \frac{\partial f_0}{\partial X} W_2 - \frac{\partial W_2}{\partial X} f_2 \tag{5.3}$$

where W_2 is a column vector

$$W_2 = \begin{bmatrix} \Psi_1' \\ \Psi_2' \\ \Psi_3' \\ \Psi_4' \end{bmatrix} \tag{5.4}$$

We choose the Ψ' in such a way that g_2 contains only long period terms. After some tedious but not very complicated calculations, we finally get

$$g_2 = \begin{bmatrix} 0 \\ 0 \\ \dfrac{3a^2}{4m\omega_0\omega_2{}^2} \\ -\dfrac{a^2}{4m\omega_0\omega_2{}^2} \end{bmatrix} \tag{5.5}$$

where

$$W_2 = 1 + \frac{kp_z{}^*}{m\omega_0} \tag{5.6}$$

Hence, the equations for J^*, $p_z{}^*$, and σ^* up to second order are given by

$$J^{*\prime} = -\frac{fa}{k}J^{*\frac{1}{2}}\,\text{Sin}\sigma \tag{5.7}$$

$$p_z{}^{*\prime} = \varepsilon a(J^*)^{\frac{1}{2}}\,\text{Sin}\sigma \tag{5.8}$$

$$\sigma^{*\prime} = 1 - kp_z{}^* - \frac{\varepsilon a}{2k}(J^*)^{-\frac{1}{2}}\,\text{Cos}\sigma - \frac{\varepsilon^2}{8\omega_2{}^2} \tag{5.9}$$

Let us now introduce the new variables

$$A = R\ \text{Cos}\sigma \quad \text{and} \quad B = R\ \text{Sin}\sigma \tag{5.10}$$

where

$$R = k\left(\frac{2J^*}{m\omega_0}\right)^{\frac{1}{2}} \tag{5.11}$$

From equation, (5.7) to (5.9) we have

$$J^{*\prime} + \frac{p_z{}^{*\prime}}{k} = I \tag{5.12}$$

where I is a constant. This result was also obtained by Coffey. Again, the set of equations (5.7) to (5.9) now reduce to equations relating only R and σ. The equations for A and B are

$$A^\prime = -B\left[1 - C + \frac{k^2}{2} - \frac{\varepsilon^2}{8\omega_2{}^2}\right] \tag{5.13}$$

$$B^\prime = -\frac{\varepsilon}{2} + A\left[1 - C + \frac{k^2}{2} - \frac{\varepsilon^2}{8\omega_2{}^2}\right] \tag{5.14}$$

where

$$C = \frac{k^2 I}{m\omega_0} \tag{5.15}$$

It is clear from equations (5.13) to (5.15) that the term proportional to ε^2 dominates when $\omega_2 \to 0$. This could not be anticipated from a first order calculation.

Equations (5.13) to (5.15) give a constant of motion given by

$$R^4 + 4(1 - C)R^2 - 4\varepsilon R \cos\sigma + \frac{2\varepsilon^2}{R^2 - 2(1 + C)} = G, \text{ say} \tag{5.16}$$

where G turns out to be given by

$$G = \frac{8k^2h}{m\omega_0} - 4C^2 - \frac{5\varepsilon^2}{2} \tag{5.17}$$

Singular points of (5.13), (5.14) are given by solbing (5.13), (5.14) with $A'=B'=0$. Remembering that

$$\omega_2 = 1 + \frac{kp_z^*}{m\omega_0} = D - \frac{k^2}{2} \tag{5.18}$$

where

$$D = 1 + \frac{2k^2 I}{m\omega_0} \tag{5.19}$$

The singular points are found by solving a seventh degree equation in A. We have obtained the roots numerically with the help of the Newton-Raphson technique and the Bairstow method [6] for various values of C using $\varepsilon = .01$. A list of some of those values are given below.

	C = 1.75	1.87	1.111
a_1	-2.345+i003	-2.396	-2.057
a_2	-2.345-i003	-2.397	-2.053
a_3	-1,221	-1.316	-.447
a_4	-1.169	-1.194	-1.004
a_5	-.007	-.006	-.045
a_6	1.228	1.322	.492
a_7	.56245	.540	.805

The nature of these singular points was also determined by putting $A=a_i+3$, and $B=b$ and then seeking solutions of the form

$$\xi = \xi_0 e^{Xt}, \quad b = b_0 e^{Xt} \tag{5.20}$$

Using (5.20), we find that a_4 and a_5 are unstable points and the rest are stable points, each of the stable points being a centre. In figures 1 and 2, several trajectories of a proton are plotted; the parameters such as the energy, background field strength, etc. have been assigned values which are the same as those used by

Coffey. Though qualitatively the results are the same, we have obtained one additional stable point (see fig. 2) at which the fluctuation of K is maximum, as will be shown later.

6. CALCULATIONS OF THE TIME DEPENDENCE OF THE MOTION.

Introducing the variable $S=R^2$ and eliminating σ from equations (5.7) to (5.9), we get the following differential equation for S:

$$S' = \tfrac{1}{4}\Big\{-[S^4 - 4(2d-1)S^3 + 6S^2(2d-1)^2 - 4(2d-1)^3 S + (2d-1)^4 + 5\varepsilon^2(2d-1)^2 - \frac{8\varepsilon^2 S(2d-1)}{S-2(1+c)} + \frac{4\varepsilon^2 S^2}{S-2(1+C)} + 5\varepsilon^2 S^2 - 10(2d-1)^2\varepsilon^2 S + \frac{4\varepsilon^2(2d-1)^2}{S-2(1+C)} - 16\varepsilon^2 S]\Big\}^{\frac{1}{2}} \tag{6.1}$$

As before, we calculate the roots of the equation obtained from (6.1) when $S'=0$ by solving the fifth degree equations numerically. Besides the roots obtained by Coffey, a rather large root is obtained which is given by $S \approx 2(1 - C)$ for small $\varepsilon(\varepsilon \neq 0)$. This root is due to the very existence of second order terms in (50) and hence cannot be anticipated from first order calculations. If we neglect this root, then Coffey's analysis of the relative fluctuation in K under resonance conditions holds. But with this root,

$$\Delta K_{max} = \frac{2(S_{max} - S_{min})}{S_{max} + S_{min}} \simeq \frac{4}{2(R_0^2 + 2)} \tag{6.2}$$

where R_0 is 2π times the ratio of the initial cyclotron radius to the wavelength of the disturbance. Hence we see that ΔK is of the order of unity for $R_0^2 < 2$, a result which could not be guessed from first order calculations.

CONCLUSION.

Using the Lie transform method, we have obtained a complete second order solution for the motion of a charged particle in a constant magnetic field on which a weak spatially periodic magnetic field is superimposed. Though our first order solutios completely agrees with that of Coffey, it is found that a second order solution is not insignificant in the sense that it modifies the relative fluctuation of the average magnetic moment by a not too negligible amount.

Our second order solution also confirms the periodic behavior of the secular motions, which means that particles travelling along the ripples sufficiently fast can chang their magnetic moment in an oscillatory manner. (Of course this does not hold true for a helical perturbation). Moreover, it shows how the secular terms are separated from the rapidly fluctuating parts in a straight forward way. As stated before, the Lie transform is very convenient in studying the oscillation of a weakly nonlinear system and the present problem provides a scope for its successful application without involving very complicated calculations.

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