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RESEARCH NOTES

BOUNDED SETS IN FAST COMPLETE INDUCTIVE LIMITS

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ABSTRACT. Let $E_1 \subset E_2 \subset \ldots$ be a sequence of locally convex spaces with all identity maps: $E_n \longrightarrow E_{n+1}$ continuous and $E = \text{ind lim } E_n$ fast complete. Then each set bounded in E is also bounded in some E_n iff for any Banach disk B bounded in E and $n \in N$, the closure of $B \cap E_n$ in B is bounded in some E_m. This holds, in particular, if all spaces E_n are webbed.

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Throughout the paper $E_1 \subset E_2 \subset \ldots$ is a sequence of Hausdorff locally convex spaces with continuous identity maps: $E_n \longrightarrow E_{n+1}$, and $E = \text{ind lim } E_n$ also Hausdorff.

For an absolutely convex set A in a locally convex space X we denote by X_A the linear hull of A equipped with the topology generated by $\{\lambda A;\ \lambda > 0\}$. If X_A is a Banach space, A is called a Banach disk, see [1]. The space X is fast complete if each set bounded in X is contained in a Banach disk which is bounded in X. Every sequentially complete locally convex space is fast complete.

If $B \subset C \subset X$, the closure of B in C is denoted by $\overline{B}^C$. For brevity we denote by DS, resp. DST, the following property: each set bounded in E is contained, resp. bounded, in some E_n.

We use the notion of webbed spaces, see [1] or [2], to derive our first criterion for DST.

THEOREM 1. If all spaces E_n are webbed and E is fast complete, then DST holds.

PROOF. Let $A \subset E$ be bounded. Then A is contained in a Banach disk B bounded in E. The space E_B is Banach and the identity map id: $E_B \longrightarrow \text{ind lim } E_n$ is continuous. By the Corollary IV.6.5 of [1], there exists $n \in N$ such that $E_B \subset E_n$ and id: $E_B \longrightarrow E_n$ is continuous. Since A is bounded in E_n, it is bounded in E_n.

REMARK. The same result is proved in [3] for strictly webbed spaces.

It is evident that if all spaces E_n are fast complete then DST implies fast completeness of E. In particular, since every Fréchet space is webbed and fast complete, we have: If all spaces E_n are Fréchet then DST holds iff E is fast complete.

PROPOSITION. Let B be a Banach disk bounded in E. Then $B = \overline{B \cap E_m}^B$ for some $m \in N$.

PROOF. Put $B_n = \overline{B \cap E_n}^B$ and $F_n = E_{B_n}$, $n \in N$. The set B_n is closed in the Banach space E_B, hence F_n is Banach and as such it is also webbed. Since each $\mathrm{id}: F_n \to E_B$ is continuous, the map $\mathrm{id}: \mathrm{ind\,lim}\, F_n \to E_B$ is continuous too and its graph is fast sequentially closed. Hence the inverse mapping $\mathrm{id}: E_B \to \mathrm{ind\,lim}\, F_n$ has also fast sequentially closed graph and, by Corollary IV.6.5 of [1], $E_B \subset F_m$ for some m.

Assume there exists $b \in B \setminus B_m$ and put $\beta = \inf\{\alpha > 0;\ b \in \alpha B_m\}$. Evidently $b \in \beta B_m$. Hence $\beta > 1$. There exists a sequence $\{b_k\} \subset B \cap E_m$ such that $b_k \to \beta^{-1} b$ is the topology of E_B. Take $\gamma \in (1, \beta)$. Then $\|\beta^{-1} b\| < \|\gamma^{-1} b\|$ and $\|b_k\| < \|\gamma^{-1} b\|$ for sufficiently large k's. For the same k's, we have $\|\gamma b_k\| < \|b\|$, which means $\gamma b_k \in B$. Further $\gamma b_k \in E_m$ and $\gamma b_k \to \gamma \beta^{-1} b \in B_m$ in the topology of E_B, i.e. $b \in \gamma^{-1} \beta B_m$, a contradiction.

THEOREM 2. Let E be fast complete. Then DS, resp. DST, holds iff for any Banach disk B bounded in E and any $n \in N$, $\overline{B \cap E_n}^B$ is contained, resp. bounded, in some E_m.

PROOF. "If" part is evident. For the "only if", take a set A bounded in E, then A is contained in a Banach disk $B \subset E$. By the Prop., there exists $n \in N$ such that $B = \overline{B \cap E_n}^B$ which is contained, resp. bounded, in some E_m.

EXAMPLE. Let FC stand for the property:

Each set bounded in E_n is contained in a bounded Banach disk in E .

And let P_1, resp. P_2, stand for:

For each bounded Banach disk B in E and $n \in N$, the set $\overline{B \cap E_n}^B$ is contained, resp. bounded, in some E_m.

It follows from Theorem 2 that: A_2 & E fast complete $\Leftrightarrow$ DST & FC, A_1 & E fast complete $\Rightarrow$ DS & FC. The last implication cannot be reversed. To show that, take a Banach space X, $\dim X = +\infty$, denote by L its underlying vector space, and choose a subspace $M \subset L$ which is dense in X. Let Y be L equipped with the finest locally convex topology, $V = \cup\{L^n \times M^N,\ n \in N\}$, $X_n = X^n \times Y^N$, and E_n be the vector space V with the topology inherited from X_n, $n \in N$.

The property DS holds, since the underlying vector spaces of all E_n are the same. We show that each E_n is quasi-complete. Hence it is fast complete, and FC trivially holds.

Let $A \subset E_n$ be bounded. Then $A \subset \Pi\ \{A_k;\ k \in N\}$, where A_k is bounded, closed, and absolutely convex in X for $k \le n$, and in Y for $k > n$. Any set bounded in Y is contained and bounded in a finite dimensional subspace of Y. Hence each A_k, $k \in N$, is complete and A is contained in the complete set $\Pi\ \{A_k;\ k \in N\}$.

The space $E = \mathrm{ind\,lim}\, E_n$, which equals V with the topology inherited from X^N, is not fast complete. Assume the contrary. If B is the closed unit ball in X, then $B_0 = B^N \cap V$ is bounded in E and a fortiori contained in a bounded Banach disk D in E .

Take $x_0 \in B \setminus M$, choose a sequence $\{x_k\} \subset B \cap M$ such that $x_k \to x_0$ in X, and put $y_k = (x_0, x_0, \dots, x_0, x_{k+1}, x_{k+2}, \dots)$, where x_0 is repeated k - times,

$k \in N$. Then $\{y_k\}$ is a Cauchy sequence in E_D and $y_k \to (x_0, x_0, \dots)$ in the topology of X^N. Since $(x_0, x_0, \dots) \notin E_D$, D is not a Banach disk.

REFERENCES

1. DE WILDE, M., Closed Graph Theorems and Webbed Spaces, Pittman, London, 1978.

2. KÖTHE, G., Topological Vector Spaces II, Springer 1979.

3. QIU, Jing-Hui, Some Results on Bounded Sets in Inductive Limits, to appear.