FUNCTION THEORY FOR A BELTRAMI ALGEBRA

B. A. CASE

Department of Mathematics
The Florida State University
Tallahassee, Florida   32306

ABSTRACT. Complex functions are investigated which are solutions of an elliptic system of partial differential equations associated with a real parameter function. The functions f associated with a particualr parameter function g on a domain D form a Beltrami algebra denoted by the pair (D,g) and a function theory is developed in this algebra. A strong conformality property holds for all functions in a (D,g) algebra. For $g \equiv |z| = r$ the algebra (D,r) is that of the analytic functions.

1. INTRODUCTION.

The idea of defining classes of functions which are generalizations of the analytic functions by means of systems of first order partial differential equations goes back at least to a paper of Picard in 1891 [1]. The solutions of that system were of the type later called "pseudo-analytic functions of the second kind" by Bers [2] where for $f = u+iv$

$$u_x = a\, v_x + b\, v_y$$
$$-u_y = b\, v_x - a\, v_y$$

(1.1)

where a, b, c and d are appropriate real-valued functions. The relationship of solutions of the elliptic system

$$u_x = a\, v_x + b\, v_y$$
$$-u_y = c\, v_x + a v_y$$

(1.2)

where a, b, c and d are real valued and $bc-a^2 = 1$, to those of system (1.1) is well developed in papers by Caldwell, [3, 4]

Two current research areas related to the early investigations of elliptic systems are the theory of quasiconformal mappings and function theoretic methods in partial differential equations. Works from the latter point of view which have

extensive references are Vekua, [5] and Gilbert and Buchanan, [6]. Ahlfors
discusses, in his introduction to [7], analytic theory generalizations and rationale
concerning quasiconformal mappings in particular. Examples from the algebras of
functions which we consider are found in very different areas of investigation,
indicating the possibility of an interesting and useful function theory. (e.g.,
Concerning quasiconformal mappings see [8], p. 71; topological analysis, [9] and
[10], p. 198; elliptic systems of partial differential equations, [5], p. 131.)

The functions considered are solutions of a Beltrami differential equation with
complex dilatation at each point z dependent on a real parameter function g . A
generalized derivative f'_g is defined. Necessary and sufficient conditions for
existence of f'_g include the Beltrami equation which reduces to the Cauchy-Riemann
equations for analytic f . Both re f and im f satisfy the same second order
partial differential equation with analytic antecedent the Laplace equation.
Although the available generalization of the Laplace equation and its applications
are of interest in any elliptic system, the parallel is not always so direct. (e.g.,
In the system of [11] a different generalized system holds for each of re f , im f.)

For functions in algebra (D,g) , a generalized conformality holds which is
invariant in the algebra, that is, for given z and two fixed arcs the angle between
the image arcs is the same for all functions in the algebra. The generalized line
integral $\,^g\!\!\int f$ shares in modified form much of the theory of the ordinary complex
line integral. Among some applications in the last section is a function continuous
on the finite complex plane, in a (D,g) algebra on $\mathbb{C}-\{(0,0)\}$, K-quasiconformal on
bounded sets, and which maps $\mathbb{C}$ onto a fixed open disk.

2. NOTATION AND DEFINITIONS.

Complex valued functions of one complex variable $z = re^{i\theta} = x+iy$ are denoted
f and re $f = u$, im $f = v$, $|f| = \rho$ and arg $f = \phi$. Partial derivatives are
denoted by subscripts. Standard definitions used are:

$$f_z = e^{-i\theta}(f_r - (i/r)f_\theta) ,$$
$$f_{\bar{z}} = e^{i\theta}(f_r + (i/r)f_\theta) , \qquad\qquad (2.1)$$
$$J = |f_z|^2 - |f_{\bar{z}}|^2 .$$

The complex plane is denoted $\mathbb{C}$ and a domain $D \subset \mathbb{C}$ is an open connected set
(and, later in the paper, is required to be simply connected). The classes C^n $[C^\infty]$
comprise the functions with continuous partial derivatives of n [all] orders.
Usage is generally consistent with [12] and [13] as concerns classical analysis;
[8], [7] and [14] as concerns quasiconformal functions; and [15] as concerns
functional analysis.

The derivative f'_g defined here is the limit of a generalized difference
quotient involving the parameter function g which reduces to the difference
quotient for analytic functions. For $f \in C'$ in a domain, existence of the limit
of this generalized difference quotient is shown in [16] to be necessary and
sufficient for our defining condition (2.2 below).

DEFINITION 1. A parameter function g in the following discussion is a real
valued positive function of $|z| = r$ for $z \in D$, such that $g' = dg/dr$ exists and
is continuous on D .

DEFINITION 2. The ordered pair (D,g) represents the set of complex valued homeomorphisms $f \in C^2(D)$ satisfying

$$f_{\bar{z}} = \frac{(1-(g/r))}{(1+(g/r))} \, e^{i2\theta} f_z \; . \tag{2.2}$$

For such functions the generalized derivative is defined

$$f'_g = e^{-i\theta} f_r \; . \tag{2.3}$$

A function is called g-analytic at a point z when there is a parameter function g so (2.2) holds on some domain containing z ; f'_g is called the g-derivative.

The Beltrami equation (2.2) is equivalent to a generalized Cauchy-Riemann condition. When stated in rectangular coordinates in the form (1.2) the functions a, b, c are: $a = xy(g^2 - r^2)/g r^3$; $b = (x^2 r^2 + y^2 g^2)/g r^3$; $c = (y^2 r^2 + x^2 g^2)/g r^3$. For computational convenience we state that condition as a system in polar form

$$(\text{im } f)_\theta = g \, (\text{re } f)_r$$

$$(\text{re } f)_\theta = -g \, (\text{im } f)_r \tag{2.4}$$

or,

$$f_\theta = ig \, f_r \; .$$

DEFINITION 3. Let f be complex valued and continuous on (piecewise) differentiable arc γ on which $r \neq 0$. A generalized line integral dependent on parameter function g is defined by:

$$\overset{g}{\underset{\gamma}{\int}} f = \int_\gamma (\omega/g)(u\cos\theta - v\sin\theta)dr - (v\cos\theta + u\sin\theta)d\theta$$
$$+ \, i\int_\gamma (\omega/g)(v\cos\theta + u\sin\theta)dr + (u\cos\theta - v\sin\theta)d\theta \tag{2.5}$$

where $\omega = \exp \int_1^r dr/g$.

The properties of the lemma follow from the definitions in a straight-forward manner.

LEMMA 1.

a)   When $f \in (D,g)$, $J = (g/r)|f_r|^2$.

b)   The constant functions are in each set (D,g) and are the only functions in $(D,g_1) \cap (D,g_2)$, $g_1 \neq g_2$; $f'_g \equiv 0$ for a constant function.

c)   If $f \in (D,g)$ and $f'_g \equiv 0$, then f is constant. Also, f is constant if either re f , im f , $|f|$ or arg f is identically constant.

d)   If $f \in (D,g)$ and c is a complex number, then $cf \in (D,g)$.

e)   If f , $h \in (D,g)$ then the sum, product and reciprocals are in (D,g) $(1/f$ for $f \neq 0)$ and the generalized derivatives are:

$$(f+h)'_g = f'_g + h'_g \; ,$$

$$(fh)'_g = f'_g h + h'_g f \; , \tag{2.6}$$

$$(1/f)'_g = -f'_g/(f)^2 \; .$$

f)   If f , $h \in (D,g)$ such that $f'_g \equiv h'_g$ on D , then f-h is a constant function on D .

g)   If nonconstant $f \in (D,g)$ and $h \in (D,r)$ then $h \circ f \in (D,g)$.

h) (D,g) is an algebra over $\mathfrak{C}$. (The multiplicative unit is f ≡ 1 .)

For $g \equiv r$, (D,g) is the set of functions analytic on D , and   (2.3) and
(2.5) are, respectively, the usual derivative and line integral. The requirement in
Definition 1 that the function g be positive is equivalent to requiring that the
Jacobians for functions $f \in (D,g)$ be non-negative and hence that the functions be
sense preserving. This is consistent with practice in the larger quasiconformal
class although, as there, many of these results hold for sense reversing functions.
Other conditions on f or g in the definitions may be weakened in some
applications. See [17], [4] for careful discussion of topological and
quasiconformal properties of Beltrami systems.

3.   MAPPING PROPERTIES

Functions which are solutions of the Beltrami equation (2.2) are locally
quasiconformal. In particular, at $z \in D$ the dilatation quotient is given

$$\rho_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} = \frac{|1 + (g/r)| + |1 - (g/r)|}{|1 + (g/r)| - |1 - (g/r)|} = \begin{array}{l} g/r, \ g/r \geq 1 , \\ 1/(g/r), \ g/r < 1. \end{array} \quad (3.1)$$

The dilatation quotient is bounded on every compact subset of D and, if it is
bounded on D then f is a regular K-quasiconformal mapping for

$$K \geq \sup_{z \in D} \max\{g/r, \ r/g\} .$$

A strong generalization of conformality holds for g-analytic functions. Given a
point z and the parameter function g , the angle between the images of two fixed
arcs is the same whichever $f \in (D,g)$ maps the arcs. This reduces to angle
preservation in the conformal case. It is more specific than the inequality
available for the larger class of quasiconformal mappings in general. (See [18],
Theorem 4, which is an equivalence to quasi-conformality.)

THEOREM 1. Arcs γ_1 and γ_2 intersect at z where they have tangents which
make angles respectively of α and β with the real axis and of σ and τ with
the radial direction through z . If $f \in C^1$ is g-analytic in a domain containing
z then the angle ψ' between $f(\gamma_1)$ and $f(\gamma_2)$ is determined by:

$\sin\psi'/\cos\psi = ((g/r)\sin(\beta-\alpha))[(\cos^2\theta+(g/r)^2\sin^2\theta)\cos\alpha\cos\beta$

$\qquad + (\sin^2\theta+(g/r)\cos^2\theta)\sin\alpha\cos\beta + \cos\theta\sin\theta(1-(g/r)^2)(\sin(\alpha+\beta)]^{-1}$ (3.2)

$\qquad = ((g/r)(\sin(\tau-\sigma))/(\cos\sigma\cos\tau+(g/r)^2 \sin\sigma\sin\tau) .$

Conversely, if $f \in C^1$, $\tau > 0$, and (3.2) holds on some domain containing z
where g satisfies Definition 1, then f is g-analytic at z .

PROOF. The second of the forms of (3.2) is most convenient to derive. The
first then follows, since $\sigma = \alpha+\Pi-\theta$, $\tau = \beta+\Pi-\theta$. Let σ' be the angle made by
$f(\gamma_1)$ at the point $f(z)$ with the radial direction through $f(z)$. Manipulating
algebraically with allowable angles, we have

$$\frac{\sin\sigma'}{\cos\sigma'} = \rho(d\phi/d\rho) = \frac{r\cos\sigma\phi_r + \sin\sigma\phi_\theta}{r\cos\sigma\rho_r + \sin\sigma\rho_\theta}$$

$$= \frac{(r\cos\sigma v_r + \sin\sigma v_\theta)u - (r\cos\sigma u_r + \sin\sigma u_\theta)v}{(r\cos\sigma v_r + \sin\sigma v_\theta)v - (r\cos\sigma u_r + \sin\sigma u_\theta)u}$$

and, using similar expressions for τ , τ' write

$$\frac{\sin\psi'}{\cos\psi'} = \frac{\sin(\tau'-\sigma')}{\cos(\tau'-\sigma')}$$

$$= \frac{(r\cos\sigma u_r+\sin\sigma u_\theta)(r\cos\tau v_r+\sin\tau v_\theta) - (r\cos\sigma v_r+\sin\sigma v_\theta)(r\cos\tau u_r+\sin\tau u_\theta)}{(r\cos\sigma v_r+\sin\sigma v_\theta)(r\cos\tau v_r+\sin\tau v_\theta) + (r\cos\sigma u_r+\sin\sigma u_\theta)(r\cos\tau u_r+\sin\tau u_\theta)} \qquad (3.3)$$

$$= \frac{r\ \text{im}(\bar{f}_r f_\theta)(\sin(\tau-\sigma))}{(r^2|f_r|^2\cos\sigma\cos\tau+\sin\sigma\sin\tau|f_\theta|^2+r(\sin(\sigma+\tau))\text{re}(\bar{f}_r f_\theta)} \quad .$$

Since $f \in (D,g)$ by hypothesis, substitution using (2.4) gives (3.2) .

Conversely, if f is a function for which the expressions (3.2) , (3.3) are equivalent for all arcs, then

$$r^2(g|f_r|^2 - \text{im}(\bar{f}_r f_\theta))+r(g(\text{re}(\bar{f}_r f_\theta))(\tan\sigma+\tan\tau))+((g|f_\theta|^2-g^2\text{im}(\bar{f}_r f_\theta)(\tan\sigma\tan\tau))=0$$

and hence we have the system

$$g|f_r|^2 - \text{im}(\bar{f}_r f_\theta) = 0$$
$$\text{re}(\bar{f}_r f_\theta) = 0 \qquad (3.4)$$

which is equivalent to the complex form of (2.4) ; since $J > 0$, f is sense preserving, hence f is g-analytic.

DEFINITION 4. If $f \in (D,g)$ and f satisfies (3.2) locally for all $z \in D$, then f is _g-conformal_ on D .

For all functions in a given (D,g) algebra, the g-conformality gives the angle between the images of intersecting arcs depending on the tangents to the arcs themselves and the point at which they intersect. In the algebra of analytic functions (D,r) the relation is not only also independent of the intersection point but, reducing to angle preservation, is the same for all pairs of arcs (γ_1,γ_2) with the same angle between them. The similarity to the analytic case is less obvious in the case of the effect of the g-analytic function on arc length. Where ds and dS are elements of arc in the z and $f(z)$ planes, respectively,

$$dS^2 = |f_g'|^2(dr^2+g^2 d\theta^2) \quad . \qquad (3.3)$$

The modulus of the generalized derivative retains position as a magnification factor, although not in the Euclidean metric.

4. PROPERTIES OF THE FUNCTION THEORY

In addition to some theory of the generalized derivative (2.3) and integral (2.5) , we give a generalized harmonicity relation which is satisfied by both re f and im f for f in a (D,g) algebra. It is helpful to first construct two examples which are the most general functions in a (D,g) algebra under certain conditions.

From now on, we assume a domain D is a simply connected.

EXAMPLE 1. If at $z \in D$

$$f(z) = \lambda(r)\ \exp(i(c_1\theta+c_2)) \qquad (4.1)$$

where c_1 , c_2 are real constants, $c_1 \neq 0$, and nonconstant real valued $\lambda(r) \in C^2(D)$, then f is g-analytic at z and $g = c_1\lambda/\lambda'$. On the other hand, when f_1

is a g-analytic function of form (4.1) then

$$f_1(r) = exp(c_1 \int dr/g + i(c_1\theta + c_2)) \ . \tag{4.2}$$

EXAMPLE 2.   If $f_2 \in (D,g)$ and $(re \ f_2)_\theta \equiv 0$, then there exist real constants c_1 , c_2 and a branch of   arg z = θ such that

$$f_2(z) = c_1 \int dr/g + i(c_1\theta + c_2) \ ; \tag{4.3}$$

when $h \in (D,g)$ and $(re \ h)_r \equiv 0$ then $h \equiv if_2$, where f_2 is as (4.3) .   The statement follows since $(im \ f_2)_\theta$ is independent of   r   and $(re \ f_2)_r$ is independent of θ , so that there exists real c_1 such that $(im \ f_2)_\theta$ = $g(re \ f_2)_r \ = c_1$.

We consider two relationships between the example functions of (4.2) and (4.3) .   Clearly $f_1 = h \circ f_2$ where   h   is the exponential function.   By Lemma 1g on composition of functions f_1 is g-analytic if and only if f_2 is g-analytic.   Also, the g-derivative of f_2 is given by $(f_2)'_g = (c_1/g)e^{-i\theta}$ which is a function of the form of   (4.1)   and hence is g_1-analytic for $g_1 = g/g'$.   Now, in general, if we require the function of form (4.1) to be in (D,g_1) we note that f'_g = $\lambda' \ exp(i((c_1-1)\theta + c_2))$ and $g_1 = c_1\lambda/\lambda'$.   It follows by mathematical induction that the   kth   derivative exists under appropriate conditions. (See Theorem 3b.)

Theorems 2 and 3 discuss the existence of higher order derivatives.

THEOREM 2.   Let $f \in (D,\lambda)$ and $z \in D$, then   f'   is g-analytic at   z   for real positive function $g = \delta$ only if at least one of the following conditions holds:

a)   There is a real number   k   such that $\lambda(r) = r+k$; for this case $\delta = r+k$.

b)   There exists real valued differentiable $h(\theta)$ such that $(im \ f)_r$ = $h(\theta)(re \ f)_r$.

c)   Either $(re \ f)_\theta$ or $(re \ f)_r$ is zero.

PROOF.   Let $f = u+iv$; then $re \ f'_\lambda = cos\theta u_r + sin\theta v_r$; $im \ f'_\lambda = cos\theta v_r - sin\theta u_r$. Requiring that   (2.4)   hold for some real $\lambda(r)$ gives the matrix equation

$$\begin{bmatrix} cos\theta & sin\theta \\ \\ sin\theta & -cos\theta \end{bmatrix} \begin{bmatrix} u_r & u_{rr} \\ \\ v_r & v_{rr} \end{bmatrix} \begin{bmatrix} \lambda'-1 \\ \\ \lambda-\delta \end{bmatrix} = \begin{bmatrix} \\ 0 \\ \\ \end{bmatrix} \ . \tag{4.4}$$

If the third matrix is the zero matrix then we have   a) .   When u_r or v_r is zero, we have   c) .   (See Example 2, (4.3).)   Otherwise, the requirement that $u_r v_{rr} - v_r u_{rr} \equiv$ 0   implies the relation of   b) .

THEOREM 3.

a)   If $f \in (D,g)$ for $g(r) = r+k$, then $f_{r+k} \in (D,g)$.   If $f \in C^\infty(D)$, all orders of higher derivatives exist in $(D,r+k)$ and are defined by

$$f_{r+k} = e^{-in\theta} \frac{\partial^n f}{\partial r^n} \ .$$

b)   If $f \in (D,g_1)$ is of the form $f = \lambda exp(i(c_1\theta + c_2)$ where g_1 = $c_1\lambda/\lambda'$, then the   nth   derivative exists for

$$n \leq \begin{cases} min\{c_1 , max\{d \in integers \ | \lambda^{(d)} \ exists\}\} , \ c_1 > 0 , \\ \\ max\{d \in integers \ | \ \lambda^{(d)} \ exists\} , \ c_1 < 0 . \end{cases}$$

In particular, where $g_n = (c_1 - n + 1)\lambda^{(n-1)}/\lambda^{(n)}$, then

$$((f'_{g_1})'_{g_2} \ldots)'_{g_n} = \lambda^{(n)} \exp(i((c_1 - n)\theta + c_2)) .$$

(When $c_1 < 0$ and $\lambda \in C^\infty$ all orders of higher derivatives exist.)

c) If nonconstant $f \in (D,\lambda)$ with $(\text{re } f)_\theta = 0$ or $(\text{re } f)_r = 0$ then the first derivative exists and reduces to case b) . (See the discussion following Example 2.)

PROOF. a) and b) can be proved using mathematical induction.

If f is in an algebra (D,g) , then both re f and im f satisfy the same second order partial differential equation which is the Laplace equation for analytic f . In a simply connected domain construction of conjugate g-analytic functions follows.

THEOREM 4. If $f \in (D,g)$, re f and im f each satisfy

$$g\phi_{rr} + g'\phi_r + (1/g)\phi_{\theta\theta} = 0 . \tag{4.5}$$

DEFINITION 5. Real valued $\phi(r,\theta) \in C^2$ satisfying (4.5) is called g-harmonic. Real functions $u(r,\theta)$ and $v(r,\theta)$ which satisfy (2.4) are called g-conjugates.

THEOREM 5. If ϕ is g-harmonic in D , then ϕ has at least one g-conjugate.

PROOF. Without loss of generality, let u satisfy (4.5) and consider

$$v(r,\theta) = \int_{r_0}^r -u_\theta(r,\theta_0)dr/g + \int_{\theta_0}^\theta g\, u_r(r,\theta)d\theta$$

where z_0 is a fixed point $z_0 = r_0 \exp(i\theta_0)$. It is immediate that $v_\theta(r,\theta) = gu_r(r,\theta)$ and we also compute

$$v_r(r,\theta) = -u_\theta(r,\theta_0)/g + \int_{\theta_0}^\theta (\lambda u_{rr}(r,\theta) + \lambda' u_r(r,\theta))d\theta$$

which after substitution of (4.5) reduces to $v_r = -u_\theta/g$, satisfying (2.4) for $f = u + iv$.

Examination of Definition 3 of the g-integral as an Riemann path integral involving the function g shows that the usual operational properties regarding multiples and path components continue to hold. We derive a few useful properties of the generalized line integral; in particular these are analogues of Cauchy's Theorem and path independence, Morera's Theorem, and some convergence theorems of the Weierstrass type. Domains are simply connected.

THEOREM 6. If $f \in (D,g)$ and γ is a simple closed curve homologous to 0 in D then $\overset{g}{\int} f = 0$.

PROOF. Using the polar form of Green's Theorem, re $\overset{g}{\int} f$ and im $\overset{g}{\int} f$ each reduce to zero after substitution of the generalized Cauchy Riemann conditions (2.4) .

The stronger form of Cauchy's Theorem, allowing exceptional points in the domain, is developed in [16], p. 62-66.

COROLLARY 1. When $f \in (D,g)$, then $\overset{g}{\underset{a}{\int}}^b f$ is independent of the path joining a and b in D .

PROOF. If two paths form a to b in D do not cross or cross a finite number of times, the result follows by Theorem 6 and the definitions and operational properties of the path integrals. If the paths cross infinitely often a limiting process is available. (See [13], p. 315).

THEOREM 7. Let $f \in (D,g)$ and z_0 be fixed in D , and define

$$F = \int_{z_0}^{g \, z} f \; ; \text{ then } F \in (D,g) \text{ and further}$$

$$F_g' = ((\exp \int_1^r dr/g)/g) \, f \; . \tag{4.6}$$

PROOF. Since the generalized integral is path independent we apply properties of line integrals in the plane to compute the partial derivatives of the real and the imaginary parts of the integral function F . System (2.4) is satisfied, so the derivative function is given by (2.3) as

$F_g' = e^{-i\theta} (\int_{z_0}^{g \, z} f)_r$ which is equivalent to (4.6) .

We thus demonstrate for the generalized case the differentiability of the integral. The analytic case $g(r) \equiv r$ is seen by (4.6) as necessary (as well as sufficient) that $F_g' \equiv f$.

THEOREM 8. Let $f \in C^1$ in D and g satisfy Definition 1. If $\int_\gamma^g f = 0$ for every simple closed curve γ in D , then f is g-analytic in D .

PROOF. In (2.5) both the real and imaginary parts of the generalized integral are represented by real line integrals $\int_\gamma (P \, dr + Q \, d\theta)$ where P and Q are both in C^1 . Since by the hypothesis we have independence of path for these integrals, $P_\theta = Q_r$, or:

$$(1/g)(u_\theta \cos\theta - v_\theta \sin\theta) = -v_r \cos\theta - u_r \sin\theta \; ,$$

$$(1/g)(v_\theta \cos\theta + u_\theta \sin\theta) = u_r \cos\theta - v_r \sin\theta \; .$$

Algebraic manipulations of this system show (2.4) for $f = u+iv$.

Before showing some Weirstrass-type convergence results, we need a bounding relation on the generalize integral which depends on inequalities for real line integrals.

LEMMA 2. When γ is rectifiable

$$| \int_\gamma^g f| \le K_g ML \tag{4.7}$$

where $K_g = 4 \max_{z \in \gamma}((\exp \int_1^r dr/g)(1 + \max_{z \in \gamma}|1/g|))$, $M = \max_{z \in \gamma}|f|$, L = length of γ .

THEOREM 9. If f_n is continuous on regular arc γ for each n , and the sequence $\{f_n\}$ converges uniformly to f on γ then

$$\lim_{n \to \infty} \int_\gamma^g f_n = \int_\gamma^g f \; .$$

PROOF. Since f is continuous there is an N such that for $n > N$

$$| \int_\gamma^g f_n - \int_\gamma^g f| = | \int_\gamma^g (f_n - f)| \le \epsilon' K_g L < \epsilon$$

by Lemma 2.

THEOREM 10. If for all n , $f_n \in C^2$ is g-analytic on open set A and $\{f_n\}$ converges uniformly to f on compact subsets of A , then f is g-analytic on A .

PROOF. For $z \in A$ there is a disk containing z with its closure in A . We

show $f \in (D,g)$ on such disks, hence f is g-analytic for $z \in A$. Let γ be any

simple closed curve in a disk contained in A . Then $\int^g f_n = 0$ for each n by

Theorem 6. By Theorem 9, $\int^g f = 0$ hence Theorem 8 gives g-analyticity in the disk.

COROLLARY 2. If to the hypothesis of Theorem 10 is added, for $f_n = u_n + iv_n$

that $\{u_n\}$ converges uniformly to u and $\{v_n\}$ converges uniformly to v where

$f = u + iv$, then $f'_g = \lim\limits_{n \to \infty}(f_n)'_g$.

PROOF. We now have $\lim\limits_{n \to \infty}(u_n)_r$ and $\lim\limits_{n \to \infty}(v_n)_r = v_r$ hence there is an N such

that for $n > N$

$$\left|(f_n)'_g - f'_g\right| = \left|e^{-i\theta}(((u_n)_r + i)v_n)_r) - (u_r + iv_r))\right| < \epsilon .$$

A version of the argument principle and of Cauchy's Integral Formula are

available from the theory of Beltrami's equation. For completeness we show the

latter for g-analytic functions.

THEOREM 11. If $f \in (D,g)$ and γ is a simple closed curve homologous to 0

in D , and z_0 is in the interior of γ , then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z - z_0} + \frac{1}{2\pi} \iint_{Int\gamma} \frac{(g-r)\, e^{i2\theta}\, f'_g(z)drd\theta}{z - z_0} .$$

See [5], p. 24-25 and p. 91 for discussion of more general cases.

5. APPLICATIONS

One interesting example of a g-analytic function with domain $\mathbb{C}-\{(0,0)\}$ and

locally quasiconformal with continuous extension at the isolated boundary point

$(0,0)$, and K-quasiconformal on bounded sets, maps the finite plane onto a fixed open

disc.

EXAMPLE 3. For fixed ρ , real and such that $0 < \rho < \infty$, define

$$f(z) = \begin{cases} (2\rho e^{i\theta}\tan^{-1}r)/\pi , & z \neq 0 , \\ 0 , & z = 0 . \end{cases}$$

The function satisfies (2.2) for $g = (1+r^2)\tan^{-1}r$ and maps the finite plane onto

the open disc $|f(z)| < \rho$.

Concentric annuli may be g-conformally mapped whatever the proportions of their

radii.

EXAMPLE 4. For annuli: $A_1 = 0 < r_1 < |z| < r_2 < \infty$,

$$A_2 = 0 < R_1 < |z| < R_2 < \infty ,$$

let

$$f(z) = e^{i\theta}(R_2(r-r_1)-R_1(r-r_2))/(r_2-r_1) .$$

Then $f: A_1 \longrightarrow A_2$ homeomorphically and f is g-analytic for

$g = r + ((R_1 r_2 - R_2 r_1)/(R_2 - R_1))$.

The convergence properties of the g-integral allow a comparison to the general

quasiconformal situation. When the generalized functions satisfying the Beltrami

equation (2.2) are quasiconformal in a domain (see Section 3), they are of the type

called regular quasiconformal. Although the uniform convergence of a sequence of K-

quasiconformal mappings to a homeomorphism implies that the limit mapping is also K-quasiconformal, in the case of a sequence of regular K-quasiconformal mappings the limit mapping need not be regular. (See [8], p. 17-19.) For g-analytic regular K-quasiconformal functions, the limit mapping is g-analytic.

COROLLARY 3.   A Beltrami algebra (D,g) is uniformly closed.

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