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ON CONVERGENCE OF (μ,ν)-SEQUENCES OF UNISOLVENT RATIONAL APPROXIMANTS TO MEROMORPHIC FUNCTIONS IN C^n

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ABSTRACT. A generalization of the classical Montessus de Ballore theorem in $\mathbb{C}^n$ and some related theorems are discussed. Non-Montessus type of convergence in $\mathbb{R}^{2n}$ dimensional Lebesgue measure is presented linking it to a result of Gonĉar.

KEY WORDS AND PHRASES. *Unisolvent rational approximants, meromorphic functions, Montessus-type convergence, non-Montessus type convergence.*
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0. INTRODUCTION.

Let f be in the class of meromorphic functions analytic at the origin, having fixed number of codimension 1 polar sets over some polydisk domain in $\mathbb{C}^n$. We investigate the modes of convergence of (μ,ν)-sequences of unisolvent rational approximants (URA) to f and we divide them into two main types; those associated with "horizontal rows" of (μ,ν)-sequences of URA (for which ν is fixed and μ is free to grow infinitely) and those linked to the "slanted rows" (for which μ and ν are related but both grow to infinity) including "diagonal rows" (cases where $\mu = \nu$). As will become clear from definition 2.1, one requirement $\mu_i \geq \nu_i$, $i = 1,2,\ldots,n$ creates a "Padé Table" of upper triangular (μ,ν)-sequences. Some study of "horizontal row" of (μ,ν)-sequences have been carried out by Karlsson and Wallin [3] using explanations expressed in terms of homogeneous polynomials in $\mathbb{C}^2$. Our investigation of the convergence behavior of the "horizontal rows" of (μ,ν) sequences constructed from non-homogeneous polynomials, show that the convergence to f is uniform on compact subsets of the domain of meromorphy except on the analytic set $\{z \in \mathbb{C}^n: 1/f = 0\}$ or on the remaining limit polar sets of (μ,ν)-sequences unattracted by f. These limit polar sets have measure zero. We refer to this type of convergence for (μ,ν)-sequences of URA as the Montessus type. The case of the "slanted rows" of (μ,ν) sequences in $\mathbb{C}^n$ has been studied by Gonĉar [2] using diagonal sequences constructed from homogeneous polynomials. Although in one variable, it is known that for certain limited classes of functions there can be locally uniform convergence for "almost diagonal rows" of (μ,ν) sequences, it is generally recognized that most meaningful ways to handle convergence of "slanted rows" of (μ,ν)-sequences is in measure or capacity. Here the measure refers to $\mathbb{C}^n$-Lebesgue measure and capacity refers to any reasonably defined capacity in $\mathbb{C}^n$ adaptable to this kind of problem. We call this type of convergence in measure (capacity) for "horizontal rows" or "slanted rows" of (μ,ν)-sequences, the non-Montessus type.

Either type of convergence for (μ,ν) sequences gives rise to a certain rapidity and over-convergence (see Walsh [9]). For the Montessus-type over-convergence means the domain of convergence of the (μ,ν)-sequences of URA, includes in its interior the domain of convergence of the local power series representative of f at the origin in $\mathbb{C}^n$. For the non-Montessus-type, according to Gonĉar [2] over-convergence in measure (capacity) means that convergence in measure (capacity) in any finite domain implies convergence in measure (capacity) in $\mathbb{C}^n$ for $f \in \mathcal{M}$.

The paper is organized to reflect the Montessus-type convergence in §3 and the non-Montessus-type in §4. In §1 we introduce the notations used in the paper and in §2 we introduce the definition of the URA's and discuss the sense in which they are unique.

The main theorems of the paper are theorems 3.1, 3.3, 3.4 and 4.1 (which examine the different cases associated with the horizontal rows) and theorem 4.2 (which examines a case of the slanted rows). There is also theorem 3.6 which is an application of theorem 3.1. The theorem yields a result about global analytic behaviour of f if all the horizontal rows of (μ,ν)-sequences are constrained in a certain way. Some extension of this idea is discussed in Lutterodt [5]. Theorem 3.5 provides some insight into the way in which $\pi_{\mu\nu}(z)$'s over-converge in the case of Montessus-type and theorem 4.4, that of the non-Montessus type.

1. NOTATION.

Let $z := (z_1,\dots,z_n)$ be an n-tuple in $\mathbb{C}^n$ and $\hat{z} := (z_1,\dots,z_{n-1}) \in \mathbb{C}^{n-1}$; $\mu := (\mu_1,\dots,\mu_n)$ and $\nu := (\nu_1,\dots,\nu_n)$ be n-tuples in $\mathbb{N}^n$ with $\hat{\nu} \in \mathbb{N}^{n-1}$. Let $\sigma > 0$ and $\Delta_\sigma := \{z_j \in \mathbb{C}: |z_j| < \sigma\}$, $1 \leqslant j \leqslant n$, then $\Delta_\sigma^n = \Delta_\sigma \times\dots\times \Delta_\sigma$ n-times, is a polydisk centered at origin. We denote by E_τ the following subset of $\mathbb{N}^n$, $E_\tau := \{\lambda \in \mathbb{N}^n: 0 \preccurlyeq \lambda \preccurlyeq \tau\}$, $\tau \in \mathbb{N}^n$ where '$\preccurlyeq$' is a partial ordering in $\mathbb{N}^n$ given by

$$0 \preccurlyeq \lambda \preccurlyeq \tau \iff 0 \leqslant \lambda_j \leqslant \tau_j,\ 1 \leqslant j \leqslant n.$$

We adopt the following short notation as well

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda} \equiv \frac{\partial^{\lambda_1+\dots+\lambda_n}}{\partial z_1^{\lambda_1}\dots\partial z_n^{\lambda_n}},\quad dz = dz_1\dots dz_n$$

$$\sum_{\alpha\in E_\tau} \equiv \sum_{\alpha_1,\dots,\alpha_n=0}^{\tau_1,\dots,\tau_n},\quad \sum_{\alpha\in\mathbb{N}^n} \equiv \sum_{\alpha_1,\dots,\alpha_n=0}^{\infty}.$$

Let Ω be a domain in $\mathbb{C}^n$ such that $0 \in \Omega$, then $\mathcal{H}(\Omega)$ is the ring of all functions holomorphic in Ω and continuous on $\bar{\Omega}$; $\mathcal{M}(\Omega)$ is the set of all functions holomorphic at the origin, but meromorphic in Ω with finite pole sets in Ω. A polynomial $P_\lambda(z)$ in $\mathbb{C}^n$ of *multiple degree* or simply '*degree*' at most $\lambda = (\lambda_1,\dots,\lambda_n)$ can be written as

$$P_\lambda(z) = \sum_{\gamma\in E} g_\gamma z^\gamma$$

where $g_\gamma \equiv g_{\gamma_1\dots\gamma_n}$ and $z^\gamma = z_1^{\gamma_1}\dots z_n^{\gamma_n}$.

Let $\mathcal{R}_{\mu\nu}$ be the class of rational functions of the form $R_{\mu\nu}(z) = P_\mu(z)/Q_\nu(z)$ where $Q_\nu(0) \neq 0$ and $P_\mu(z)$, $Q_\nu(z)$ are polynomials of 'degree' at most μ and ν respectively; moreover $(P_\mu(z), Q_\nu(z)) = 1$ on Δ^n_ρ, except on a subvariety of codimension $\geqslant 2$ for some $\rho > 0$ and for all (μ,ν).

2. RATIONAL APPROXIMANTS.

Let U be an open neighborhood of the origin in $\mathbb{C}^n$.

DEFINITION 2.1: Suppose $f \in \mathcal{H}(U)$ with $f(0) \neq 0$, a rational function $R_{\mu\nu}(z) \in \mathcal{R}_{\mu\nu}$ is said to be a *rational approximant* to f at z = 0 if

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda}(Q_\nu(z)f(x) - P_\mu(z))\Big|_{z=0} = 0 \tag{2.1}$$

for $\lambda \in E^{\mu\nu} \subset \mathbb{N}^n$, an index interpolation set with the following properties:

(i) $0 \in E^{\mu\nu}$

(ii) $\lambda \in E^{\mu\nu} \Rightarrow \gamma \in E^{\mu\nu}$, $0 \preccurlyeq \gamma \preccurlyeq \lambda$

(iii) $E_\mu \subset E^{\mu\nu}$

(iv) Each projected variable has the Padé indexing set.

(v) For each pair (μ,ν), $\nu \preccurlyeq \mu$

(vi) $|E^{\mu\nu}| \leqslant \prod_{j=1}^n (\mu_j + 1) + \prod_{j=1}^n (\nu_j + 1) - 1$ where $|E^{\mu\nu}|$ is the cardinality of $E^{\mu\nu}$.

REMARK 1. Padé index set is the index set that is used in the one variable case to define Padé Approximants.

REMARK 2. If the function f used in definition 2.1 is such that f(0) = 0, then depending on the order of regularity and in which variable, z_j say, Weierstrass Preparation Theorem can be employed to write $f = W \cdot g$ where W is a Weierstrass polynomial $W(\hat{z},z_j)$ with $W(0) = 0$ and g is a unit with $g(0) \neq 0$. Definition 2.1 may then apply to g in $f = W \cdot g$.

The rational approximants defined above are not in general unique. The question of uniqueness is firmly tied to the cardinality of $E^{\mu\nu}$ employed. Uniqueness as known in Padé case seems only possible if $E^{\mu\nu}$ is maximal in the following sense.

DEFINITION 2.2. The interpolation set $E^{\mu\nu}$ is said to be *maximal* if $|E^{\mu\nu}| \geqslant \prod_{j=1}^n (\mu_j + 1) + \prod_{j=1}^n (\nu_j + 1) - 1$.

It should be pointed out that there are many maximal $E^{\mu\nu}$ that can be chosen for defining unique approximants. This is a feature peculiar to several variables, unknown in the Padé case in one variable. We note that when $E^{\mu\nu}$ is maximal, there is a one-to-one correspondence between $E^{\mu\nu}$ and its "Padé table" equivalent.

PROPOSITION 2.1. Let the pair $\langle P_\mu(z), Q_\nu(z)\rangle$ define a (μ,ν)-rational approximant to $f \in \mathcal{H}(U)$ w.r.t. maximal $E^{\mu\nu}$. Suppose $\langle P^*_\mu(z), Q^*_\nu(z)$ is another pair that satisfies definition 2.1 w.r.t. the same $E^{\mu\nu}$. Then

$$\frac{P_\mu(z)}{Q_\nu(z)} \equiv \frac{P^*_\mu(z)}{Q^*_\nu(z)} \text{ w.r.t. } E^{\mu\nu}.$$

PROOF: Since $f \in \mathcal{H}(U)$ and P_μ, Q_ν are polynomials $Q_\nu f - P_\mu \in \mathcal{H}(U)$ and has a Taylor development in U. Now by definition 2.1 and with $E^{\mu\nu}$ maximal we get

$$Q_\nu(z)f(z) - P_\mu(z) = \sum_{\lambda\in\mathbb{N}^n\setminus E^{\mu\nu}} g_{\mu\nu\lambda} z^\lambda \tag{2.2}$$

where $g_{\mu\nu\lambda}$ are the coefficients of the expansion. Similarly from the pair $\langle P^*_\mu, Q^*_\nu\rangle$ we again have

$$Q^*_\nu(z)f(z) - P^*_\mu(z) = \sum_{\lambda\in\mathbb{N}^n\setminus E^{\mu\nu}} g^*_{\mu\nu\lambda} z^\lambda . \tag{2.3}$$

Multiplying (2.2) by $Q^*_\nu(z)$ and (2.3) by $Q_\nu(z)$ and subtracting the former from the latter we get

$$Q_\nu(z)P^*_\mu(z) - Q^*_\nu(z)P_\mu(z) = \sum_{\lambda\in\mathbb{N}^n\setminus E^{\mu\nu}} r_{\mu\nu\lambda} z^\lambda \tag{2.4}$$

where $r_{\mu\nu\lambda}$ is the coefficient of the expansion after taking the desired difference. Now the R.H.S. of (2.4) is a power series regular with at least order $\mu_j + 1$ in each z_j-variable. Therefore the L.H.S. of (2.4) must vanish w.r.t. the maximal $E^{\mu\nu}$ chosen, since the L.H.S. is a polynomial of 'degree' at most $\mu + \nu$. Thus we obtain

$$Q_\nu(z)P^*_\mu(z) \equiv Q^*_\nu(z)P_\mu(z) \quad \text{w.r.t. } E^{\mu\nu}. \tag{2.5}$$

Recalling that maximal $E^{\mu\nu}$ by construction, has maximal Padé indexing in each variable and thus on projecting (2.5) on z_n-axis, the uniqueness of Padé approximants then yields

$$\frac{P_\mu(\underline{0},z_n)}{Q_\nu(\underline{0},z_n)} \equiv \frac{P^*_\mu(\underline{0},z_n)}{Q^*_\nu(\underline{0},z_n)} \tag{2.6}$$

where $Q_\nu(\underline{0},z_n)$, $Q^*_\nu(\underline{0},z_n)$ do not vanish in some neighborhood $U_0 \subset U$ since $Q_\nu(\underline{0},0) \neq 0$ and $Q^*_\nu(\underline{0},0) \neq 0$. (2.6) holds not only for $\hat{z} = \underline{0}$ but also for each fixed $\hat{z} \in \mathbb{C}^{n-1} \cap U$. The desired result then follows w.r.t. $E^{\mu\nu}$.

From equation (2.1), we separate out the following linear equations:

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda}(Q_\nu(z)f(z) - P_\mu(z))\Big|_{z=0} = 0; \ \lambda \in E_\mu \tag{2.1a}$$

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda}(Q_\nu(z)f(z))\Big|_{z=0} = 0; \ \lambda \in E^{\mu\nu}\setminus E_\mu \tag{2.1b}$$

whose solutions for coefficients of $Q_\nu(z)$ and then those of $P_\mu(z)$ give rise to a (μ,ν) rational approximant. Now (μ,ν)-sequences of rational approximants will be called *unisolvent* if the underlying $E^{\mu\nu}$ is maximal and a certain determinatal or rank condition is satisfied for the system of linear equations that arises from (2.1b). (see also Lutterodt [4].).

For the rest of this paper we shall assume that we have (μ,ν)-sequences of unisolvent rational approximants (URA). The latter is denoted by $\pi_{\mu\nu}(z) = P_{\mu\nu}(z)/Q_{\mu\nu}(z)$ with respect to some chosen $E^{\mu\nu}$ maximal. We then normalize $Q_{\mu\nu}(z)$, dividing $P_{\mu\nu}(z)$ and $Q_{\mu\nu}(z)$ in $\pi_{\mu\nu}(z)$ by the modulus of the largest coefficient of $Q_{\mu\nu}(z)$. This operation leaves $\pi_{\mu\nu}(z)$ unchanged but since $P_{\mu\nu}(z)$, $Q_{\mu\nu}(z)$ change we adopt the following denotation of $\pi_{\mu\nu}(z)$ with $\tilde{Q}_{\mu\nu}(z)$ normalized,

$$\pi_{\mu\nu}(z) = \tilde{P}_{\mu\nu}(z)/\tilde{Q}_{\mu\nu}(z) \tag{2.7}$$

3. MONTESSUS-TYPE CONVERGENCE.

This section gives the generalized Montessus de Ballore theorem, some related theorems covering the Montessus-type convergence and an application of Montessus theorem to the generalized Padé Table.

THEOREM 3.1 (MONTESSUS). Let $\rho > 0$ and $\nu = (\nu_1,\ldots,\nu_n)$ be fixed. Suppose $f \in \mathfrak{M}(\Delta^n)$ with finite polar set defined by $G_\nu := \{z \in \mathbb{C}^n: q_\nu(z) = 0\}$ and $q_\nu f \in C(\bar{\Delta}^n_\rho)$ where $q_\nu(z)$ is a polynomial of exact minimal 'degree' ν and $\Delta^n_\rho \cap G_\nu \neq \phi$. Suppose $\pi_{\mu\nu}(z)$ is an (μ,ν)-unisolvent rational approximant to $f(z)$ with its polar set $Q^{-1}_{\mu\nu}(0)$ satisfying for μ sufficiently large, $Q^{-1}_{\mu\nu}(0) \cap \Delta^n_\rho \neq \phi$. Then as $\mu' = \min_{1\leqslant j\leqslant n}(\mu_j) \to \infty$

(i) $\Delta^n_\rho \cap Q^{-1}_{\mu\nu}(0) \to \Delta^n_\rho \cap G_\nu$

(ii) $\pi_{\mu\nu}(z) \to f(z)$ uniformly on compact subsets of $\Delta^n_\rho \backslash G_\nu$.

REMARK: The degree of convergence in (ii) of theorem 3.1 is geometric and it depends on $\mu' = \min_{1\leqslant j\leqslant n}(\mu_j)$. The 'degree' of $\tilde{Q}_{\mu\nu}(z)$ in $\pi_{\mu\nu}(z)$ has to be exactly ν.

The following lemma is used in the proof of the theorem.

LEMMA 3.2. For $0 < \rho' < \rho$ and $\mu' = \min_{1\leqslant j\leqslant n}(\mu_j)$

$$\sum_{\lambda\in\mathbb{N}^n\setminus E_\mu} \left(\frac{\rho'}{\rho}\right)^{|\lambda|} \leqslant \frac{n\left(\frac{\rho'}{\rho}\right)^{\mu'+1}}{\left(1-\frac{\rho'}{\rho}\right)^n}.$$

The proofs of Theorem 3.1 and Lemma 3.2 were given in Lutterodt [6]; that of Lemma 3.2 being in the appendix.

The following two theorems are closely related to theorem 3.1 not only in statement but also in proof. We shall therefore state them together and then briefly indicate where in their proofs they differ from theorem 3.1. (see, [6]).

THEOREM 3.3. Let $\omega = (\omega_1,\ldots,\omega_n)$ and $\rho > 0$ be fixed. Suppose $f \in \mathfrak{M}(\Delta^n_\rho)$) with a finite polar set defined by $G_\omega := \{z \in \mathbb{C}^n: q_\omega(z) = 0\}$ and $q_\omega f \in C(\bar{\Delta}^n_\rho)$ where $q_\omega(z)$ is a polynomial of minimal 'degree' ω in $\mathbb{C}^n$ and $G_\omega \cap \Delta^n_\rho \neq \phi$.

Suppose $\pi_{\mu\nu}(z)$ is a (μ,ν)-unisolvent rational approximant to $f(z)$ with ν fixed but $\nu \lneqq \omega$ and $Q^{-1}_{\mu\nu}(0)$ is the polar set of $\pi_{\mu\nu}(z)$. Then as $\mu' \to \infty$

(i) $\Delta^n_\rho \cap Q^{-1}_{\mu\nu}(0)$ tends to a subset of $\Delta^n_\rho \cap G_\omega$

(ii) $\pi_{\mu\nu}(z) \to f(z)$ uniformly on compact subsets of $\Delta^n_\rho \backslash G_\omega$.

THEOREM 3.4. Suppose the hypothesis of theorem 3.3 is satisfied with $\omega \lneqq \nu$. Then as $\mu' \to \infty$

(i) $\Delta^n_\rho \cap Q^{-1}_{\mu\nu}(0)$ tends to a set containing $\Delta^n_\rho \cap G_\omega$ as a proper subset.

(ii) $\pi_{\mu\nu}(z) \to f(z)$ uniformly on compact subsets of Δ^n_ρ except on $G_\omega \cup Z_q$ of $\mathbb{C}^n$-Lebesgue measure zero where Z_q contains the remaining limiting polar set of $\pi_{\mu\nu}$.

REMARK ON THE PROOFS OF THE THEOREMS 3.3 & 3.4.

The proofs of the two theorems 3.3 and 3.4 as already indicated are essentially the same as that of theorem 3.1 in our paper [6] except for minor changes in the second half of their (i)-parts. For theorem 3.4 we show further that the set $G_\omega \cup Z_q$, the exceptional set has $\mathbb{C}^n$-Lebesgue measure zero.

The proofs, for second half of the (i)-parts for theorems 3.3 and 3.4, which determine the relations between $G_\omega \cap \Delta^n_\rho$ and $Q^{-1}_\nu(0) \cap \Delta^n_\rho = \lim_{\mu'\to\infty} Q^{-1}_{\mu\nu}(0) \cap \Delta^n_\rho$, focus on their corresponding versions of equation (3.8) in our paper [6]. i.e.

$$\tilde{Q}_\nu(z)q_\omega(z)f(z) = q_\omega(z)\tilde{P}(z). \tag{3.1}$$

Dealing with theorem 3.3 first, we find that $a \in \Delta_\rho^n \cap Q_\nu^{-1}(0)$ implies $\tilde{Q}_\nu(a) = 0$ so that R.H.S. of (3.1) must give $q_\omega(a)\tilde{P}(a) = 0$. Since $(\tilde{P}(z), \tilde{Q}_\nu(z)) = 1$ for $z \in \Delta_\rho^n$ (except for some subvariety of codim $\geqslant 2$), we must have that $q_\omega(a) = 0$. Here $\nu \preccurlyeq \omega$ and hence as $\mu' \to \infty$

$$Q_{\mu\nu}^{-1}(C) \cap \Delta_\rho^n \to Q_\nu^{-1}(0) \cap \Delta_\rho^n \subset G_\omega \cap \Delta_\rho^n. \tag{3.2}$$

The validity of equation (3.1) remains in tact on passing from subsequences to the full sequence as discussed in the proof of theorem 3.1. Thus the polar set of $\pi_{\mu\nu}(z)$ in Δ_ρ^n gets completely attracted by the polar set of $f(z)$ with similar multiplicity on Δ_ρ^n.

In the case of theorem 3.4, the equation (3.1) yields the following: Suppose $a \in G_\omega \cap \Delta_\rho^n$, then $q_\omega(a) = 0$. But since $q_\omega(z)f(z) \neq 0$, except on a sub-variety of co-dimension $\geqslant 2$ therefore $\tilde{Q}_\nu(a) = 0$. With $\omega \precneqq \nu$ we must have as $\mu' \to \infty$

$$Q_{\mu\nu}^{-1}(0) \cap \Delta_\rho^n \to Q_\nu^{-1}(0) \cap \Delta_\rho^n \supset G_\omega \cap \Delta_\rho^n. \tag{3.3}$$

This remains true on passing to the full sequence from subsequences as done in the proof of theorem 3.1 of [6]. Next we proceed to the final part of theorem 3.4 which begins with a modified version of the inequality (3.7) from our paper [6]. The inequality in question is

$$|f(z) - \pi_{\mu\nu}(z)| \leqslant \frac{M}{|\tilde{Q}_{\mu\nu}(z)|\ |q_\omega(z)|} \Big(\sum_{\lambda \in \mathbb{N}^n \setminus E_\mu} \frac{|z^\lambda|}{\rho^{|\lambda|}} \Big).$$

Let K be any compact subset of Δ_ρ^n, and choose $\rho' > 0$ such that $0 < \rho' < \rho$ implies $\Delta_{\rho'}^n \subsetneqq \Delta_\rho^n$ and $K \subset \Delta_{\rho'}^n$. Then using Lemma 3.2 we can tighten the above inequality to get $\forall\ z \in K$

$$|f(z) - \pi_{\mu\nu}(z)| \leqslant \frac{C_1(\frac{\rho'}{\rho})^{\mu'+1}}{|\tilde{Q}_{\mu\nu}(z)q_\omega(z)|}\ . \tag{3.4}$$

Now if we let $d = (d_1,\ldots,d_n) = (\nu_1 + \omega_1,\ldots,\nu_n + \omega_n)$ and define $F_d(z) = \tilde{Q}_\nu(z)q_\omega(z)$, then the zero set of $F_d(z)$ in Δ_ρ^n, $Q_\nu^{-1}(0) \cap \Delta^n \supseteq (G_\omega \cup Z_q) \cap \Delta_\rho^n$ from (3.3). We claim that $Q_\nu^{-1}(0) \cap \Delta_\rho^n$ is a set of $\mathbb{R}^{2n}$-Lebesgue measure zero. This consequently will imply that $(G_\omega \cup Z_q) \cap \Delta_\rho^n$ is of $\mathbb{R}^{2n}$-Lebesgue measure zero. To prove the result we first note that $Q_\nu^{-1}(0) \cap \Delta_\rho^n$ must have empty interior. For if not then since Δ_ρ^n is connected, $\tilde{Q}_\nu(z) \equiv 0$ on Δ_ρ^n which we know is not the case. So $\Delta_\rho^n \setminus Q_\nu^{-1}(0)$ must be dense in Δ_ρ^n. Therefore we can choose a countable sequence $\{\zeta_m\}_m \subset \Delta_\rho^n \setminus Q_\nu^{-1}(0)$ with corresponding polydisks $U_m \subset \Delta_\rho^n$, $m = 1,2,\ldots$ centered at ζ_m where $\{U_m\}_m$ forms a covering of Δ_ρ^n and each $U_m \cap Q_\nu^{-1}(0) \neq \phi$, $m = 1,2,\ldots$ Invoking Jensen's inequality we have

$$\int_{U_m} \log|F_d(z)|d\mu > -\infty \tag{3.5}$$

where $d\mu$ is a $\mathbb{R}^{2n}$-Lebesgue volume measure for U_m. The inequality (3.5) tells us that the set on which F_d vanishes in U_m cannot have a positive measure, i.e. it must have measure zero. Since there is a countable number of U_m that cover Δ_ρ^n, the claim is proved and hence the desired result.

Now for $z \in K\setminus(G_\omega \cup Z_q)$ and μ' sufficiently large we can find $\delta > 0$ such that $|\tilde{Q}_{\mu\nu}(z)| > \delta$ and $|q_\omega(z)| > \delta$. Thus on passing to sup-norm in (3.3) we can find $C_2 = C_2(\delta,C_1) > 0$ such that

$$||f(z) - \pi_{\mu\nu}(z)|| \leq C_2(\frac{\rho'}{\rho})^{\mu'+1}, \; 0 < \frac{\rho'}{\rho} < 1. \tag{3.6}$$

From (3.6) the desired result on degree of convergence follows, showing the uniform convergence.

The next theorem compares the rate of convergence of (μ,ν) sequences of URA $\pi_{\mu\nu}(z)$ to $f \in \mathfrak{M}(\Lambda_\rho^n)$ where ν is fixed, with the rate of convergence of the Taylor polynomials $\pi_{\mu 0}(z)$ to f. The following definition shows that 'degree' of $\pi_{\mu\nu}$ is the same as that of $\pi_{\mu 0}$.

DEFINITION 3.1. A rational runction $R_{\mu\nu} \in \mathcal{R}_{\mu\nu}$ is said to have 'degree' $\mu^* = (\mu_1^*,\ldots,\mu_n^*)$ if in each z_j-variable, $R_{\mu\nu}(z)$ expressed as a quotient of two pseudo-polynomials in z_j has degree given by $\mu_j^* = \max(\mu_j,\nu_j)$.

COROLLARY 3.1a. The 'degree' of (μ,ν) - URA in this paper is always μ. This follows from property (v) of $E^{\mu\nu}$ in definition 2.1 where each pair satisfies $\nu \lneq \mu$, making the "Padé Table" upper-triangular.

THEOREM 3.5. Suppose the hypothesis of theorem 3.1 is satisfied. Let K be any compact subset of $\Lambda_\rho^n \backslash G_\nu$ then in terms of sup-norm on K, for sufficiently large μ' we get

$$||f(z) - \pi_{\mu\nu}(z)||_K \leq ||f(z) - \pi_{\mu 0}(z)||_K$$

where $\pi_{\mu 0}(z)$ is the Taylor polynomial of 'degree' μ to f at the origin.

PROOF. 1°. Let $0 < r < \rho$ and let $U = \Delta_r^n$ (so that $\Delta_r^n \subset \Delta_\rho^n$) be a neighborhood of the origin such that $f \in \mathcal{H}(U)$ and let the $\pi_{\mu 0}(z)$ be the partial sum of f(z) of 'degree' μ in U where $\bar{U} \subset V$, $V \cap G_\nu = \phi$; V being open and connected in Δ_ρ^n and $\bar{U}$ being the closure of U. Let

$$F_{\mu\nu}(z) = \tilde{Q}_{\mu\nu}(z)\pi_{\mu 0}(z) - \tilde{P}_{\mu\nu}(z).$$

Then $F_{\mu\nu}(z) \in C(\bar{U})$ and $F_{\mu\nu}(z) \in \mathcal{H}(U)$ and therefore by Cauchy's Integral formula

$$F_{\mu\nu}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 U} \frac{F_{\mu\nu}(t)}{\prod_{j=1}^{n}(t_j - z_j)} dt_1 \ldots dt_n$$

where $\partial_0 U$ is the distinguished boundary of U. Using arguments similar to those employed in the proof of theorem 3.1 of our paper [6] we can write

$$F_{\mu\nu}(z) = \sum_{\lambda\in E_{\mu+\nu}} z^\lambda f_{\mu\nu\lambda}$$

where

$$f_{\mu\nu\lambda} = \frac{1}{(2\pi i)^n} \int_{\partial_0 U} \frac{F_{\mu\nu}(t)}{t^{\lambda+1}} dt_1 \ldots dt_n$$

we claim that

$$F_{\mu\nu}(z) = \sum_{\lambda\in E_{\mu+\nu}\backslash E_\mu} z^\lambda f_{\mu\nu\lambda} \tag{3.6a}$$

where

$$f_{\mu\nu\lambda} = \frac{1}{(2\pi i)^n} \int_{\partial_0 U} \frac{\tilde{Q}_{\mu\nu}(t)\, \pi_{\mu 0}(t)}{t^{\lambda+1}} dt_1 \ldots dt_n. \tag{3.6b}$$

In order to substantiate this claim, we write $f(z) = \pi_{\mu 0}(z) + g_\mu(z)$, where $\pi_{\mu 0}$ is the partial sum of f(z) and $g_\mu(z)$ is the remainder of the power series expansion of f in U. This makes the function $g_\mu(z)$ analytic but regular in each variable z_j with order

$\mu_j + 1$. Now observe from equations (2.1a) and (2.1b) that

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda} [\tilde{Q}_{\mu\nu}(z)\ \pi_{\mu 0}(z) - \tilde{P}_{\mu\nu}(z)]\Big|_{z=0} = - \frac{\partial^{|\lambda|}}{\partial z^\lambda} (\tilde{Q}_{\mu\nu}(z) g_\mu(z))\Big|_{z=0};\ \lambda \in E_\mu \tag{3.7a}$$

and

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda} (\tilde{Q}_{\mu\nu}(z)\ \pi_{\mu 0}(z))\Big|_{z=0} = - \frac{\partial^{|\lambda|}}{\partial z^\lambda} (\tilde{Q}_{\mu\nu}(z) g_\mu(z))\Big|_{z=0};\ \lambda \in E_{\mu+\nu}\backslash E_\mu . \tag{3.7b}$$

However, the regularity conditions on $g_\mu(z)$ in each variable imply that $\frac{\partial^{|\lambda|}}{\partial z^\lambda} g_\mu(z)\Big|_{z=0} = 0$, $\forall\ \lambda \in E_\mu$ and hence invoking Leibnitz rule in the R.H.S. of (3.7a) we get the R.H.S. to vanish, yielding the claim.

We remark that even though for $\lambda \in E_{\mu+\nu}\backslash E_\mu$ equation (3.7b) provides an alternative formula for (3.6b): there is no real advantage gained in adopting the second form for $\lambda \in E_{\mu+\nu}\backslash E_\mu$.

Recall that $\tilde{Q}_{\mu\nu}(t)$ is a normalized polynomial of fixed multiple 'degree' ν, thus it is uniformly bounded on U. We then let $M_Q = \max_{t\in\partial_0 U} |\tilde{Q}_{\mu\nu}(t)|$, independent of μ.

$\pi_{\mu 0}(t)$, being the partial sum of an absolute convergent development of f on $\partial_0 U$, must satisfy $|\pi_{\mu 0}(t)| \leqslant M$, $\forall\ t \in \partial_0 U$ since $V \cap G_\nu = \phi$ and $\bar{U} \subset V$. Here M is independent μ. Thus from (3.6a) we get using Cauchy inequality on $f_{\mu\nu\lambda}$ that

$$|F_{\mu\nu}(z)| \leqslant MM_Q \sum_{\lambda\in E_{\mu+\nu}\backslash E_\mu} \frac{|z^\lambda|}{r^{|\lambda|}}$$

Thus

$$|F_{\mu\nu}(z)| \leqslant MM_Q \sum_{\lambda\in \mathbb{N}^n\backslash E_\mu} \frac{|z^\lambda|}{r^{|\lambda|}}. \tag{3.8}$$

The R.H.S. of (3.8) is the tail of a geometric series in $\mathbb{R}^n$ and clearly as $\mu' \to \infty$, R.H.S. of (3.8) tends to zero. Thus given $\varepsilon_1 > 0$, $\exists\ N_{\varepsilon_1}$ such that $\mu' > N_{\varepsilon_1} > |F_{\mu\nu}(z)| < \varepsilon_1$ for $z \in U$. This result holds everywhere in the complete Reinhardt domain of convergence of f(z) denoted by U_R where $V \subset U_R \subset \Delta^n_\rho$ and $U_R \cap G_\nu = \phi$. Thus on any compact subset $K_0 \subset U_R$, we must have for $\mu' > N_{\varepsilon_1}$

$$||\tilde{Q}_{\mu\nu}(z)\pi_{\mu 0}(z) - \tilde{P}_{\mu\nu}(z)||_{K_0} < \varepsilon_1 .$$

2°. Now since $K_0 \subset U_R$ and $U_R \cap G_\nu = \phi$, $\exists\ \delta_0 > 0$ such that $|q_\nu(z)| > \delta_0$ for all $z \in K_0$. By Theorem 3.1 $Q^{-1}_{\mu\nu}(0) \cap \Delta^n_\rho \to G_\nu \cap \Delta^n_\rho$ as $\mu' \to \infty$. Thus for μ' sufficiently large, we get by way of Hurwitz theorem in $\mathbb{C}^n$ that $|\tilde{Q}_{\mu\nu}(z)| > \delta_0$ for all $z \in K_0$ and furthermore

$$||\pi_{\mu 0}(z) - \pi_{\mu\nu}(z)||_{K_0} < \frac{\varepsilon_1}{\delta_0} = \varepsilon .$$

Using triangular inequality for sup-norms on K_0, we get

$$||f(z) - \pi_{\mu\nu}(z)||_{K_0} \leqslant ||f(z) - \pi_{\mu 0}(z)||_{K_0} + ||\pi_{\mu 0}(z) - \pi_{\mu\nu}(z)||_{K_0}$$

and for $\mu' > N_{\varepsilon_1}$, we obtain

$$||f(z) - \pi_{\mu\nu}(z)||_{K_0} \leqslant ||f(z) - \pi_{\mu 0}(z)||_{K_0} + \varepsilon$$

since $\varepsilon > 0$ is arbitrary, we get on K_0

$$||f(z) - \pi_{\mu\nu}(z)||_{K_0} \leqslant ||f(z) - \pi_{\mu 0}(z)||_{K_0} . \tag{3.9}$$

This inequality is not violated if K_0 is extended to any compact subset K where $K_0 \subset \bar{U}_R \subset K \subset \Delta^n_\rho \setminus G_\nu$. Theorem 3.1 guarantees the L.H.S. of (3.9) to remain as small as possible on $K \subset \Delta^n_\rho \setminus G_\nu$ whereas R.H.S. of (3.9) cannot be made arbitrarily small in $\Delta^n_\rho \setminus \bar{U}_R$ for sufficiently large μ'. Thus $\mu' > N_\varepsilon$ implies

$$||f(z) - \pi_{\mu\nu}(z)||_K \leq ||f(z) - \pi_{\mu 0}(z)||_K.$$

THEOREM 3.6. Suppose f(z) is analytic at the origin and is at most meromorphic with a finite polar set in $\mathbb{C}^n$. Suppose for each fixed $\nu = (\nu_1,\ldots,\nu_n)$, the polar set of each (μ,ν) unisolvent rational approximant $\pi_{\mu\nu}(z)$ to f(z) tends to infinity as $\mu' \to \infty$. Then f(z) must be entire in $\mathbb{C}^n$.

We need the following lemma in order to prove the above theorem.

LEMMA 3.7. Let $\nu = (\nu_1,\ldots,\nu_n)$ be fixed. The polar set $Q^{-1}_{\mu\nu}(0)$ of $\pi_{\mu\nu}(z)$ tends to infinity as $\mu' \to \infty$ if and only if given any $\rho > 0$ and a polydisk Δ^n_ρ

$$Q^{-1}_{\mu\nu}(0) \cap \Delta^n_\rho = \phi \tag{3.10}$$

for μ' sufficiently large.

PROOF. For fixed ν, suppose the polar set $Q^{-1}_{\mu\nu}(0)$ tends to infinity as $\mu' \to \infty$. Then it is immediate that for any given $\rho > 1$ and a polydisk Δ^n_ρ, $\exists\, N_\rho$ such that $\mu' > N_\rho \Rightarrow \Delta^n_\rho \cap Q^{-1}_{\mu\nu}(0) = \phi$.

To prove the converse, we assume its opposite, i.e. we can choose $\rho_0 > 1$ for which $\mu' > N_{\rho_0}$ we have

$$\Delta^n_{\rho_0} \cap Q^{-1}_{\mu\nu}(0) \neq \phi.$$

Now on $\Delta^n_{\rho_0}$, $\{Q_{\mu\nu}(z)\}_\mu$ is a normalized sequence of polynomials with fixed 'degree' ν and therefore we can choose a subsequence $\{\tilde{Q}_{\mu_\kappa\nu}(z)\}_\kappa$ for which $\tilde{Q}_{\mu_\kappa\nu}(z) \to \tilde{Q}_\nu(z)$ as $\kappa \to \infty$ uniformly on a compact subset $K_1 \subset \Delta^n_{\rho_0}$ so that

$$\Delta^n_{\rho_0} \cap Q^{-1}_{\mu_\kappa\nu}(0) \neq \phi.$$

However, fix κ and suppose $\{a_m\}_m$ is a sequence of points in $Q^{-1}_{\mu_\kappa\nu}(0)$ for which $\lim_{m\to\infty} a_m = a \in K_1$. Then $\tilde{Q}_{\mu_\kappa\nu}(a_m) = 0$ for all m and therefore by continuity we get $\tilde{Q}_{\mu_\kappa\nu}(a) = 0$. Now let $\kappa \to \infty$, $\tilde{Q}_{\mu_\kappa\nu}(a) \to \tilde{Q}_\nu(a)$ on K_1. Consequently $\tilde{Q}_\nu(a) = 0$. Since $a \in K_1 \subset \Delta^n_{\rho_0}$, it follows that

$$\Delta^n_{\rho_0} \cap Q^{-1}_\nu(0) \neq \phi.$$

Thus the converse holds and this concludes the proof.

The proof of theorem 3.6 was given in Lutterodt [4], so we merely provide an outline for the present paper: Suppose f is analytic at the origin and meromorphic in $\mathbb{C}^n$ with at most a finite polar set. By theorems 3.1 or 3.4, the polar set of f(z) attracts the whole polar set $\pi_{\mu\nu}(z)$ or a subset of it as the case may be. But for each fixed ν we know by hypothesis that the polar set of $\pi_{\mu\nu}(z)$, $Q^{-1}_{\mu\nu}(0)$, tends to infinity as $\mu' \to \infty$. Thus the polar set of f must be drawn to infinity, making f entire in $\mathbb{C}^n$.

4. NON-MONTESSUS TYPE CONVERGENCE.

In this section, we consider convergence of (μ,ν)-sequences of unisolvent rational approximants with ν not necessarily fixed as in §3. The convergence is weakly expressed in terms of measure for the cases examined, using a lemma of Bishop [1]. It should be noted that $\nu = (\nu_1,\ldots,\nu_n)$ is replaced by $\underline{\nu} = (\nu,\ldots,\nu)$ where in the latter ν is no longer an n-tuple but a mere natural number. Similarly $\underline{\omega} = (\omega,\ldots,\omega)$ with $\omega \in \mathbb{N}$ but μ remains an n-tuple.

THEOREM 4.1. Let $\varepsilon > 0$ and $0 < \eta < 1$, and let $\omega, \nu \in \mathbb{N}$ be fixed. Suppose $f \in \mathfrak{M}(\Delta^n_\rho)$ and its finite polar set over Δ^n_ρ is $G_\omega := \{z \in \mathbb{C}^n\colon\ q_{\underline{\omega}}(z) = 0\}$ and $q_{\underline{\omega}} f \in C(\bar{\Delta}^n_\rho)$ where $q_{\underline{\omega}}(z)$ is a polynomial of minimal 'degree' $\underline{\omega} = (\omega,\ldots,\omega)$. Suppose $\mu = (\mu_1,\ldots,\mu_n)$ and $\underline{\nu} = (\nu,\ldots,\nu)$ are such that $0 < \omega \leqslant \nu < \mu' = \min\limits_{1\leqslant j\leqslant n} (\mu_j)$.

Suppose $\pi_{\mu\underline{\nu}}(z)$ is a $(\mu,\underline{\nu})$-unisolvent rational approximant of f. Then for any compact subset $K \subset \Delta^n_\rho$, $\exists\, c > 0$, $0 < \delta < 1$ and $\mu'_0 \in \mathbb{N}$, positive such that $\mu' > \mu'_0 >$

$$|f(z) - \pi_{\mu\underline{\nu}}(z)|^{1/\mu'} < \delta$$

for all $z \in K\backslash Z^{\mu\nu}_\eta$ where $Z^{\mu\nu}_\eta = \{z \in K\colon |\tilde{Q}_{\mu\underline{\nu}}(z) q_{\underline{\omega}}(z)| < \eta^{\nu+\omega}\}$ and

$$m[\{z \in K\colon\ |f(z) - \pi_{\mu\underline{\nu}}(z)|^{1/\mu'} \geqslant \varepsilon\}] < \varepsilon$$

where m is $\mathbb{C}^n$-Lebesgue measure.

The difference between the above theorem 4.1 and the next one theorem 4.2 lies in ν not being fixed in Theorem 4.2.

THEOREM 4.2. Suppose the hypothesis of theorem 4.1 is satisfied, except for $\underline{\nu} = (\nu,\ldots,\nu)$ is not fixed, with $0 < \omega < \nu < \min\limits_{1\leqslant j\leqslant n} (\mu_j) = \mu'$, $\nu = \nu(\mu') \to \infty$ as $\mu' \to \infty$ but $\nu = o(\mu')$.

Suppose $\pi_{\mu\nu}(z)$ is a (μ,ν)-unisolvent rational approximant to f(z). Then for any $K \subset \Delta^n_\rho$, compact, $\exists\, c > 0$, $0 < \delta < 1$ and μ'_0 such that $\mu' > \mu'_0 \Rightarrow$

$$|f(z) - \pi_{\mu\underline{\nu}}(z)| < \delta^{\mu'}$$

for all $z \in K\backslash Z^{\mu\underline{\nu}}_\eta$ where $m(Z^{\mu\underline{\nu}}_\eta) < \varepsilon$ and m is a $\mathbb{C}^n$-Lebesgue measure.

LEMMA 4.3 (Bishop). Let $F_{\underline{d}}(z)$ be a normalized polynomial of degree $\underline{d} = (d,\ldots,d)$ in $\mathbb{C}^n$. Let $\rho > 0$ and $0 < \eta < 1$ be given. Then there exists a constant $c = c(n,\rho)$ such that the set

$$Z_\eta = \{z \in \bar{\Delta}^n_\rho\colon\ |F_{\underline{d}}(z)| < \eta^d\}$$

has $\mathbb{C}^n$-Lebesgue measure satisfying

$$m(Z_\eta) \leqslant c\eta^{2/n}.$$

REMARK. The proof of this Lemma uses an induction argument and it is presented in Narasimhan [8].

PROOF OF THE THEOREM. 1°. $f(z) \in \mathfrak{M}(\Delta^n_\rho)$ has a polar set $G_\omega = \{z \in \mathbb{C}^n\colon q_{\underline{\omega}}(z) = 0\}$. Without loss of generality we assume that $q_{\underline{\omega}}(z)$ is normalized in the same was as $\tilde{Q}_{\mu\underline{\nu}}(z)$. Since $\tilde{P}_{\mu\underline{\nu}}(z)$ and $\tilde{Q}_{\mu\underline{\nu}}(z)$ are both polynomials in $\mathbb{C}^n$ and $q_{\underline{\omega}} f \in \mathcal{H}(\Delta^n_\rho)$ and therefore $q_{\underline{\omega}}\tilde{Q}_{\mu\underline{\nu}} f - q_{\underline{\omega}}\tilde{P}_{\mu\underline{\nu}} \in \mathcal{H}(\Delta^n_\rho)$. Following the same presentation given in the first part of the proof of theorem 3.1 of our paper [6], we extract the inequality (3.7) and transplant it to this paper as

$$|H_{\mu\underline{\nu}}(z)| \leq M\left(\sum_{\lambda\in\mathbb{N}^n\setminus E_\mu} \frac{|z^\lambda|}{\rho^{|\lambda|}}\right)$$

where M is as before. From the above inequality we get,

$$|f(z) - \pi_{\mu\underline{\nu}}(z)| \leq \frac{M}{|Q_{\mu\underline{\nu}}(z)\, q_\omega(z)|}\left(\sum_{\lambda\in\mathbb{N}^n\setminus E_\mu} \frac{|z^\lambda|}{\rho^{|\lambda|}}\right). \tag{4.1}$$

Let K be any compact subset of Λ_ρ^n and let $\rho' > 0$ be chosen so that $0 < \rho' < \rho \Rightarrow \Lambda_{\rho'}^n \subsetneq \Lambda_\rho^n$ and $K \subset \Lambda_{\rho'}^n$. Then appealing to Lemma 3.2 for $z \in K$, (4.1) becomes

$$|f(z) - \pi_{\mu\underline{\nu}}(z)| \leq \frac{M\kappa(\rho,n)}{|\tilde{Q}_{\mu\underline{\nu}}(z)\, q_{\underline{\omega}}(z)|}\left(\frac{\rho'}{\rho}\right)^{\mu'+1} \tag{4.2}$$

where $\kappa(\rho,n)$ is a constant depending on ρ and n.

2°. Given $0 < \eta < 1$, let $\underline{d} = \underline{\nu} + \underline{\omega} = (\nu + \omega,\dots,\nu + \omega)$ and $F_{\underline{d}}(z) = \tilde{Q}_{\mu\underline{\nu}}(z)q_{\underline{\omega}}(z)$. Let $Z_\eta^{\mu\underline{\nu}} = \{z \in K: |F_{\underline{d}}(z)| \leq \eta^d\}$ with $d = \eta + \omega$. The polynomial $F_{\underline{d}}(z)$ of degree $(d,\dots,d)$ is normalized in $\mathbb{C}^n$. Thus by Lemma 4.3, we must have the $\mathbb{C}^n$-Lebesgue measure satisfying

$$m(Z_\eta^{\mu\underline{\nu}}) \leq c\eta^{2/n} \tag{4.3}$$

R.H.S. being independent of the degree of $F_{\underline{d}}$. For any $z \in K\setminus Z_\eta^{\mu\underline{\nu}}$ we must therefore have $|F_{\underline{d}}(z)| \geq \eta^d > 0$, so that

$$|f(z) - \pi_{\mu\underline{\nu}}(z)| \leq \frac{M\kappa(\rho,n)}{\eta^{\nu+\omega}}\left(\frac{\rho'}{\rho}\right)^{\mu'+1}. \tag{4.4}$$

Since $0 < \frac{\rho'}{\rho} < 1$ and $\mu' \to \infty$, we can find $\delta \in (0,1)$ and μ_0' such that $\mu' > \mu_0' \Rightarrow$

$$|f(z) - \pi_{\mu\underline{\nu}}(z)| < \delta^{\mu'} \tag{4.5}$$

for $z \in K\setminus Z_\eta^{\mu\nu}$. However, for $\varepsilon > 0$ and $c\eta^{2/n} < \varepsilon$; the set

$\Omega_\varepsilon^{\mu\underline{\nu}} = \{z \in K: |f(z) - \pi_{\mu\underline{\nu}}(z)|^{1/\mu'} \geq \varepsilon\}$ is included in the set $Z_\eta^{\mu\nu}$ and consequently from (4.3).

$$m(\Omega_\varepsilon^{\mu\underline{\nu}}) < \varepsilon.$$

The proof of Theorem 4.2 is identical to that of Theorem 4.1. But one has to recognize that even though $\nu \to \infty$ as $\mu' \to \infty$, because $\nu = o(\mu')$, this does not preclude us finding $\delta \in (0,1)$ for which the inequality (4.5) holds.

The Theorem 4.2 and some generalization for the case of meromorphic maps has been discussed in Lutterodt [7].

The next theorem establishes over-convergence for the non-Montessus type.

THEOREM 4.4 (Gončar). Let $\rho > 1$ be fixed and let $f \in \mathfrak{M}(D)$ where D is a domain in $\mathbb{C}^n$ such that $\Lambda_\rho^n \subset D$. Suppose the conditions of either Theorem 4.1 or Theorem 4.2 are satisfied and for any $K \subset \Delta_\rho^n$ compact $\pi_{\mu\underline{\nu}}(z) \overset{m}{\to} f(z)$ on K as $\mu' \to \infty$ and $\nu = o(\mu')$. Then $\pi_{\mu\underline{\nu}}(z) \overset{m}{\to} f(z)$ on any K compact in D as $\mu' \to \infty$.

REMARK. The statement $\pi_{\mu\underline{\nu}}(z) \overset{m}{\to} f(z)$ as $\mu' \to \infty$ means that $\pi_{\mu\underline{\nu}}(z)$ converges to f(z) in $\mathbb{C}^n$-Lebesgue measure m as $\mu' \to \infty$.

PROOF OF THEOREM 4.4. We shall assume without loss of generality that $0 \in \mathcal{K}$. The pole set G_ω of f in $D \subset \mathbb{C}^n$ has $\mathbb{C}^n$-Lebesgue measure zero (cf. final part of Theorem 3.4 proof); since D is an open connected set, $(D \supset \mathcal{K})$ so $D \backslash G_\omega$ must be dense in D. Thus we can select distinct countable points $0 = a_1, a_2, \ldots$ in $D \backslash G_\omega$ and polydisk neighborhoods $\Delta^n_{\rho_j}$ centered at a_j such that $\bigcup_{j=1}^{\omega} \Delta^n_{\rho_j} \supset \mathcal{K}$. Since $\mathcal{K}$ is compact, we choose overlapping polydisks $\Delta^n_\rho, \Delta^n_{\rho_{j_1}}, \ldots, \Delta^n_{\rho_{j_\kappa}}$ whose respective centers $0, a_{j_1}, \ldots, a_{j_\kappa}$ are linked by a polygonal path which does not intersect the codimension 1 polar set G_ω of f such that $\mathcal{K} \subset \bigcup_{\ell=0}^{\kappa} \Delta^n_{\rho_{j_\ell}}$. The polygonal path then becomes a path along which f can be analytically continued in D. Here we have identified Δ^n_ρ with $\Delta^n_{\rho_{j_0}}$ so that from theorem 4.1, for μ' sufficiently large we must have, for $\varepsilon > 0$.

$$m\{z \in \mathcal{K} \cap \Delta^n_\rho : \ |f(z) - \pi_{\mu\underline{\nu}}(z)|^{1/\mu'} \geq \varepsilon\} < \varepsilon. \tag{4.6}$$

For each of the overlapping polydisk neighborhoods $\Delta^n_{\rho_{j_1}}, \ldots, \Delta^n_{\rho_{j_\kappa}}$ a result identical to (4.6) in fact holds. Since $\mathcal{K} = \mathcal{K} \cap \bigcup_{\ell=0}^{\kappa} \Delta^n_{\rho_j}$ we obtain the following: for μ' sufficiently large

$$m\{z \in \mathcal{K}: \ |f(z) - \pi_{\mu\underline{\nu}}(z)|^{1/\mu'} \geq \varepsilon\} < \kappa\varepsilon$$

which gives the desired result.

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