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A NOTE ON SUBORDINATION

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ABSTRACT. Suffridge showed a result for subordinate functions. The object of the present paper is to show some subordinate theorems with the aid of the result by Suffridge.

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1. INTRODUCTION.

Let $f(z)$ and $g(z)$ be analytic in the unit disk $U = \{z: |z| < 1\}$. A function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $\phi(z)$ analytic in the unit disk U satisfying $\phi(0) = 0$ and $|\phi(z)| < 1$ $(z \in U)$ such that $f(z) = g(\phi(z))$ for $z \in U$. We denote by $f(z) \prec g(z)$ this relation. In particular, if $g(z)$ is univalent the unit disk U the subordination is equivalent to $f(0) = g(0)$ and range $f(z) \subset$ range $g(z)$.

This concept of subordination can be traced to Lindelöf [1], but Littlewood [2],[3] and Rogosinski [4], [5] introduced the term and discovered the basic properties. Recently Suffridge [6] and Hallenbeck and Ruscheweyh [7] studied the subordinate functions and showed many interesting results for subordinations.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk U. Further let S be the subclass of A consisting of analytic and univalent functions in the unit disk U. Then a function $f(z)$ of S is said to be starlike of order α if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U)$$

for some α $(0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α. Further a function $f(z)$ of S is said to be convex of order α if and only if

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U)$$

for some α $(0 \leq \alpha < 1)$. And we denote by $K(\alpha)$ the class of convex functions of order α. it is well-known that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, $S^*(\alpha) \subset S^*$, $K(a) \subset K$, and $S^*(0) \equiv S^*$, $K(0) \equiv K$ for $\alpha = 0$.

The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [8], and latter studied by Schild [9], MacGregor [10] and Pinchuk [11]. Further, recently, some] classes defined by using the extremal function $z/(1-z)^{2(1-\alpha)}$ for $S^*(\alpha)$ were studied by Ruscheweyh [12], Sheil-Small, Silverman and Silvia [13], Silverman and Silvia [14], and Ahuja and Silverman [15].

Our main tool in this paper is the following result by Suffridge [15].

LEMMA. Let the function $f(z) = \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disk U and the function $g(z)$ be in the class S^*. If $f(z)$ is subordinate to $g(z)$, that is, $f(z) \prec g(z)$, then

$$\int_0^z \frac{f(t)}{t}\,dt \prec \int_0^z \frac{g(t)}{t}\,dt$$

for $z \in U(r) = \{z: |z| \leqq r, 0 \leqq r < 1\}$.

2. SUBORDINATION THEOREMS.

In this section, we show some subordination theorems with the aid of Lemma.

THEOREM 1. Let the function $f(z)$ defined by (1.1) be in the class of $K(\alpha)$. Then $f'(re^{i\theta})$ $(0 \leqq r < 1)$ is contained in the image domain of the closed disk $U(r)$ under the function $e^{4(\alpha-1)/(1-z)}$. Further it lies for $r \neq 0$ on the boundary of of this image domain if and only if

$$f(z) = \int_0^z e^{4(1-\alpha)/(1-\varepsilon t)}dt. \tag{2.1}$$

where $|\varepsilon| = 1$.

PROOF. Since $f(z)$ is in the class $K(\alpha)$, $f(z)$ satisfies that

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \alpha - 1 \qquad (z \in U).$$

Note that $zf''(z)/f'(z) = 2a_2 z + \ldots$ is analytic in the unit disk U, and that the function $z/(1-z)^2$ is starlike with respect to the orgin and $\operatorname{Re}\{z/(1-z)^2\} > -1/4$. Hence we have that

$$\frac{zf''(z)}{f'(z)} \prec \frac{4(1-\alpha)z}{(1-z)^2} \qquad (z \in U).$$

Consequently, by using Lemma, it follows that $\log f'(re^{i\theta})$ is contained in the image domain of $U(r)$ under the function $4(\alpha-1)/(1-z)$, where log is understood to be that branch which vanishes at the point one. Thus we can see that $f'(re^{i\theta})$ lies for $r \neq 0$ on the boundary of the image domain of $U(r)$ under $e^{4(\alpha-1)/(1-z)}$. Further $f'(re^{i\theta})$ lies for $r \neq 0$ on the boundary of the image domain $U(r)$ under $e^{4(\alpha-1)/(1-z)}$ if and only if

$$\frac{zf''(z)}{f'(z)} = \frac{4(1-\alpha)\varepsilon z}{(1-\varepsilon z)^2} \qquad (|\varepsilon| = 1),$$

hence further, $f(z)$ is the function of the form (2.1). This completes the proof of the theorem.

THEOREM 2. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha)$. Then $f(re^{i\theta})/re^{i\theta}$ $(0 \leq r < 1)$ is contained in the image domain of the closed disk $U(r)$ under the function $e^{4(\ -1)/(1-z)}$. Further it lies for $r \neq 0$ on the boundary of this image domain if and only if $f(z) = ze^{4(1-\alpha)/(1-\varepsilon z)}$, where $|\varepsilon| = 1$.

PROOF. Since $f(z) \in S^*(\alpha)$, $f(z)$ satisfies that

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - 1\right\} > \alpha - 1 \qquad (z \in U)$$

and the function $zf'(z)/f(z) - 1 = a_2 z + \ldots$ is analytic in the unit disk U. Hence $zf'(z)/f(z) - 1$ takes values in the image domain of the unit disk U under the function $4(1-\alpha)z/(1-z)^2$, that is,

$$\frac{zf'(z)}{f(z)} - 1 \prec \frac{4(1-\alpha)z}{(1-z)^2} \qquad (z \in U).$$

By virtue of Lemma, we observe that $\log f(re^{i\theta})/re^{i\theta}$ $(0 \leq r < 1)$ is contanined in in the image domain of $U(r)$ under $4(\alpha-1)/(1-z)$ and it lies for $r \neq 0$ on the boundary of this image domain if and only if

$$\frac{zf'(z)}{f(z)} - 1 = \frac{4(1-\alpha)\varepsilon z}{(1-\varepsilon z)^2} \qquad (|\varepsilon| = 1),$$

hence further, $f(z) = ze^{4(1-\alpha)/(1-\varepsilon z)}$. This gives the result we require.

Finally we show a theorem for functions $f(z)$ satisfying $\operatorname{Re}\{zf'(z)\} > \alpha(\alpha > 0)$.

THEOREM 3. Let the function $f(z)$ defined by (1.1) satisfy $\operatorname{Re}\{zf'(z)\} > \alpha$ $(\alpha>0)$. Then $f(re^{i\theta})$ $(0 \leq r < 1)$ is contained in the image domain of the closed disk $U(r)$ under the function $-4\alpha/(1-z)$. Further it lies for $r \neq 0$ on the boundary of this domain if and only if $f(z) = -4\alpha/(1-\varepsilon z)$, where $|\varepsilon| = 1$.

PROOF. We note that the function $zf'(z) = z + 2a_2 z^2 + \ldots$ takes values in the image domain of the unit disk U under the function $-4\alpha z/(1-z)^2$ which belongs to the class S^*. Therefore we can prove the theorem by using the same technique as in the one of Theorem 1 with the aid of Lemma.

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