

CATEGORICAL CONSTRUCTIONS IN GRAPH THEORY

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ABSTRACT. This paper presents some graph-theoretic questions from the viewpoint of the portion of category theory which has become common knowledge. In particular, the reader is encouraged to consider whether there is only one natural category of graphs and how theories of directed graphs and undirected graphs are related.

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1. INTRODUCTION

The use of category theory in graph theory is no longer novel (P. Hell [1]), and methods of category theory have been made generally available to the "Working Mathematician". The main category theoretic topics that we need and the section number of MacLane [2] in which they may be found are: adjoint functors (Chap. IV); representable functors (Sect. III.2); Kan extensions (Chap. IX); functor categories (Sect. II.4); monads (Chap. VI); and cartesian closed categories (Sect. IV.6). Nevertheless, one can find examples in the literature of errors, imprecision, or laborious constructions of special cases. As a result of reviewing some of this work for Mathematical Reviews, we set ourselves the task of presenting a sound description of some of the material which seemed most susceptible to these flaws.

In particular, we wish to answer the question in the title of the paper of Harary and Read [3] with a firm "No!", and elaborate on defects noted in the reviews of Farzan and Waller [4] and Ribenboim [5] (see also Ribenboim [6]). The concepts of "involutorial graph" and "hom-graph" are defined by Ribenboim without citing any earlier work. This makes it difficult to assess their pedigree. We shall present evidence that, as originally presented, these definitions were wrong since they did not account for the category theoretical properties which have been recognized as essential parts of the algebraic structure and cartesian closedness. The difficulty with a study of a "category of graphs" is that graph theory was born as a branch of

combinatorics which leans towards defining graphs in terms of adjacency properties of a set of vertices, which restrict the available objects and ignores the relations between them, while category theory emphasizes the structure preserving mappings (morphisms) and gives its best results where one is able to construct objects "freely determined by some properties".

Our first observation is that directed graphs are easier to describe than undirected graphs. In Section 2, we construct categories which give satisfactory descriptions of a category of directed graphs and which have simple categorical descriptions (as functor categories). The construction of a category of undirected graphs is discussed in Section 3.

Second, vertices and edges have traditionally been considered to be different things, but Ribenboim treated vertices as being degenerate edges. In Section 2, we study the relationship between the categories of directed graphs which model these two definitions. In Section 4, we observe that the "Cartesian" structure has more intuitively satisfying properties when the vertices are considered as degenerate edges.

Section 3 is devoted to recapturing an undirected graph from one of its canonically generated directed graphs. This process uses the theory of monads which stems from the definition of "algebraic structure" in the context of category theory.

2. TWO POSSIBLE DEFINITIONS OF "DIRECTED GRAPH"

In order to describe a directed graph G , one first specifies a set V of vertices and a set E of edges. Each edge is considered as starting at a vertex, called its origin and going to another vertex, called its terminal. Actually, these assignments define functions o (for origin) and t (for terminal) from E to V . This definition allows oriented graphs to have multiple edges ($e_1, e_2 \in E$ with $e_1 \neq e_2$, $o(e_1) = o(e_2)$ and $t(e_1) = t(e_2)$) and loops ($e \in E$ with $o(e) = t(e)$). Often, such graphs are excluded in combinatorial graph theory problems.

This definition is equivalent to defining a directed graph as a functor from the category $\underline{A}^{\text{OP}}$ (see Figure 1) to the category Ens , of sets.

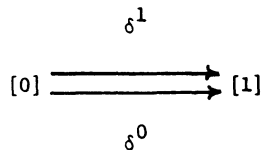


Figure 1. The category \underline{A}

Clearly, $G[0]$ corresponds to V and $G[1]$ to E . Then, without loss of generality, the functions $G(\delta^1):G[1] \rightarrow G[0]$ and $G(\delta^0):G[1] \rightarrow G[0]$ are the maps origin and terminal, respectively.

If graphs are functors, then the appropriate definition of "graph morphism" should be a natural transformation between these functors. Such a natural transformation

$$f:G_1 \xrightarrow{\quad} G_2:A^{\text{OP}} \rightarrow \text{Ens}$$

is given by a map of vertices $f[0]:G_1[0] \rightarrow G_2[0]$ and a map of edges $f[1]:G_1[1] \rightarrow G_2[1]$ which "commute with" origin and terminal. Thus a category of directed graphs can be modeled by the functor category $[\underline{A}^{\text{OP}}, \text{Ens}]$.

By contrast, considering vertices to be "degenerate edges" leads one to view the category of graphs as the functor category $[\underline{\underline{B}}^{OP}, \text{Ens}]$ (see Figure 2).

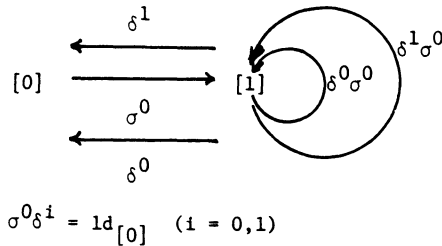


Figure 2. The Category $\underline{\underline{B}}$

If $H: \underline{\underline{B}}^{OP} \rightarrow \text{Ens}$, then $H[0]$ represents the vertices of H and $H[1]$, the edges of H . The functions $H(\delta^1): H[1] \rightarrow H[0]$ and $H(\delta^0): H[1] \rightarrow H[0]$ are again said to be origin and terminal, respectively. In addition, the identities $\sigma^0 \delta^i = \text{Id}_{[0]}$, $i = 0, 1$ in $\underline{\underline{B}}$ insure that the function $H(\sigma^0): H[0] \rightarrow H[1]$ is an embedding of the set of vertices into the set of edges of H , mapping each vertex to a loop based at that vertex. Thus, each vertex can be thought of as a "degenerate edge".

Since $[\underline{\underline{A}}^{OP}, \text{Ens}]$ and $[\underline{\underline{B}}^{OP}, \text{Ens}]$ are both examples of functor categories, we are able to use general properties of functor categories to describe the fundamental category-theoretic constructions within each category, and to relate these two candidates for a category of directed graphs. The first principle which we use is that limits and colimits (including the terminal object as a special case of a limit and the initial object as a special case of a colimit) are computed "pointwise".

For example, the null set \emptyset is an initial object in Ens since there is a unique (null) map of \emptyset into every set. An initial object in $[\underline{\underline{A}}^{OP}, \text{Ens}]$ or $[\underline{\underline{B}}^{OP}, \text{Ens}]$ is a graph with no vertices, no edges, and all structural maps null. The null graph is thus essential to a categorical approach.

Similarly, since a terminal object of Ens is any one pointed set, the terminal object of $[\underline{\underline{A}}^{OP}, \text{Ens}]$ must have one vertex and one edge which is a loop. In $[\underline{\underline{B}}^{OP}, \text{Ens}]$, the unique edge of the terminal object T is degenerate. Thus T has the property of representing the vertex set of a graph, i.e., $H[0]$ and the hom-set $[\underline{\underline{B}}^{OP}, \text{Ens}](T, H)$ are isomorphic as sets, for each graph H . In fact, $[_][0]: [\underline{\underline{B}}^{OP}, \text{Ens}] \rightarrow \text{Ens}$ and $[\underline{\underline{B}}^{OP}, \text{Ens}](T, _): [\underline{\underline{B}}^{OP}, \text{Ens}] \rightarrow \text{Ens}$ are functors which are naturally equivalent.

Since a product in a category is a limit, products in $[\underline{\underline{A}}^{OP}, \text{Ens}]$ and $[\underline{\underline{B}}^{OP}, \text{Ens}]$ are also computed "pointwise". The product $P = G \times G'$ of $\underline{\underline{A}}$ -graphs G and G' has vertices $P[0] = G[0] \times G'[0]$ and edges $P[1] = G[1] \times G'[1]$. The origin and terminal maps from $P[1]$ to $P[0]$ are constructed canonically from these maps in G and G'

EXAMPLE. Suppose G and G' each have one edge connecting two distinct vertices (e from a to b and e' from a' to b' , respectively). Then $P[0] = \{(a, a'), (a, b'), (b, a'), (b, b')\}$; $P[1] = \{(e, e')\}$; $P(\delta^1)(e, e') = (a, a')$; and $P(\delta^0)(e, e') = (b, b')$ (see Figure 3).

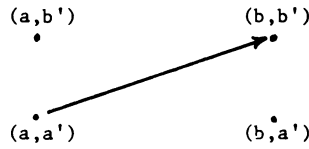


Figure 3. The $\underline{\underline{A}}$ -graph $G \times G'$

EXAMPLE. The product Q of $\underline{\underline{B}}$ -graphs H and H' has $Q[0] = H[0] \times H'[0]$ and $Q[1] = H[1] \times H'[1]$ with $Q(\delta^0)$, $Q(\delta^1)$ and $Q(\sigma^0)$ constructed from the corresponding maps in H and H' . In particular, if H and H' each have two vertices and one nondegenerate edge, then the product graph Q has four vertices and nine edges, four of which are degenerate (see Figure 4).

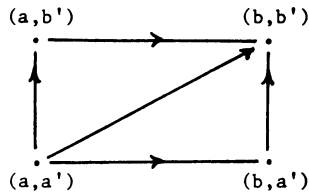


Figure 4. The $\underline{\underline{B}}$ -graph $H \times H'$

Since the same construction can product very different results in $[\underline{\underline{A}}^{OP}, \text{Ens}]$ and $[\underline{\underline{B}}^{OP}, \text{Ens}]$, it is desirable to be able to relate these two categories so that constructions in one would be naturally "induced" in the other.

The inclusion $u: \underline{\underline{A}} \hookrightarrow \underline{\underline{B}}$ of $\underline{\underline{A}}$ as a subcategory of $\underline{\underline{B}}$ induces a (forgetful) functor

$$U: [\underline{\underline{B}}^{OP}, \text{Ens}] \rightarrow [\underline{\underline{A}}^{OP}, \text{Ens}].$$

For a $\underline{\underline{B}}$ -graph H , the $\underline{\underline{A}}$ -graph $U(H)$ is just the restriction of H to the subcategory $\underline{\underline{A}}^{OP}$ of $\underline{\underline{B}}^{OP}$. In the $\underline{\underline{A}}$ -graph $U(H)$, the degenerate edges $H(\sigma^0)v$, for various vertices $v \in H[0]$, are no longer distinguished. The restriction of natural transformations gives no difficulty, and it is easy to verify that $U: [\underline{\underline{B}}^{OP}, \text{Ens}] \rightarrow [\underline{\underline{A}}^{OP}, \text{Ens}]$ is actually a functor.

Because $U: [\underline{\underline{B}}^{OP}, \text{Ens}] \rightarrow [\underline{\underline{A}}^{OP}, \text{Ens}]$ is defined by composition with the inclusion $u^{OP}: \underline{\underline{A}}^{OP} \rightarrow \underline{\underline{B}}^{OP}$, it induces more structure: U has both left and right adjoints which are the Kan extensions L and R . These are described below.

For each $\underline{\underline{A}}$ -graph G , $L(G)$, the " $\underline{\underline{B}}$ -graph freely generated by G ", is obtained by adding to $G[1]$ a new degenerate edge \hat{v} , one for each vertex $v \in G[0]$, and setting $L(G)(\sigma^0)(v) = \hat{v}$. If $f: G \rightarrow G'$ is an $\underline{\underline{A}}$ -morphism (i.e., a natural transformation from the functor G to the functor G'), then $L(f): L(G) \rightarrow L(G')$ is defined to agree with f when "restricted to G " and to satisfy

$$(L(f)[1])(\hat{v}) = \widehat{(f[0])(v)}.$$

To see that $L: [\underline{\underline{A}}^{OP}, \text{Ens}] \rightarrow [\underline{\underline{B}}^{OP}, \text{Ens}]$ is the left adjoint of $U: [\underline{\underline{B}}^{OP}, \text{Ens}] \rightarrow [\underline{\underline{A}}^{OP}, \text{Ens}]$, one shows that there is a natural one-to-one correspondence

$$\psi(G,H):[\underline{B}^{OP},\text{Ens}](L(G),H) \rightarrow [\underline{A}^{OP},\text{Ens}](G,U(H))$$

for each \underline{A} -graph G and each \underline{B} -graph H .

This adjoint relationship may also be described in terms of the unit natural transformation

$$\eta: \text{Id} \rightarrow \text{UL}: [\underline{A}^{OP},\text{Ens}] \rightarrow [\underline{A}^{OP},\text{Ens}]$$

or the counit natural transformation

$$\varepsilon: \text{LU} \rightarrow \text{Id}: [\underline{B}^{OP},\text{Ens}] \rightarrow [\underline{B}^{OP},\text{Ens}]$$

of the adjoint pair $\langle L,U \rangle$. We now construct both natural transformations. The construction, given above, for the \underline{B} -graph $L(G)$ may be restricted to \underline{A}^{OP} to give the \underline{A} -graph $UL(G)$. This construction yields G as a specific subgraph of $UL(G)$; the collection of these inclusions $\eta_G: G \rightarrow UL(G)$ is natural in G . In fact, this collection is the unit natural transformation of the adjunction $\langle L,U \rangle$. Similarly, the counit $\varepsilon: \text{LU} \rightarrow \text{Id}$ is given by describing the \underline{B} -graph morphism $\varepsilon_H: \text{LU}(H) \rightarrow H$ for each \underline{B} -graph H . The graph $\text{LU}(H)$ differs from H by the addition of new degenerate edges for each vertex, the old ones having lost their distinction in $U(H)$. The \underline{B} -graph epimorphism ε_H is the identity on $H \subseteq \text{LU}(H)$ and maps each new degenerate \hat{v} added by L to the original degenerate edge $H(\sigma^0)(v)$ in $H[1]$. Next, the adjoint relationship $\langle L,U \rangle$ may be verified by demonstrating that the following composites are identity natural transformations:

$$\begin{array}{ccccc} U & \xrightarrow{\eta U} & ULU & \xrightarrow{U\varepsilon} & U \\ \cdot & & \cdot & & \cdot \\ L & \xrightarrow{L\eta} & LUL & \xrightarrow{\varepsilon L} & L \\ \cdot & & \cdot & & \cdot \end{array}$$

Lastly, note that neither ε_H nor η_G can be an isomorphism unless H or G , respectively, is a null graph.

The right adjoint

$$R: [\underline{A}^{OP},\text{Ens}] \rightarrow [\underline{B}^{OP},\text{Ens}]$$

is given as follows: for each \underline{A} -graph G , $R(G)$ is the \underline{B} -graph whose vertex set $R(G)[0]$ is the collection of all loops of G (i.e. edges $l \in G[1]$ with $G(\delta^1)(l) = G(\delta^0)(l)$). The set of edges $R(G)[1]$ is the collection of all triples $\langle l_0, e, l_1 \rangle$ of edges of G with l_0, l_1 loops and e an edge having origin the vertex of l_0 and terminal the vertex of l_1 , i.e. $G(\delta^1)(l_0) = G(\delta^0)(l_0) = G(\delta^1)(e)$ and $G(\delta^1)(l_1) = G(\delta^0)(l_1) = G(\delta^0)(e)$. The origin and terminal of $\langle l_0, e, l_1 \rangle$ are l_0, l_1 , respectively; i.e. $(R(G))(\delta^1)\langle l_0, e, l_1 \rangle = l_0$ and $(R(G))(\delta^0)\langle l_0, e, l_1 \rangle = l_1$. Lastly, the distinguished $R(G)$ -loop at the $R(G)$ -vertex l is $\langle l, l, l \rangle$; i.e.

$(R(G))(\sigma^0)(l) = \langle l, l, l \rangle$. The unit natural transformation

$$\eta: \text{Id} \rightarrow \text{RU}: [\underline{B}^{OP},\text{Ens}] \rightarrow [\underline{B}^{OP},\text{Ens}]$$
 consists of the \underline{B} -morphisms $\eta_H: H \rightarrow \text{RU}(H)$.

Particularly, η_H carries a vertex v of the \underline{B} -graph H to the vertex of $\text{RU}(H)$ which is the degenerate loop $H(\sigma^0)(v)$, i.e. $\eta_H(v) = H(\sigma^0)(v)$; and η_H maps an edge e of H with origin v and terminal w to the edge $\langle H(\sigma^0)v, e, H(\sigma^0)w \rangle$, i.e. $\eta_H(e) = \langle H(\sigma^0\delta^1)e, e, H(\sigma^0\delta^0)e \rangle$. Clearly, $\eta_H: H \rightarrow \text{RU}(H)$ is the \underline{B} -monomorphism which identifies H with the full subgraph of $\text{RU}(H)$ generated by vertices $\{H(\sigma^0)v \mid v \in H[0]\}$.

The counit $\varepsilon: \text{UR} \rightarrow \text{ld}: [\underline{\underline{A}}^{\text{OP}}, \underline{\underline{E}}\text{ns}] \rightarrow [\underline{\underline{A}}^{\text{OP}}, \underline{\underline{E}}\text{ns}]$ is analogously defined to be a family of $\underline{\underline{A}}$ -graph morphisms $\varepsilon_G: \text{UR}(G) \rightarrow G$. The vertices of $\text{UR}(G)$ are the loops of G . The $\underline{\underline{A}}$ -morphism ε_G maps such a loop to its vertex $G(\delta^0)(\ell) = G(\delta^1)(\ell)$ and carries an edge $\langle \ell_0, e, \ell_1 \rangle$ to the edge e ; i.e. $\varepsilon_G[0](\ell) = G(\delta^0)(\ell)$ and $\varepsilon_G[1]\langle \ell_0, e, \ell_1 \rangle = e$. Thus ε_G "compresses" $\text{UR}(G)$ onto the full $\underline{\underline{A}}$ -subgraph of G generated by those vertices having loops.

Note that ε_G is an $\underline{\underline{A}}$ -graph isomorphism if and only if at each vertex of G , there is precisely one loop. Similarly, η_H is a $\underline{\underline{B}}$ -graph isomorphism exactly when each loop of H is a degenerate loop $H(\sigma^0)v$. Therefore U and R induce an isomorphism between the full subcategories $\underline{\underline{A}}$ of $\underline{\underline{A}}$ -graphs with exactly one loop per vertex and $\underline{\underline{B}}$ of $\underline{\underline{B}}$ -graphs with only degenerate loops.

These ideas above illustrate the first part of the following theorem whose second part we use later.

THEOREM 2.1 (Lambek and Rattray). Let $S: \underline{\underline{X}} \rightleftarrows \underline{\underline{Y}}: T$ be a pair of adjoint functors with unit $\eta: \text{ld} \rightarrow \text{TS}: \underline{\underline{X}} \rightarrow \underline{\underline{X}}$ and counit $\varepsilon: \text{ST} \rightarrow \text{ld}: \underline{\underline{Y}} \rightarrow \underline{\underline{Y}}$. Then T and S induce an equivalence between $\text{Fix}(\text{TS}, \eta) = \{X \in \text{obX} \mid \eta_X: X \xrightarrow{\cong} \text{TS}(X)\}$ and $\text{Fix}(\text{ST}, \varepsilon) = \{Y \in \text{obY} \mid \varepsilon_Y: \text{ST}(Y) \xrightarrow{\cong} Y\}$. Moreover, the following statements are equivalent:

- (i) the triple $(\text{TS}, \eta, \text{TS})$ on $\underline{\underline{X}}$ is idempotent;
- (ii) ηT is a natural isomorphism;
- (iii) the cotriple $(\text{ST}, \varepsilon, \text{ST})$ on $\underline{\underline{Y}}$ is idempotent;
- (iv) εS is a natural isomorphism;
- (v) $\text{TS}: \underline{\underline{X}} \rightarrow \underline{\underline{X}}$ factors through the subcategory $\text{Fix}(\text{TS}, \eta)$;
- (vi) $\text{ST}: \underline{\underline{Y}} \rightarrow \underline{\underline{Y}}$ factors through the subcategory $\text{Fix}(\text{ST}, \varepsilon)$.

If these conditions hold, $\text{Fix}(\text{TS}, \eta)$ is a reflective subcategory of $\underline{\underline{X}}$ with reflector the factorization (v) and $\text{Fix}(\text{ST}, \varepsilon)$ is a coreflective subcategory of $\underline{\underline{Y}}$ with coreflector the factorization (vi).

PROOF. See Lambek and Rattray [8], Theorem 1.1.

In the example just discussed, $S: \underline{\underline{X}} \rightarrow \underline{\underline{Y}}$ is $U: [\underline{\underline{B}}^{\text{OP}}, \underline{\underline{E}}\text{ns}] \rightarrow [\underline{\underline{A}}^{\text{OP}}, \underline{\underline{E}}\text{ns}]$ and $T: \underline{\underline{Y}} \rightarrow \underline{\underline{X}}$ is the right adjoint $R: [\underline{\underline{A}}^{\text{OP}}, \underline{\underline{E}}\text{ns}] \rightarrow [\underline{\underline{B}}^{\text{OP}}, \underline{\underline{E}}\text{ns}]$. The condition for $\text{Fix}(\text{RU}, \eta) \subseteq [\underline{\underline{B}}^{\text{OP}}, \underline{\underline{E}}\text{ns}]$ to be reflective and $\text{Fix}(\text{UR}, \varepsilon) \subseteq [\underline{\underline{A}}^{\text{OP}}, \underline{\underline{E}}\text{ns}]$ to be coreflective fail. Consider $\eta_{\text{RG}}: \text{RG} \rightarrow \text{RU}(\text{RG})$ where G is a graph with one vertex v and two loops ℓ_0, ℓ_1 at v . Then RG has two vertices $\{\ell_0, \ell_1\}$, each with two loops

$$\{\langle \ell_0, \ell_1, \ell_0 \rangle, \langle \ell_1, \ell_1, \ell_1 \rangle\}, \quad i = 0, 1. \quad (2.1)$$

However, $\text{RU}(\text{RG})$ has four vertices, the four loops (2.1). Thus η_{RG} fails to be an isomorphism and condition (ii) above does not hold. In fact, $\text{Fix}(\text{RU}, \eta)$ is not coreflective since the inclusion does not preserve colimits.

Suppose we compose the adjunction $S: \underline{\underline{X}} \rightleftarrows \underline{\underline{Y}}: T$ with an adjoint pair $M: \underline{\underline{Y}} \rightleftarrows \underline{\underline{Z}}: N$ then Theorem 2.1 applies to the composite adjunction $MS: \underline{\underline{X}} \rightleftarrows \underline{\underline{Z}}: NT$. In particular, if $\underline{\underline{Z}}$ is a reflective subcategory of $\underline{\underline{Y}}$ with idempotent cotriple $(\text{MSTN}, \varepsilon, \text{MSTN})$, then there is a reflective subcategory $\underline{\underline{X}}_1$ of $\underline{\underline{X}}$, guaranteed by Theorem 2.1 which is equivalent to a coreflective subcategory $\underline{\underline{Z}}_1$ of $\underline{\underline{Z}}$.

As an illustration, let Γ_1 be the full reflective subcategory of $[\underline{\underline{A}}^{\text{OP}}, \underline{\underline{E}}\text{ns}]$ generated by all $\underline{\underline{A}}$ -graphs which have at most one loop at each vertex. The left

adjoint (usually called the reflector) $M: [\underline{A}^{\text{OP}}, \text{Ens}] \rightarrow \Gamma_1$ of the inclusion $N: \Gamma_1 \hookrightarrow [\underline{A}^{\text{OP}}, \text{Ens}]$ maps each \underline{A} -graph G to the quotient \underline{A} -graph G_1 with the same vertices as G , the same edges between distinct vertices, but having only one loop at each vertex at which G has loops and having no new loops. Applying $UR: [\underline{A}^{\text{OP}}, \text{Ens}] \rightarrow [\underline{A}^{\text{OP}}, \text{Ens}]$ to a graph NG_1 with $G_1 \in \Gamma_1$, yields the full subgraph $UR(G_1)$ of NG_1 generated by the vertices of NG_1 at which there is a loop. Note that $UR(G_1)$ is already in the subcategory Γ_1 ; hence the function $M: [\underline{A}^{\text{OP}}, \text{Ens}] \rightarrow \Gamma_1$, has no further effect and $MURN(G_1) = URN(G_1) \hookrightarrow NG_1$. Iterating this construction yields the same graph, up to isomorphism. Thus condition (vi) of Theorem 2.1 holds, and all the other equivalent properties (i)-(v) and the conclusions of Theorem 2.1 follow. Particularly, the coreflective subcategory Γ of Γ_1 consisting of \underline{A} -graphs with exactly one loop per vertex is equivalent to the reflective subcategory Δ of \underline{B} -graphs with only degenerate loops. Note that $\Gamma = \text{Fix}(UR, \varepsilon)$ and $\Delta = \text{Fix}(RU, \eta)$ for the original adjunction $U: [\underline{B}^{\text{OP}}, \text{Ens}] \rightleftarrows [\underline{A}^{\text{OP}}, \text{Ens}]: R$. Thus it is possible for $\text{Fix}(TS, \eta)$ to be a reflective subcategory of \underline{X} without the existence of a factorization of $TS: \underline{X} \rightarrow \underline{X}$ through $\text{Fix}(TS, \eta)$ (i.e. condition (v) of Theorem 2.1 fails even though part of the conclusion is still true).

A second important subcategory Γ_2 of $[\underline{A}^{\text{OP}}, \text{Ens}]$ is the full reflective subcategory generated by the simple \underline{A} -graphs G with at most one edge from any particular vertex to another. The reflector $M: [\underline{A}^{\text{OP}}, \text{Ens}] \rightarrow \Gamma_2$, the left adjoint of the full inclusion $N: \Gamma_2 \hookrightarrow [\underline{A}^{\text{OP}}, \text{Ens}]$, maps each \underline{A} -graph G to the quotient \underline{A} -graph G_2 having the same vertices as G , but with edges e and e' identified whenever $G(\delta^1)e = G(\delta^1)e'$ and $G(\delta^0)e = G(\delta^0)e'$. Again, the cotriple on Γ_2 is easily seen to be idempotent. Using an argument similar to the one for Γ_1 , it follows that there exists a reflective subcategory Δ_2 of $[\underline{B}^{\text{OP}}, \text{Ens}]$ which is equivalent to a coreflective subcategory $\hat{\Gamma}_2$ of Γ_2 .

Theorem 2.1 thus establishes equivalences between subcategories of $[\underline{A}^{\text{OP}}, \text{Ens}]$ and $[\underline{B}^{\text{OP}}, \text{Ens}]$ which introduce the combinatorially useful concepts of "absence of (unnecessary) loops" or "characterization of edges by their origin and terminal vertices". The "combinatorially interesting graphs" are extracted in a natural way from either of the functor categories $[\underline{A}^{\text{OP}}, \text{Ens}]$ or $[\underline{B}^{\text{OP}}, \text{Ens}]$. Thus either of the two possible general definitions of "directed graph" leads to the same theory. Along the way, though, we have found that the interplay of these generalizations leads to deeper properties than one might have expected.

3. UNDIRECTED GRAPHS

In combinatorial problems, a graph is simple considered to be one-dimensional complex. Each graph U has a set $V(U)$ of vertices, a set $E(U)$ of edges, and an incidence relation $I(U) \subseteq V(U) \times E(U)$ with each edge incident to at most two vertices. Clearly, a directed \underline{A} -graph G can be given this structure, denoted by $P(G)$: the vertices are $G[0]$; the edges are $G[1]$; and the incidence relation is the collection of pairs

$$\{(G(\delta^0)e, e), (G(\delta^1)e, e) \mid e \in G[1]\} \subseteq G[0] \times G[1].$$

A morphism $f: U_1 \rightarrow U_2$ of undirected graphs is given by a pair of functions,

$$V(f): V(U_1) \rightarrow V(U_2) \quad \text{and} \quad E(f): E(U_1) \rightarrow E(U_2),$$

such that incidence is preserved; i.e.

$$(V(f) \times E(f)I(U_1)) \subseteq I(U_2).$$

Any $\underline{\underline{A}}$ -graph morphism $f:G_1 \rightarrow G_2$ "induces" such a pair

$$P(f):P(G_1) \rightarrow P(G_2)$$

of functions between undirected graphs since f commutes with origin and terminal. Hence, this construction induces a functor

$$P:[\underline{\underline{A}}^{\text{op}}, \text{Ens}] \rightarrow \mathcal{T}$$

from the category of $\underline{\underline{A}}$ -graphs to the category \mathcal{T} of undirected graphs.

The category \mathcal{T} is not easy to describe, since it depends on using, within category theory, the notation of a set with at most two elements (to describe the incidence relation $I(U)$). This difficulty illustrates a difference between the combinatorial and categorical viewpoints in graph theory. The categorical approach searches for general concepts which may be simply described and proceeds from there to particular or special examples; while the combinatorial approach assumes that such things as multiple edges or loops will be complicated. The "purity" of the categorical approach aside, it appears necessary to make some arbitrary choices in order to obtain a category to model \mathcal{T} . However, we are able to avoid this difficulty by using external relationships (which we expect to exist) between $[\underline{\underline{A}}^{\text{op}}, \text{Ens}]$ and \mathcal{T} to help "internally" construct \mathcal{T} .

The functor $P:[\underline{\underline{A}}^{\text{op}}, \text{Ens}] \rightarrow \mathcal{T}$ forgets orientation; hence it is reasonable to investigate the possibility that P has a right adjoint

$$O:\mathcal{T} \rightarrow [\underline{\underline{A}}^{\text{op}}, \text{Ens}].$$

Thus we search for objects OU in $[\underline{\underline{A}}^{\text{op}}, \text{Ens}]$ such that they agree with our "intuition" about the category \mathcal{T} and such that there is a natural equivalence

$$\psi(G,U):T(PG,U) \xrightarrow{\cong} [\underline{\underline{A}}^{\text{op}}, \text{Ens}](G,OU). \quad (3.1)$$

Every $\underline{\underline{A}}$ -graph G is determined by the sets $G[0]$ and $G[1]$ of vertices and edges, respectively, and by the functions $G(\delta^1)$ and $G(\delta^0)$ of origin and terminal. Since $[\underline{\underline{A}}^{\text{op}}, \text{Ens}]$ is a functor category, the Yoneda Lemma guarantees that these sets and functions can be naturally described in terms of representable functors; in particular,

$$G[i] = [\underline{\underline{A}}^{\text{op}}, \text{Ens}](\underline{\underline{A}}(_, [i]), G), \quad i = 0, 1$$

and

$$G(\delta^i) = [\underline{\underline{A}}^{\text{op}}, \text{Ens}](\underline{\underline{A}}(_, \delta^i), G), \quad i = 0, 1. \quad (3.2)$$

The object (actually functor)

$$V \equiv \underline{\underline{A}}(_, [0]):\underline{\underline{A}}^{\text{op}} \rightarrow \text{Ens}$$

which represents vertices has one vertex since $\underline{\underline{A}}([0], [0])$ consists only of the identity, and no edges since $\underline{\underline{A}}([1], [0])$ is empty. The object

$$E \equiv \underline{\underline{A}}(_, [1]):\underline{\underline{A}}^{\text{op}} \rightarrow \text{Ens}$$

which represents edges similarly has two vertices and one edge joining them. The morphisms (natural transformations)

$$B \equiv \underline{\underline{A}}(_, \delta^1):\underline{\underline{A}}(_, [0]) \rightarrow \underline{\underline{A}}(_, [1])$$

and

$$T \cong \underline{A}(_, \delta^0): \underline{A}(_, [0]) \rightarrow \underline{A}(_, [1])$$

from V to E select the origin and terminal vertices, respectively, of the unique edge in E . The subcategory \underline{A}' of $[\underline{A}^{OP}, \text{Ens}]$ consisting of objects V and E with morphisms B and T is isomorphic to \underline{A} .

For each graph U in T , we can determine the structure of OU by using both (3.1) and (3.2). The set of vertices of OU is $[\underline{A}^{OP}, \text{Ens}](V, OU)$ which is naturally isomorphic to $T(PV, U)$:

$$\begin{aligned} (OU)[0] &\cong [\underline{A}^{OP}, \text{Ens}](\underline{A}(_, [0]), OU) \\ &= [\underline{A}^{OP}, \text{Ens}](V, OU) \\ &\cong T(PV, U). \end{aligned}$$

Since graph V has one vertex and no edges, and $P: [\underline{A}^{OP}, \text{Ens}] \rightarrow T$ is a forgetful functor, PV is the graph in T with one vertex and no edges. A T -morphism $PV \rightarrow U$ is given by selecting a vertex of U to be the image of the unique vertex of PV . Thus $(OU)[0]$ is identified with the set of vertices of U .

Similarly, the edge set of OU is given by the set $T(PE, U)$. The \underline{A} -morphism $T: V \rightarrow E$ induces a map $T(PT, U): T(PE, U) \rightarrow T(PV, U)$. The undirected graph PE has one edge and two vertices; thus $T(PT, U)$ restricts a morphism with domain PE to one of these vertices. If vertex v of U is incident to edge e , then there is a T -morphism

$$f_e: PE \rightarrow U$$

such that the unique edge of PE is mapped to e and such that the vertex of PE designated by $PT: PV \rightarrow PE$ is mapped to v .

We expect that this description above would determine the morphism f_e uniquely. First, if e is a proper edge, then e has precisely one other vertex incident with it; and $f_e: PE \rightarrow U$ should have this vertex as the image of the second vertex of PE . Second, if e is a loop, then there is only one vertex v incident with e ; and both vertices of PE must be mapped to v . Note that these assignments establish a natural equivalence between

$$(OU)[1] \cong T(PE, U) \text{ and } I(U) \subseteq V(U) \times E(U).$$

In addition, the terminal map

$$(OU)(\delta^0): (OU)[1] \rightarrow (OU)[0]$$

is given by the natural projection of $I(U)$ on $V(U)$.

To complete the description of the directed graph OU , it is necessary to give the origin map

$$(OU)(\delta^1): (OU)[1] \rightarrow (OU)[0].$$

The argument given above to justify the identification of $T(PE, U)$ with $I(U)$ is a construction of $(OU)(\delta^1)$; i.e.

$$(OU)(\delta^1)(v, e) = \begin{cases} v'; & \text{if } (v', e) \in I(U), v' \neq v \\ v; & \text{otherwise.} \end{cases}$$

This construction uses the difficult to naturally describe notion "is incident with either one or two vertices". The existence of the adjoint relationship $\langle P, 0 \rangle$

demands that the construction of the origin be natural. Hence we will exploit the simple functor category $[\underline{A}^{OP}, \text{Ens}]$ of \underline{A} -graphs, use the assumed existence of an adjoint pair $\langle P, 0 \rangle$, and apply the general theory of monads to construct a category of algebras Γ which will be a functor category model $[\underline{X}^{OP}, \text{Ens}]$ for T .

As a first step, we give a description of the monad (or triple) $M = \langle T, \eta, \mu \rangle$ in $[\underline{A}^{OP}, \text{Ens}]$ which results from the assumed existence of the adjunction $\langle P, 0 \rangle$. For each \underline{A} -graph G , $TG \equiv OP(G)$ is described using the above discussion. Hence, $TG[0] = G[0]$ and $TG[1]$ is given as the pushout

$$\begin{array}{ccc}
 & G[1] & \xrightarrow{z_0} \\
 \lambda(G) & \swarrow & \searrow \\
 & G[1] & \xrightarrow{z_1} \\
 & & TG[1]
 \end{array} \tag{3.3}$$

where $\lambda(G) = \{e \in G[1] \mid G(\delta^0)(e) = G(\delta^1)(e)\}$, the set of loops of G . The universal properties of the pushout in Ens are used to give a description of origin $TG(\delta^1)$ and terminal $TG(\delta^0)$ to reflect the above construction of

$$T \equiv OP: [\underline{A}^{OP}, \text{Ens}] \rightarrow [\underline{A}^{OP}, \text{Ens}].$$

Thus $TG(\delta^0): TG[1] \rightarrow TG[0]$ is the composition of the identification $\gamma_G[0]: G[0] \xrightarrow{\cong} TG[0]$ with the function ω in

$$\begin{array}{ccccc}
 & & G(\delta^0) & & \\
 & & \curvearrowright & & \\
 & G[1] & \xrightarrow{z_0} & TG[1] & \xrightarrow{\omega} & G[0]; \\
 \lambda(G) & \swarrow & \searrow & \dashrightarrow & \\
 & G[1] & \xrightarrow{z_1} & TG[1] & \xrightarrow{\omega} & G[0]; \\
 & & \curvearrowleft & & \\
 & & G(\delta^1) & &
 \end{array} \tag{3.4}$$

while $TG(\delta^1): TG[1] \rightarrow TG[0]$ is defined similarly by interchanging $G(\delta^0)$ and $G(\delta^1)$ in (3.4).

The unit of the adjunction $\langle P, 0 \rangle$,

$$\eta: \text{Id} \rightarrow T: [\underline{A}^{OP}, \text{Ens}] \rightarrow [\underline{A}^{OP}, \text{Ens}],$$

is given by

$$\eta_G[0] \equiv \text{Id}: G[0] \rightarrow TG[0], \text{ and } \eta_G[1] \equiv z_0: G[1] \rightarrow TG[1]$$

of diagram (3.3), for each \underline{A} -graph G . Clearly, from the definition of $TG(\delta^0)$ and $TG(\delta^1)$, η_G is an \underline{A} -graph morphism natural in G .

The multiplication

$$\mu: T^2 \rightarrow T: [\underline{A}^{OP}, \text{Ens}] \rightarrow [\underline{A}^{OP}, \text{Ens}]$$

of the monad M is constructed from the counit $\varepsilon: PO \rightarrow \text{Id}: T \rightarrow T$ of the adjoint pair $\langle P, 0 \rangle$. For each graph U , $\varepsilon_U: PO(U) \rightarrow U$ is the identifying map on vertices, $V(\varepsilon_U) \equiv \text{Id}: V(PO(U)) \rightarrow V(U)$. The set $E(PO(U))$ of edges of $PO(U)$ is, from the above discussion, just the incidence relation $I(U) \subseteq V(U) \times E(U)$. A sketch of the construction of the equivalence

$$\psi(G, U): T(PG, U) \xrightarrow{\cong} [A^{OP}, \text{Ens}](G, OU)$$

leads to the function $E(\varepsilon_U): E(PO(U)) \rightarrow E(U)$ being the projection $I(U) \rightarrow E(U)$. The incidence relation $I(PO(U))$ in $PO(U)$ can be given as a pushout with a map to $I(K)$:

Diagram (3.5) is a commutative diagram with nodes $I(U)$ (top), $I(U)$ (bottom), and $I(POU)$ (middle). Solid arrows form a diamond shape: $I(U) \xrightarrow{\ell(U)} I(U)$ (left), $I(U) \xrightarrow{\ell(U)} I(POU)$ (top), $I(U) \xrightarrow{\tau_U} I(POU)$ (bottom), and $I(U) \xrightarrow{\tau_U} I(U)$ (right). Dashed arrows connect $I(POU)$ to the top and bottom $I(U)$ nodes.

where τ_U is a "twist" map which chooses the other element of $I(U)$ over the same edge in $E(U)$. In the context of the monad M , the above constructions lead to the pushout description (3.6) for the multiplication $\eta_G: T^2G \rightarrow TG$:

Diagram (3.6) is a commutative diagram with nodes $TG[1]$ (top), $TG[1]$ (bottom), and $T^2G[1]$ (middle). Solid arrows form a diamond shape: $TG[1] \xrightarrow{\ell(TG)} TG[1]$ (left), $TG[1] \xrightarrow{\ell(TG)} T^2G[1]$ (top), $TG[1] \xrightarrow{t} T^2G[1]$ (bottom), and $TG[1] \xrightarrow{t} TG[1]$ (right). Dashed arrows connect $T^2G[1]$ to the top and bottom $TG[1]$ nodes. A label $\mu_G[1]$ is placed above the dashed arrow from $T^2G[1]$ to $TG[1]$.

where $t: TG[1] \rightarrow TG[1]$ is defined by

Diagram (3.7) is a commutative diagram with nodes $G[1]$ (top), $G[1]$ (bottom), and $TG[1]$ (middle). Solid arrows form a diamond shape: $G[1] \xrightarrow{\ell(G)} G[1]$ (left), $G[1] \xrightarrow{\ell(G)} TG[1]$ (top), $G[1] \xrightarrow{z_1} TG[1]$ (bottom), and $G[1] \xrightarrow{z_0} TG[1]$ (right). Dashed arrows connect $TG[1]$ to the top and bottom $G[1]$ nodes. A label f is placed above the dashed arrow from $TG[1]$ to $TG[1]$.

Note that $t: TG[1] \rightarrow TG[1]$ together with $ld: G[0] \rightarrow G[0] \cong TG[0]$ on vertices does not define an \underline{A} -graph morphism, but $\mu_G = \langle ld, \mu_G[1] \rangle: T^2G \rightarrow TG$ is in $[\underline{A}^{op}, \text{Ens}]$.

The axioms for a monad (see MacLane [2], VI.1) are now easily verified; in particular, the constructions are clearly "natural in G ", $G \in [\underline{A}^{op}, \text{Ens}]$.

An algebra for the monad $M = \langle T, \eta, \mu \rangle$ is defined as an object X of the underlying category, which is $[\underline{A}^{op}, \text{Ens}]$ in this case, together with a morphism $h: TX \rightarrow X$ such that diagrams (3.8) commute:

Diagram (3.8) consists of two commutative diagrams. The left diagram has nodes T^2X (top), TX (middle), TX (bottom), and X (right). Solid arrows: $T^2X \xrightarrow{Th} TX$, $TX \xrightarrow{h} X$, $TX \xrightarrow{h} TX$, and $TX \xrightarrow{\mu_X} TX$. Dashed arrows: $T^2X \xrightarrow{\mu_X} TX$ and $TX \xrightarrow{h} X$. The right diagram has nodes X (top), TX (middle), and X (bottom). Solid arrows: $X \xrightarrow{\eta_X} TX$, $TX \xrightarrow{h} X$, and $X \xrightarrow{ld} X$. Dashed arrows: $X \xrightarrow{ld} TX$ and $TX \xrightarrow{h} X$.

Since $\eta_X[0] = ld: X[0] \rightarrow (TX[0] \cong X[0])$ and $\eta_X[1]: X[1] \rightarrow TX[1]$ is the map z_0 of (3.3), we construct $h: TX \rightarrow X$ with $h\eta_X = ld_X$ from the pushout (3.9):

Diagram (3.9) is a commutative diagram with nodes $X[1]$ (top), $X[1]$ (bottom), and $TX[1]$ (middle). Solid arrows form a diamond shape: $X[1] \xrightarrow{\ell(X)} X[1]$ (left), $X[1] \xrightarrow{\ell(X)} TX[1]$ (top), $X[1] \xrightarrow{z_1} TX[1]$ (bottom), and $X[1] \xrightarrow{b} TX[1]$ (right). Dashed arrows connect $TX[1]$ to the top and bottom $X[1]$ nodes. A label $h[1]$ is placed above the dashed arrow from $TX[1]$ to $X[1]$. A label $z_0 = \eta_X[1]$ is placed above the dashed arrow from $X[1]$ to $TX[1]$.

The requirement that $h: TX \rightarrow X$ be an \underline{A} -graph morphism forces the map $b: X[1] \rightarrow X[1]$ to reverse orientation (i.e., $X(\delta^i)b = X(\delta^{1-i}): X[1] \rightarrow X[0]$, $i = 0, 1$). Furthermore, the restriction $b/\ell(X)$ must be the "identity" inclusion $\ell(X) \rightarrow X[1]$. The commuting of the first diagram of (3.8) reduces, in this case, to the statement that

$b: X[1] \rightarrow X[1]$ has

$$b^2 \equiv \text{ld}: X[1] \rightarrow X[1];$$

i.e., b is an involution.

Thus the concept of graph with involution arises naturally from the concept of unoriented graph; in fact, it provides an algebraic realization of this concept. The category T_1 of M -algebras always has a pair of adjoint functors which induce the monad M in the base category.

In addition, the Kleisli construction uses the monad M to describe a category T_2 which is essentially the category of free algebras. The objects of T_2 are the same as those of $[A^{OP}, Ens]$, but there are additional morphisms in T_2 . Any pair of functions $f = \langle f_0, f_1 \rangle$, $f_0: G[0] \rightarrow G'[0]; f_1: G[1] \rightarrow G'[1]$ which preserve incidence (rather than origin and terminal) is a morphism of T_2 .

In general, the category of all algebras (in our case, T_1) is a terminal object in the category $Adj(M)$ of adjoint pairs which induce the monad M , and the Kleisli construction (in our case T_2) is initial in $Adj(M)$. The unique $\psi: T_2 \rightarrow T_1$ functor of MacLane [2] (VI.5, Theorem 3) is both full and faithful. Furthermore, every algebra in T_1 is isomorphic to a free algebra in $\psi(T_2)$; hence, the two categories are equivalent. Thus, T_1 and T_2 are, essentially, equally good models of T , and they are canonically chosen from all categories inducing the monad M .

The category T_2 is perhaps closer to our original idea of unoriented graphs T ; but the category T_1 has other advantages. Observe that A-graphs with involution can be naturally extended to a C-graph (object in $[C^{OP}, Ens]$) with C pictured in Fig. 5.

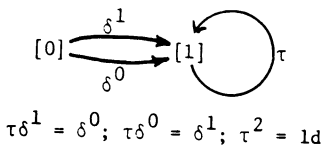


Figure 5. The category C

The category of algebras T_1 defined by the monad M is the full subcategory of $[C^{OP}, Ens]$ of C-graphs U satisfying $\{e | U(\tau)e = e\} = \{e | U(\delta^0)e = U(\delta^1)e\}$; i.e., the set of edges fixed by $U(\tau)$ is precisely the set of all loops of U . It is easy to show that T_1 is a reflective subcategory of $[C^{OP}, Ens]$. Construct the reflector, the left adjoint to the inclusion of T_1 into $[C^{OP}, Ens]$, by passing from a given C-graph U to a new C-graph U_1 , a quotient of U , having the same vertices as U , having the same edges between distinct vertices of U , but having $U(\tau)e$ identified with e for each loop $e \in \mathcal{L}(U)$.

Note that equivalence of categories is weaker than isomorphism. In particular, the comparison $\psi: T_2 \rightarrow T_1$ is not an isomorphism of categories, i.e., the object map $|\psi|: |T_2| \rightarrow |T_1|$ is not one-to-one and onto. However, the notion of equivalence appears to be more appropriate than isomorphism for modeling a theory. Hence, there is only one "useful" model up to equivalence, but two different approaches to that model.

We may also form a corresponding theory for B-graphs starting from an orientation-forgetful functor

$$p': [B^{OP}, Ens] \rightarrow T'.$$

The category of algebras T'_1 for a monad M' in $[\underline{B}^{OP}, \text{Ens}]$, with similar properties to the monad M in $[\underline{A}^{OP}, \text{Ens}]$, may be described by introducing an "involution". These algebras approximate the "involutorial graphs" of Ribenboim [5]. Of course, if a graph admits more than one involution, each involution defines a different algebra. (The original definition required only the existence of an involution in the hope of defining a subcategory of $[\underline{B}^{OP}, \text{Ens}]$; but the involution was incorporated into the algebraic structure in Ribenboim [6].) Curiously, the definition in Ribenboim [5,6] also requires that

$$\text{an edge fixed by involution be degenerate.} \tag{3.10}$$

The requirement (3.10) is incorporated naturally into the program developing the monad M' by having $H(\sigma^0): H[0] \leftrightarrow H[1]$ play the role of the inclusion $\lambda(G) \leftrightarrow G[1]$ of (3.3) in the definition of $T'H[1]$. With this change, $T'H(\sigma^0)$ is the common composition $H[0] \rightarrow T'H[1]$ in the analog of (3.3); and $T'H(\delta^1): T'H[1] \rightarrow T'H[0]$, $i = 0, 1$, is given as in (3.4). Again, $\eta'_H[1]: H[1] \rightarrow T'H[1]$ is the analog of z_0 of (3.3), and $\mu'_H[1]: T'T'H[1] \rightarrow T'H[1]$ is constructed as in (3.6) and (3.7). (Note that the mapping μ'_H , in general, is not an isomorphism (cf. Ribenboim [5], p. 159).)

The category T'_1 of all algebras for the monad M' in $[\underline{B}^{OP}, \text{Ens}]$ is the category $[\underline{D}^{OP}, \text{Ens}]$, where \underline{D} is depicted in Figure 6.

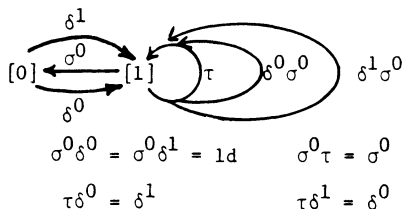


Figure 6. The category \underline{D}

The corresponding Kleisli construction yields a category T'_2 which is not equivalent to T'_1 ; rather the free algebras in T'_2 are characterized by (3.10). Thus the original definition of involutorial graph was designed to select a subcategory of T'_1 equivalent to T'_2 . It is also curious that (3.10) defines a subcategory of $T'_1 \equiv [\underline{D}^{OP}, \text{Ens}]$ which is both reflective and coreflective.

Also, consider the Kan extensions of the functors induced by the inclusions $\underline{B} \hookrightarrow \underline{D}$ and $\underline{A} \hookrightarrow \underline{C}$. In particular, the forgetful functor $U: [\underline{D}^{OP}, \text{Ens}] \rightarrow [\underline{B}^{OP}, \text{Ens}]$ and its left adjoint (i.e., left-Kan extension) induce the monad M .

4. CATEGORIES OF GRAPHS ARE CARTESIAN CLOSED

A benefit of viewing a category of graphs as a functor category is that any functor category $[\underline{X}^{OP}, \text{Ens}]$, for small \underline{X} is cartesian closed (see Freyd [9], p. 8). The proof of this result is constructive; i.e. an algorithm is provided for computing the "internal-Hom" functors. If Y, Z are objects in $[\underline{X}^{OP}, \text{Ens}]$, then $\text{Hom}(Y, Z)$ is also an object in $[\underline{X}^{OP}, \text{Ens}]$, whose evaluation at object p of \underline{X} is defined by:

$$\text{Hom}(Y, Z)(p) \equiv [\underline{X}^{OP}, \text{Ens}](\underline{X}(_, p) \times Y, Z). \tag{4.1}$$

Thus, for our examples, we give the representable functors

$$\underline{X}(_, [i]): \underline{X}^{OP} \rightarrow \text{Ens}$$

for $i = 0, 1$ and $X = A, B, C$ or D . Figure 7 depicts the directed graphs given by these representable functors.

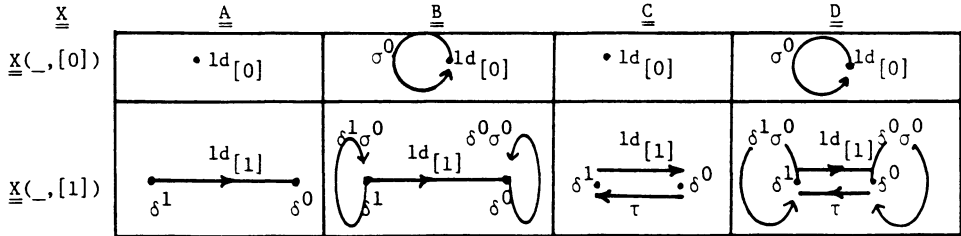


Figure 7. Representable functors

As an example of the use of formula (4.1), we construct $\text{Hom}(Y, Z)$ in $[\underline{A}^{OP}, \text{Ens}]$. The set of vertices is given by

$$\text{Hom}(Y, Z)[0] \cong [\underline{A}^{OP}, \text{Ens}](\underline{A}(_, [0]) \times Y, Z).$$

But $\underline{A}(_, [0])$ is a graph with one vertex and no edges. Since products are computed "coordinatewise", $\underline{A}(_, [0]) \times Y$ has vertices $Y[0]$ and no edges. Thus

$$\text{Hom}(Y, Z)[0] \cong \text{Ens}(Y[0], Z[0]).$$

The graph $\underline{A}(_, [1])$ has one edge joining two vertices. Thus $\underline{A}(_, [1]) \times Y$ has the same number of edges as Y , but twice as many vertices. The structural maps satisfy

$$\delta^1(\langle \text{Id} \rangle, e) = (\langle \delta^1 \rangle, \delta^1 e).$$

Thus $\text{Hom}(Y, Z)[1]$ is identified with the set of all triples of functions $\langle r: Y[1] \rightarrow Z[1], s: Y[0] \rightarrow Z[0], t: Y[0] \rightarrow Z[0] \rangle$ for which diagrams (4.2) commute

$$\begin{array}{ccc}
 Y[1] & \xrightarrow{r} & Z[1] \\
 \downarrow Y(\delta^1) & & \downarrow Z(\delta^1) \\
 Y[0] & \xrightarrow{s} & Z[0]
 \end{array}
 \quad
 \begin{array}{ccc}
 Y[1] & \xrightarrow{r} & Z[1] \\
 \downarrow Y(\delta^0) & & \downarrow Z(\delta^0) \\
 Y[0] & \xrightarrow{t} & Z[0]
 \end{array}
 \tag{4.2}$$

The origin, $\text{Hom}(Y, Z)(\delta^1)$, is given by

$$\langle r, s, t \rangle \mapsto s$$

the terminal, $\text{Hom}(Y, Z)(\delta^0)$, by

$$\langle r, s, t \rangle \mapsto t.$$

EXAMPLE. If Y is discrete, i.e. $Y[1] = \emptyset$, then $\text{Hom}(Y, Z)$ has all functions $Y[0] \rightarrow Z[0]$ as vertices and has a unique edge joining each pair of vertices.

In general, the set of edges joining points $s, t \in \text{Hom}(Y, Z)[0]$ is given by the set of functions $r: Y[1] \rightarrow Z[1]$ satisfying (4.2). Although this construction satisfies the adjoint property (4.1), it seems to have little relation to graph homomorphisms.

Next, consider $[\underline{B}^{OP}, \text{Ens}]$. From Figure 7, it is clear that $\underline{B}(_, [0])$ has one vertex and one edge; so that

$$\underline{B}(_, [0]) \times Y \cong Y.$$

Thus, $\text{Hom}(Y, Z)[0]$ is the set of \underline{B} -graph morphisms:

$$\begin{aligned} \text{Hom}(Y, Z)[0] &\equiv [\underline{B}^{\text{OP}}, \text{Ens}] (\underline{B}(_, [0]) \times Y, Z) \\ &\cong [\underline{B}^{\text{OP}}, \text{Ens}] (Y, Z). \end{aligned}$$

Each "edge" of $\text{Hom}(Y, Z)[1]$ is represented by a triple

$\langle r: Y[1] \rightarrow Z[1]; s: Y \rightarrow Z; t: Y \rightarrow Z \rangle$, where s and t are \underline{B} -graph morphisms for which (4.3) is commutative:

$$\begin{array}{ccccc} Y[0] & \xleftarrow{Y(\delta^1)} & Y[1] & \xrightarrow{Z(\delta^0)} & Y[0] \\ \downarrow s[0] & & \downarrow r & & \downarrow t[0] \\ Z[0] & \xleftarrow{Z(\delta^1)} & Z[1] & \xrightarrow{Z(\delta^0)} & Z[0]. \end{array} \tag{4.3}$$

As before,

$$\langle r, s, t \rangle \mapsto s$$

gives the origin of $\text{Hom}(Y, Z)$ and

$$\langle r, s, t \rangle \mapsto t$$

defines the terminal.

If s and t are given, then the set of edges of $\text{Hom}(Y, Z)$ with origin s and terminal t is indexed by those functions $r: Y[1] \rightarrow Z[1]$ satisfying (4.3). The definition of Hom graph in Ribenboim [5], p. 165, was in this spirit. However, for no apparent reason, only the restriction of r to $Y(\sigma^0)Y[0]$ entered into that definition. In Ribenboim [6], p. 110, a totally different definition was given. Again, it was an explicit construction, and again there was no claim of naturality. The idea was that the edges should be indexed by the functions $r: Y[1] \rightarrow Z[1]$ of our definition. The difficulty is that (4.3) determines only $s[0]$ and $t[0]$, so that the vertices could only be the equivalence classes of "graph homomorphisms restricting to the same functions on vertices". By contrast, our construction is no more cumbersome than these two attempts, but the underlying general principle is quite simple and guarantees the Hom will have the proper adjoint relation to cartesian product.

Our internal Hom-functor has the usual properties of Hom in Ens in particular, there is a composition

$$\circ: \text{Hom}(Y, Z) \times \text{Hom}(W, Y) \rightarrow \text{Hom}(W, Z). \tag{4.4}$$

The realizations of vertices in $\text{Hom}(Y, Z)[0]$ as graph homomorphisms and of $\text{Hom}(Y, Z)[1]$ as triples $\langle r, s, t \rangle$ are compatible with the composition (4.4).

The composition (4.4) induces a natural monoid structure on $\text{Hom}(W, W)$. If the invertible elements of $\text{Hom}(W, W)$ are selected, the resulting subgraph is a group called $\text{Aut}(W)$.

The various reflective subcategories constructed in Section 2 and Section 3 in the discussion of categories of graphs could be cartesian closed. In fact, the following proposition gives an easy computation of Hom in many cases.

PROPOSITION. Suppose Γ is a cartesian closed category and $I: \Gamma' \rightarrow \Gamma$ is a full reflective inclusion with $R: \Gamma \rightarrow \Gamma'$ the reflector (i.e., $\langle R, I \rangle$ adjoint pair). Then R preserves products iff for all $A \in \Gamma$ and $B \in \Gamma'$, $\text{Hom}(A, B)$ is in Γ' .

PROOF. See Freyd [9; p. 13].

REMARK. The reflective subcategory A_1 of $[\underline{A}^{\text{OP}}, \text{Ens}]$ constructed in Section 2 does not satisfy this property; however A_2 does.

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