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INCLUSIONS OF HARDY ORLICZ SPACES

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ABSTRACT. Let ϕ be a continuous positive increasing function defined on $[0,\infty)$ such that $\phi(x + y) \leq \phi(x) + \phi(y)$ and $\phi(0) = 0$. The Hardy-Orlicz space generated by ϕ is denoted by $H(\phi)$. In this paper, we prove that for $\phi \neq \psi$, if $H(\phi) = H(\psi)$ as sets, then $H(\phi) = H(\psi)$ as topological vector spaces. Some other results are given.

KEY WORDS AND PHRASES. Modulus function, Orlicz spaces.
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1. INTRODUCTION.

Let $\phi:[0,\infty) \longrightarrow [0,\infty)$ be a continuous increasing function such that $\phi(x + y) \leq \phi(x) + \phi(y)$ and $\phi(0) = 0$. Let T be the unit circle, and m be the Lebesgue measure on T. A complex valued measurable function f defined on T is called ϕ-integrable if $\int_I \phi|f(t)|dm(t) < \infty$. The space of all ϕ-integrable functions on T will be denoted by $L(\phi)$. This space was first introduced by Orlicz, [8]. Subsequent papers were written to study different aspects of $L(\phi)$. Examples of these papers are Cater, [4], Gramsch, [5] and Pallashke [9].

In [6] and [7], Lesniewicz introduced the so called Hardy-Orlicz spaces $H(\phi)$ for a given such function ϕ. The space $H(\phi)$ was defined to be the space of all functions $f \in L(\phi)$ such that f is the radial limit of some function g analytic in the open unit disc and belongs to the Nevalinna class N. The relation between different $H(\phi)$-spaces was studied by Deeb, Khalil and Marzug [3]. In this paper, we show that the inclusion map between two $H(\phi)$-spaces is always continuous. Some other results are given. It should be remarked that in the work of Lesniewicz, [6], [7] and many other authors, ϕ is assumed to be a ϕ-convex function. In this paper it is not assumed so.

2. PRELIMINARIES AND NOTATIONS.

A function $\phi:[0,\infty) \longrightarrow [0,\infty)$ is called a modulus function if

(i) ϕ is continuous and increasing

(ii) $\phi(x) = 0$ if and only if $x = 0$

(iii) $\phi(x + y) \leq \phi(x) + \phi(y)$.

The functions $\phi(x) = x^p$, $0 < p \leq 1$ and $\phi(x) = \ln(1 + x)$ are examples of modulus functions. Further, if ϕ_1 and ϕ_2 are modulus functions, then $\phi_1 + \phi_2$ and $\phi_1 \circ \phi_2$

are modulus functions. Further, $\psi = \frac{\phi}{1+\phi}$ is a modulus function if ϕ is.

Let $T = \{z: |z| = 1\}$, $\Delta = \{z: |z| < 1\}$. The space of analytic functions on Δ is denoted by $H(\Delta)$. Let $H^+(\Delta) = \{f \in H(\Delta): \lim_{r\to 1} f(re^{i\theta})$ exists a.e.$\theta\}$. We will consider $H^+(\Delta)$ as a space of functions on T. For a given modulus function ϕ, we define:

$$H(\phi) = \{f \in H^+(\Delta): \sup_{0 \le r < 1} \int_0^{2\pi} \phi|f(re^{i\theta})|d\theta = \int_0^{2\pi} \phi|f(e^{i\theta})|d\theta < \infty\}.$$

The function $d: H(\phi) \times H(\phi) \longrightarrow [0,\infty)$, $d(f,g) = \int_0^{2\pi} \phi|f(e^{i\theta}) - g(e^{i\theta})|d\theta$ defines a metric on $H(\phi)$, under which $H(\phi)$ becomes a topological vector space. If one assumes that $\phi|u|$ is subharmonic for $u \in H(\Delta)$, then $H(\phi)$ turns out to be complete [2]. For $f \in H(\phi)$, we write $||f||_\phi = \int_T \phi|f(e^{i\theta})|d\theta$. If $\phi(x) = x^p$, $0 < p \le 1$, then $H(\phi) = H^p$ and for $\phi(x) = \ln(1+x)$, $H(\phi) = N^+ = \{f \in N: \int_T \ln(1+|f|) < \infty\}$, where N is the Nevalinna class.

3. I: $H^1 \longrightarrow H(\phi)$ IS CONTINUOUS.

In [2], it was shown that $H^1 \subseteq H(\phi)$ for all modulus functions ϕ. The authors in [3] were not able to show that the inclusion map $I: H^1 \longrightarrow H(\phi)$ is continuous. In this section we prove that $I: H^1 \longrightarrow H(\phi)$ is continuous. Some other related questions are discussed.

THEOREM 2.1. Let ϕ and ψ be two modulus functions such that $\lim_{x\to\infty} \frac{\phi(x)}{\psi(x)} = \lambda$ exists. Then:

(i) $H(\phi) = H(\psi)$ if $\lambda \neq 0$ and λ is finite

(ii) $H(\phi) \subseteq H(\psi)$ if $\lambda = 0$

(iii) $H(\psi) \subseteq H(\phi)$ if $\lambda = \infty$.

PROOF. (i) Let $\lambda \neq 0$ be finite. Then there exists $a_1, b_1, a_2, b_2 \in [0,\infty)$ such that

$\phi(x) \le a_1\psi(x)$ for $x \in [a_2,\infty)$ ··· (*)

$\psi(x) \le b_1\phi(x)$ for $x \in [b_2,\infty)$ ··· (**).

Let $f \in H(\psi)$. Set $E(a_2) = \{t \in T : |f(t)| \ge a_2\}$. Then

$$||f||_\phi = \int_{E(a_2)} \phi|f(e^{i\theta})|d\theta + \int_{E^c(a_2)} \phi|f(e^{i\theta})|d\theta$$

$$\le a_1 ||f||_\psi + \phi(a_2) < \infty.$$

Hence $f \in H(\phi)$ and $H(\psi) \subseteq H(\phi)$. Similarly we show $H(\phi) \subseteq H(\psi)$. Consequently, $H(\phi) = H(\psi)$. Case (ii) and (iii) are proved similarly and details are omitted. This ends the proof.

THEOREM 2.2. Let $\lim_{x\to\infty} \frac{\phi(x)}{\psi(x)} = \lambda > 0$. Then the inclusion map $I: H(\phi) \longrightarrow H(\psi)$ is continuous.

PROOF. From the proof of Theorem 2.1, there exists $a,b > 0$ such that $||f||_\psi \le \psi(a) + b\ ||f||_\psi$ for all $f \in H(\phi)$.

Let $f_n \longrightarrow 0$ in $H(\phi)$. Thus the sequence (f_n) is bounded in the metric of $H(\phi)$ and consequently bounded in $H(\psi)$. If possible let there exist a subsequence (f_{n_k})

such that $\|f_{n_k}\| \to \alpha > 0$. Since $\|f_{n_k}\|_\phi \longrightarrow 0$, (f_{n_k}) has a subsequence which converges pointwise to the zero function. With no loss of generality, we can assume that $f_{n_k} \longrightarrow 0$ a.e. Another application of the proof of Theorem 2.1, yields $\psi(x) \leq \psi(a) + b \cdot \phi|x|$ for all $x \in [0,\infty)$. Hence

$$\psi\, |f_{n_k}(t)| \leq \psi(a) + b \cdot \phi|f_{n_k}(t)| \; .$$

The sequence of functions $g_{n_k} = \psi(a) + b\; \phi|f_{n_k}|$ converges a.e. to $\psi(a)$ and

$$\int_T g_{n_k}(t)dt \longrightarrow \psi(a).$$

Consequently, by the generalized Lebesgue convergence theorem, [10], we have

$$\lim_{n_k} \int_T \psi|f_{n_k}(t)|dt = \int_T \lim_{n_k} \psi|f_{n_k}(t)|dt = 0.$$

This is a contradiction. Thus, the point $w = 0$ is the only limit point of the bounded sequence $(\|f_n\|_\psi)$. Consequently, [11], the sequence $\|f_n\|_\psi$ converges to zero. Hence $I\colon H(\phi) \longrightarrow H(\psi)$ is continuous. This ends the proof.

COROLLARY 2.3. If $\lim_{x\to\infty} \frac{\phi(x)}{\psi(x)} = \lambda \in (0,\infty)$, then $H(\phi) = H(\psi)$ as topological vector spaces.

PROOF. By Theorem 2.1, $H(\phi) = H(\psi)$ as sets. Theorem 2.2 implies that $I\colon H(\phi) \longrightarrow H(\psi)$ is an isomorphism. This ends the proof.

A linear map $A\colon H(\phi) \longrightarrow H(\psi)$ is called metrically bounded if $\|Af\|_\psi \leq \lambda\, \|f\|_\phi$ for all $f \in H(\phi)$ and some $\lambda > 0$. Clearly every metrically bounded map is continuous. The converse need not be true. However, for the inclusion map, we have the following:

THEOREM 2.4. Let ϕ be any modulus function. Then there exists $\lambda > 0$ such that for all $f \in H^1$, $\|f\|_1 \geq 1$, $\|f\|_\phi \leq \lambda\, \|f\|_1$.

PROOF. It is know, [2] (and easy to show) that $H^1 \subseteq H(\phi)$ for all modulus functions ϕ. If $f \in H^1$ and $\|f\|_1 = 1$, then using the argument in Theorem 2.1, we have $\|f\|_\phi \leq \lambda\|f\|_1$.

Let $f \in H^1$, $\|f\|_1 > 1$. Then there exists $0 < \alpha < 1$ such that $\|\alpha f\|_1 = 1$. Since $\alpha < 1$, there exists a natural number n such that $\frac{1}{n+1} \leq \alpha \leq \frac{1}{n}$. Hence

$$\|\alpha f\|_\phi \leq \lambda\, \|\alpha f\|_1 = \lambda\alpha\, \|f\|_1 \; .$$

But $\|\frac{1}{n+1} f\|_\phi \leq \|\alpha f\|_\phi$, and $\|\frac{1}{k} f\|_\phi \geq \frac{1}{k}\|f\|_\phi$ for any modulus function ϕ. It follows that:

$$\frac{1}{n+1}\, \|f\|_\phi \leq \lambda \cdot \alpha\, \|f\|_1 \leq \frac{\lambda}{n}\, \|f\|_1 \; ,$$

and consequently

$$\|f\|_\phi \leq \lambda\, \frac{n+1}{n}\, \|f\|_1 \leq 2\lambda\, \|f\|_1 \, .$$

This ends the proof.

THEOREM 2.5. Let ϕ be a given modulus function such that $H^1 = H(\phi)$. If metric and topological bounded sets coincide in $H(\phi)$, then $\|f\|_1 \leq \lambda\|f\|_\lambda$ for all $f \in H(\phi)$, $\|f\|_\phi \leq 1$ for some $\lambda > 0$.

PROOF. Applying Corollary 2.3, $I: H(\phi) \to H^1$ is an isomorphism of topological vector spaces. If possible, let $\| f \|_1 \leq \lambda \| f \|_\phi$ be not true on the unit sphere of $H(\phi)$. Then, for each n, there exists $f_n \in H(\phi)$, $\| f_n \|_\phi = 1$ such that

$$\| f_n \|_1 \geq n \| f_n \|_\phi = n$$

Consider the sequence $\frac{f_n}{n} = g_n$. By the assumption on bounded sets of $H(\phi)$, we have, [12], $g_n \to 0$ in $H(\phi)$. But $\| g_n \|_1 = \| \frac{f_n}{n} \|_1 \geq 1$ for all n. This contradicts the continuity of the identity map $I: H(\phi) \to H^1$. Hence there exists $\lambda > 0$ such that:

$$\| f \|_1 \leq \lambda \| f \|_\phi \quad \ldots \qquad (*) ,$$

for all $f \in H(\phi)$, $\| f \|_\phi = 1$.

Let $f \in H(\phi)$, $\| f \|_\phi < 1$. Consider the map $K: [0,\infty) \to [0,\infty)$, $K(t) = \| tf \|_\phi$. It can be easily seen that K is continuous. Hence there exists $a > 1$ such that $K(a) = 1$. Thus for every $f \in H(\phi)$, $\| f \|_\phi < 1$, we can find $a > 1$ such that $\| af \|_\phi = 1$. Hence, from equation $(*)$, we get:

$$\| af \|_1 \leq \lambda \| af \|_\phi \leq 2a\lambda \| f \|_\phi .$$

Consequently, $\| f \|_1 \leq 2\lambda \| f \|_\phi$. This end the proof.

4. FURTHER RESULTS

The concept of metrically bounded linear operator was introduced in Section 3. A linear map $A: H(\phi) \to H(\psi)$ is called metrically bounded if there exists $\lambda \in (0,\infty)$ such that $\| Af \|_\psi \leq \lambda \| f \|_\phi$. In general, a continuous linear map need not be metrically bounded. In this section we prove a result which is a generalization of Theorem 3.1 in [3].

THEOREM 4.1. Let ϕ and ψ be any two modules functions. Then the following are equivalent:

(i) $\lim\limits_{x\to 0} \frac{\phi(x)}{\psi(x)} = \delta$, $\lim\limits_{x\to \infty} \frac{\phi(x)}{\psi(x)} = \varepsilon$, for some $\varepsilon,\delta \in (0,\infty)$.

(ii) $H(\phi) = H(\psi)$, and the identity map I is metrically bounded.

PROOF. (i) $\to$ (ii). From the assumption in (i), one can choose a and b in $(0,\infty)$ such that

$$\frac{\phi(x)}{\psi(x)} \geq r \quad \text{on} \quad [0,a]$$

$$\frac{\phi(x)}{\psi(x)} \geq s \quad \text{on} \quad (b,\infty)$$

for some $r,s \in (0,\infty)$. Theorem 3.2 implies that $H(\phi) = H(\psi)$.

Let $f \in H(\phi)$. Consider the following sets:

$$E(a) = \{t: \; 0 \leq |f(e^{it})| < a\}$$
$$E(b) = \{t: \; |f(e^{it})| > b\}$$
$$E(a,b) = \{t: \; a \leq |f(e^{it})| \leq b\} .$$

Then:

$$\| f \|_\psi = \int_{E(a)} \psi|f(e^{it})|\,dt + \int_{E(a,b)} \psi|f(e^{it})|\,dt + \int_{E(b)} \psi|f(e^{it})|\,dt$$
$$\leq \frac{1}{r}\| f \|_\phi + \int_{E(a,b)} \psi|f(e^{it})|\,dt + \frac{1}{s}\| f \|_\phi .$$

On the closed interval $[a,b]$, the continuity of $\frac{\phi(x)}{\psi(x)}$ implies the existence of $\lambda > 0$ such that $\psi(x) \leq \lambda\phi(x)$. Hence

$$\int_{E(a,b)} \psi|f(e^{it})|\,dt \leq \frac{1}{\lambda}\| f \|_\phi .$$

Thus, $\| f \|_\psi \leq \beta \| f \|_\phi$ where $\beta = \max(\frac{1}{r},\frac{1}{s},\frac{1}{\lambda})$. In a similar way one can show that $\| f \|_\phi \leq \gamma \| f \|_\phi$ for all $f \in H(\phi) = H(\psi)$. Hence the identity map is metrically bounded.

Conversely, (ii) $\rightarrow$ (i). Assume $H(\phi) = H(\psi)$ and $I: H(\phi) \leftrightarrow H(\psi)$ is metrically bounded. Then there exists α and β in $(0,\infty)$ such that

$$\| f \|_\phi \leq \alpha \| f \|_\psi \leq \| f \|_\phi .$$

Hence $\frac{\alpha}{\beta} \leq \frac{\| f \|_\phi}{\| f \|_\psi} \leq \alpha$ for all $f \in H(\phi) = H(\psi)$. Consider the function $f(z) = xz$ for $z = e^{it}$, $x \in (0,\infty)$. Then

$$\| f \|_\phi = \phi(x) \quad \text{and} \quad \| f \|_\psi = \psi(x) .$$

Consequently $\frac{\alpha}{\beta} \leq \frac{\phi(x)}{\psi(x)} \leq \alpha$. Since $\alpha,\beta \in (0,\infty)$, (i) then follows. This end the proof.

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