QUANTUM RELATIVISTIC TODA CHAIN

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Received 19 September 2000 and in revised form 16 March 2001

Investigated is the quantum relativistic periodic Toda chain, to each site of which the ultra-local Weyl algebra is associated. Weyl's q we are considering here is restricted to be inside the unit circle. Quantum Lax operators of the model are intertwined by six-vertex R-matrix. Both independent Baxter's Q-operators are constructed explicitly as seria over local Weyl generators. The operator-valued Wronskian of Qs is also calculated.

1. Introduction

Long ago, Baxter in his famous papers [1, 2, 3, 4, 5] has introduced the object which is known now as Q-operator. This operator does satisfy the so-called Baxter (or $T-Q$) equation and besides has many interesting properties. Recently Q-operator was intensively discussed in the series of papers [7, 8] in connection with continuous quantum field theory. In [14, 21] it was pointed out the relation of Q-operator with quantum Bäklund transformations. In [17], was discovered the relation of Q-operator with Block solutions of quantum linear problem.

Q-operator was used initially for the solution of the eigenvalue problem of XYZ-spin chain, where usual Bethe ansatz fails. The reason is that $T-Q$ equation, together with an appropriate boundary conditions, provides a one-dimensional multiparameter spectral problem which allows one to determine the spectra of both the auxiliary transfer matrix T and the operator Q. In the case of the quantum mechanical integrable chains (e.g., the periodic Toda chain) the appropriate solution of the Baxter equation plays the prominent role in the functional Bethe ansatz and the quantum separation of variables.

Journal of Applied Mathematics 1:2 (2001) 47–68
2000 Mathematics Subject Classification: 82B23, 37K15
URL: http://jam.hindawi.com/volume-1/S1110757X01000055.html

Thus the reader should distinguish two approaches to Q-operator: the first one is Baxter's functional equation as the spectral problem, while the second one is the investigation of Q, obeying Baxter's equation, as the operator defined in terms of quantum observables. This paper is devoted to the investigation of Q as an *operator*.

In quite recent papers explicit constructions of Q-operators were obtained for several models, like the isotropic Heisenberg spin chain, [18], and the periodic Toda chain and other models with the rational R-matrix, [17]. In these papers Q-operator was obtained as the trace of monodromies of the appropriate local operators. It is well known that with free boundary conditions for Q, $T-Q$ equation provides a one-parametric family of solutions, so that one may extract two independent solutions with nonzero discrete Wronskian (see [7, 8, 19]). In [17, 18] both independent Q operators were obtained for the models considered.

In this paper, we investigate the exactly integrable model known as "quantum relativistic Toda chain," [13, 15, 20]. Local L operator for the model is constructed with the help of the Weyl algebra generators, commuting on q, and we deal with the case $|q| < 1$. Here, we do not consider the Jacoby partners to the Weyl algebra, dealing thus with the compact q-dilogarithms (investigation of the modular formulation of the quantum relativistic Toda chain is the subject of the forthcoming paper, [12]). Quantum space of our model is a formal module of an enveloping of the tensor product of several copies of Weyl algebras. The only thing we suggest for the Weyl generators is their invertibility and a q-equidistant spectrum for one of them. Both independent operators Q_+ and Q_- and their Wronskian are calculated locally as the operators acting in the ultra-local Weyl algebra. *Actually all our results are to be understood as the well-defined series expansions for functions from the enveloping mentioned.*

2. The model and the results

This section consists of two parts. We formulate the model first, actually just defining the transfer matrix, and then we give the final formulae for $Q_\pm$ operators and their q-Wronskian. All the sections beyond the introduction are the QUISM-type derivation of these results.

2.1. Problem

First of all, we define the relativistic Toda chain L-operator, associated with fth site of a chain, as

$$L_f(x) = \begin{pmatrix} x\mathbf{u}_f - (x\mathbf{u}_f)^{-1} & \boldsymbol{v}_f \\ q^{-1/2}\lambda\boldsymbol{v}_f^{-1} & 0 \end{pmatrix}, \tag{2.1}$$

where $x \in \mathbb{C}$ is the spectral parameter, $\lambda \in \mathbb{C}$ is an extra parameter, common for all sites (i.e., λ is a module), the set of elements $\mathbf{u}_f$, $\mathbf{v}_f$ form the "half-integer" ultra-local Weyl algebra:

$$\mathbf{u}_f \cdot \mathbf{v}_f = q^{1/2} \mathbf{v}_f \cdot \mathbf{u}_f, \tag{2.2}$$

and elements with different fs commute. As usual, the whole quantum space is the tensor product of some copies of Weyl modules, and f marks the "number" of given Weyl algebra in this tensor product. Recall, we will always imply $|q| < 1$.

The correspondence between the relativistic Toda chain and usual Toda chain may be established, for example, in the following parameterization:

$$\begin{gathered} q = e^{-i\epsilon}, \qquad \lambda = -\epsilon^2, \qquad x = e^{\epsilon\theta/2}, \\ \mathbf{u}_f = e^{-\epsilon \mathbf{p}_f/2}, \qquad \mathbf{v}_f = \epsilon e^{\mathbf{q}_f}, \end{gathered} \tag{2.3}$$

where

$$[\mathbf{p}, \mathbf{q}] = i, \tag{2.4}$$

in the limit

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} L_f(x) = \begin{pmatrix} \theta - \mathbf{p}_f & e^{\mathbf{q}_f} \\ -e^{-\mathbf{q}_f} & 0 \end{pmatrix}. \tag{2.5}$$

The right-hand side of this relation is known as the L-operator for the quantum Toda chain.

In the L-operator as well as in all other objects the spectral parameter x will always couple with $\mathbf{u}_f$. So we introduce the useful notation

$$x^2 \mathbf{u}_f^2 \stackrel{\text{def}}{=} q^{\mathbf{s}_f}, \tag{2.6}$$

so that for any formal function $g(\mathbf{s}_f)$,

$$g(\mathbf{s}_f) \cdot \mathbf{v}_f^n = \mathbf{v}_f^n \cdot g(\mathbf{s}_f + n) \quad \forall n. \tag{2.7}$$

We define the transfer matrix for the chain with F sites, $f = 1, \ldots, F$, as

$$T(x^2) = \left((-x)^F \prod_f \mathbf{u}_f \right) \cdot \operatorname{tr} \left(L_1(x) \cdot L_2(x) \cdots L_F(x) \right). \tag{2.8}$$

The matrix $T(x^2)$ becomes a polynomial of x^2 with operator-valued coefficients:

$$T(x^2) = \sum_{j=0}^{F} \left(-x^2 \right)^{F-j} t_j. \tag{2.9}$$

Here it is implied that $t_F = 1$ and

$$t_0 = \prod_f \mathbf{u}_f^2. \tag{2.10}$$

Later we will argue that $\{t_j\}$ is the commutative set. Equivalently, this means that

$$T(x^2)T(y^2) = T(y^2)T(x^2) \quad \forall x, y. \tag{2.11}$$

Note that apart from the trivial $t_F = 1$ all other F coefficients are independent. For a given set $\{t_j\}$ one can define another set $\{\overline{t_j}\}$ by

$$\overline{t_j} = t_0^{-1} t_{F-j}. \tag{2.12}$$

This means simply that

$$T(x^2) = (-x^2)^F t_0 \overline{T}(x^{-2}). \tag{2.13}$$

Baxter's operator $Q(x^2)$, by the definition, is an operator commuting with the set of t_j,

$$Q(x^2)T(y^2) = T(y^2)Q(x^2) \quad \forall x, y, \tag{2.14}$$

and obeying the Baxter $T-Q$ relation

$$T(x^2)Q(x^2) = ((-\lambda x^2)^F t_0)Q(qx^2) + Q(q^{-1}x^2), \tag{2.15}$$

where t_0 is given by (2.10). The model-dependent coefficients of $Q(qx^2)$ and $Q(q^{-1}x^2)$ in the right-hand side of (2.15) are the subject of separate calculations, and this particular form will be argued later in this paper.

In what follows, we will see that with this normalization of the coefficients in (2.15) the Baxter equation has a solution entire on x^2. We will call this solution

$$Q_+(x^2) = J(x^2, \lambda, \{t\}). \tag{2.16}$$

Proposition 2.1. *The entire on x^2 solution of (2.15) as a series on λ^F is*

$$J(x^2, \lambda, \{t\}) = \left(\prod_{k=1}^{\infty} T(q^k x^2)\right) \cdot \left(\sum_{k=0}^{\infty} (-\lambda^F)^k c_k(x^2)\right), \tag{2.17}$$

where $c_{-1} \equiv 0$, $c_0 \equiv 1$, *and recursively*

$$c_k(x^2) = \sum_{j=1}^{\infty} \frac{(q^j x^2)^F c_{k-1}(q^{1+j}x^2)}{T(q^j x^2)T(q^{1+j}x^2)}. \tag{2.18}$$

Note that $J(x^2,\lambda,\{t\})$ is the entire function on all its arguments. The proof of this proposition is a rather simple exercise.

The other solution $Q_-(x^2)$ must contain a cut with respect to the variable x, and up to this cut we guess $Q_-(x^2)$ to be entire on x^{-2}. More exactly, with the $\mathbf{s}_f$-notation introduced by (2.6), one may check that

$$Q_-\left(x^2\right)=\lambda^{-\sum_f \mathbf{s}_f}\cdot J\left(x^{-2},\lambda,\{\bar{t}\}\right) \tag{2.19}$$

also solves (2.15). Obviously, Q_+ and Q_- may be considered as two independent solutions of (2.15). The last definition we need is the q-Wronskian of these two solutions

$$W\left(x^2\right)\stackrel{\text{def}}{=}Q_+\left(q^{-1}x^2\right)Q_-\left(x^2\right)-Q_+\left(x^2\right)Q_-\left(q^{-1}x^2\right). \tag{2.20}$$

2.2. Solution

In this paper, we give explicit expressions for both functions $Q_\pm$. The natural question arises: we have got yet the form (2.17) and (2.18), what one may otherwise do. The aim of this paper is to investigate the relativistic Toda chain by QUISM method, to construct local operators $M_f(x^2)$ such that a trace of their monodromy gives $Q_\pm(x^2)$, to prove the commutativity of the transfer matrices and $Q_\pm$ and to calculate the Wronskian. Note that in QUISM approach we construct $Q_\pm(x^2)$ not as functions of $\{t\}$, but as functions of local $\mathbf{u}_f$, $\mathbf{v}_f$. This is in some sense a factorization, the simplest analogue of this is the well-known q-exponential formula

$$(\mathbf{x}+\mathbf{y};q)_\infty=(\mathbf{x};q)_\infty\cdot(\mathbf{y};q)_\infty;\qquad \mathbf{x}\mathbf{y}=q\mathbf{y}\mathbf{x}, \tag{2.21}$$

where conventionally

$$(x;q)_n\stackrel{\text{def}}{=}\prod_{k=0}^{n-1}\left(1-q^k x\right),\qquad (x;q)_\infty\stackrel{\text{def}}{=}\prod_{n=0}^{\infty}\left(1-q^k x\right), \tag{2.22}$$

and as the series expansions

$$(x;q)_\infty=\sum_{n=0}^{\infty}q^{n(n-1)/2}\frac{(-x)^n}{(q;q)_n},\qquad (x;q)_\infty^{-1}=\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}. \tag{2.23}$$

The right-hand side of (2.21) we call the local form of its "global" left-hand side.

Now we describe the local form of all solutions. First of all, we introduce a universal function

$$\mathbf{g}_{\alpha,\beta}(n,m)\stackrel{\text{def}}{=}q^{nm}\alpha^n\beta^m\frac{\left(q^{1+n};q\right)_\infty\left(q^{1+m};q\right)_\infty}{(q;q)_\infty}, \tag{2.24}$$

where α and β are complex numbers, and elements q^n and q^m commute.

Proposition 2.2. *The operator* $Q_+(x^2)$, *defined by (2.16)*, *(2.17)*, *and (2.18)*, *in the local form is*

$$Q_+(x^2) = \sum_{\{n_f \geq 0\}} \left(\prod_f g_{1,\lambda}(n_f + s_f, n_f) \right) \cdot \left(\prod_f (\mathbf{uv})_f^{n_{f+1}-n_f} \right), \quad (2.25)$$

and the operator $Q_-(x^2)$, *defined by (2.17)*, *(2.18)*, *and (2.19)*, *in the local form is*

$$Q_-(x^2) = \sum_{\{n_f \geq 0\}} \left(\prod_f g_{1,\lambda}(n_f, n_f - s_f) \right) \cdot \left(\prod_f (\mathbf{uv})_f^{n_f - n_{f-1}} \right). \quad (2.26)$$

Their Wronskian, *defined by (2.20)*, *is*

$$W(x^2) = \left(\prod_f (q^{s_f}; q)_\infty (q^{1-s_f}; q)_\infty \lambda^{-s_f} \right) \cdot \left(\prod_f \left(\frac{\lambda (\mathbf{uv})_f}{(\mathbf{uv})_{f+1}}; q \right)_\infty \right). \quad (2.27)$$

Note that the operators Q_+, Q_-, and W ((2.25), (2.26), and (2.27)) are the series with respect to $\mathbf{u}_f$ up to the simple common multiplier $\lambda^{-\sum_f s_f}$. Thus (2.6) is indeed just the useful notation, and actually we do not need the notion of the logarithm of the operator $\mathbf{u}_f$.

3. Intertwiners

3.1. Integrability

First of all, the integrability of the relativistic Toda chain follows from the commutativity of the transfer matrices (2.8). The origin of it is the famous six-vertex R-matrix. The following relation holds:

$$R_{1,2}\left(\frac{x}{y}\right) \cdot L_{1,f}(x) \cdot L_{2,f}(y) = L_{2,f}(y) \cdot L_{1,f}(x) \cdot R_{1,2}\left(\frac{x}{y}\right), \quad (3.1)$$

where $L_{1,f}(x) = L_f(x) \otimes 1$, $L_{2,f}(y) = 1 \otimes L_f(y)$, and so forth, the cross product implies the tensor product of the 2×2 matrices, and the six-vertex R-matrix has the form

$$R(x) = \begin{pmatrix} 1 - x^{-2} q & 0 & 0 & 0 \\ 0 & q^{1/2}(1 - x^{-2}) & x^{-1}(1-q) & 0 \\ 0 & x^{-1}(1-q) & q^{1/2}(1 - x^{-2}) & 0 \\ 0 & 0 & 0 & 1 - x^{-2} q \end{pmatrix}. \quad (3.2)$$

The Yang-Baxter relation (3.1) provides the commutativity of the traces of the monodromies for $L_{1,f}(x)$ and $L_{2,y}(x)$, and as the extra multiplier in our definition of the transfer matrix (2.8) is the total shift operator, our modified transfer matrices (2.8) also form the commutative family, (2.11).

3.2. Origin of Baxter's equation

The appearance of the six-vertex R-matrix is the criterion of the existence of Baxter's "$TQ = Q' + Q''$" relation for our transfer matrix. Here we give a brief description of the method of obtaining Baxter's equation.

So, for a given quantum Lax operator, obeying (3.1) with six-vertex R-matrix, let there exists another auxiliary relation

$$\tilde{L}_h\left(\frac{x}{y}\right) * L_f(x) \cdot M_{h,f}(y) = M_{h,f}(y) \cdot L_f(x) * \tilde{L}_h\left(\frac{x}{y}\right), \tag{3.3}$$

where "$*$" means the 2×2 matrix multiplication, 2×2 matrix $\tilde{L}_h(z)$ with entries acting in some space h is an auxiliary L-operator, and a scalar with respect to the matrix structure of L_f and $\tilde{L}_h$ operator $M_{h,f}$ acts in the tensor product of the spaces f and h. In the case of the usual quantum Toda chain, $\tilde{L}$ is a Sklyanin's Dimer self-trapping L-operator. As usual, the monodromies of L and M

$$\widehat{t}(x) = L_1(x)L_2(x)\cdots L_F(x), \tag{3.4}$$

$$\widehat{Q}_h(y) = M_{h,1}(y)M_{h,2}(y)\cdots M_{h,F}(y), \tag{3.5}$$

obey the same relation (3.3), this provides the commutativity of

$$t(x^2) = \operatorname{tr}\widehat{t}(x), \qquad Q(y^2) = \operatorname{tr}_h \widehat{Q}_h(y). \tag{3.6}$$

Suppose next that the matrix $\tilde{L}_h(z)$ has the degeneration point, without loss of generality,

$$\tilde{L}_h(1) = \psi_h\overline{\psi}_h, \tag{3.7}$$

where ψ_h and $\overline{\psi}_h$ are column and row two components vectors with operator-valued entries. These vectors are known as Baxter's vacuum vectors [6]. Analogously, the inverse matrix $\tilde{L}^{-1}(z)$, being normalized appropriately, has the orthogonal decomposition,

$$\tilde{L}_h(1)^{-1} = \phi_h\overline{\phi}_h. \tag{3.8}$$

Obviously,

$$\overline{\psi}_h\phi_h = \overline{\phi}_h\psi_h = 0. \tag{3.9}$$

It is easy to obtain, the following elements are two orthogonal projectors,

$$P' = \psi_h\left(\overline{\psi}_h\psi_h\right)^{-1}\overline{\psi}_h, \qquad P'' = \phi_h\left(\overline{\phi}_h\phi_h\right)^{-1}\overline{\phi}_h, \tag{3.10}$$

and because L-operators have the matrix dimension 2, $P' + P'' = 1$.

Consider (3.3) and its inverse partner in the degeneration point. The projector structure of $\tilde{L}_h(1)^{\pm 1}$ means that there exist two scalar (with respect to the 2×2 structure of L-operators and ψ and ϕ-vectors) operators M' and M'', defined by the following relations:

$$\begin{aligned} \overline{\psi}_h L_f(x) M_{h,f}(x) &= M'_{h,f}(h)\overline{\psi}_h, \\ L_f(x) M_{h,f}(x)\phi_h &= \phi_h M''_{h,f}(h). \end{aligned} \tag{3.11}$$

Equation (3.3) for the monodromies may be written as

$$\begin{aligned} t(x^2)Q(y^2) = \mathrm{tr}_h\left(\overline{\psi}_h \hat{t}(x)\hat{Q}(x)\psi_h(\overline{\psi}_h\psi_h)^{-1}\right) \\ + \mathrm{tr}_h\left(\overline{\phi}_h \hat{t}(x)\hat{Q}(x)\phi_h(\overline{\phi}_h\phi_h)^{-1}\right). \end{aligned} \tag{3.12}$$

Due to (3.11), in the right-hand side there appear the traces of $M'_{h,f}$ and $M''_{h,f}$, so if

$$\begin{aligned} Q' &= \mathrm{tr}_h\left(M'_{h,1}M'_{h,2},\dots,M'_{h,F}\right), \\ Q'' &= \mathrm{tr}_h\left(M''_{h,1}M''_{h,2},\dots,M''_{h,F}\right), \end{aligned} \tag{3.13}$$

then the fusion relation appears

$$t(x^2)Q(x^2) = Q'(x^2) + Q''(x^2). \tag{3.14}$$

In most particular cases, when h is an infinite-dimensional space, in (3.14) it appears that $Q'(x^2) \sim Q(qx^2)$ and $Q''(x^2) \sim Q(q^{-1}x^2)$. All these are the subject of detailed investigation.

4. Operator $M_{h,f}$

4.1. Triangle relations for the quantum relativistic Toda chain

Now we try to guess a form of (3.7) and (3.8) and try to solve (3.11).

Taking (3.9) into account, let

$$\tilde{L}_h(1) = \begin{pmatrix} -\mathbf{a}^+ \\ 1 \end{pmatrix} k_1(-\mathbf{a},1), \qquad \tilde{L}_h^{-1}(1) = \begin{pmatrix} 1 \\ \mathbf{a} \end{pmatrix} k_2(1,\mathbf{a}^+), \tag{4.1}$$

where $\mathbf{a}$ and $\mathbf{a}^+$ are two essential elements for h-algebra. Writing this decomposition, we do not impose no conditions on $\mathbf{a}$, $\mathbf{a}^+$ and the unknown factors k_1 and k_2. The experience of usual Toda chain [17] says that when $q \mapsto 1$, $\mathbf{a}$ and $\mathbf{a}^+$ become usual bosonic annihilation and creation operators, this inspires our notations. In the next derivations we will suggest the invertibility of $\mathbf{a}$ and $\mathbf{a}^+$, so that the decompositions (4.1) are written without loss of generality. To get something applicable, we introduce an element $\mathbf{N}$, such that a sort of q-oscillator relations hold: first, for any function g let

$$\mathbf{a}\cdot g(\mathbf{N}) = g(\mathbf{N}+1)\cdot\mathbf{a}, \tag{4.2}$$

and second let there exists a function $[\mathbf{N}]$,

$$\boldsymbol{a}^+ \cdot \boldsymbol{a} = [\mathbf{N}], \qquad \boldsymbol{a} \cdot \boldsymbol{a}^+ = [\mathbf{N}+1]. \tag{4.3}$$

The element $\mathbf{N}$ is introduced, without loss of generality, as a pair to $\boldsymbol{a}$. Operators $\tilde{L}$ must form an integrable chain, this provides relations (4.3), and therefore $k_1 = k_1(\mathbf{N})$ and $k_2 = k_2(\mathbf{N})$ in (4.1). Thus, the parameterization (4.1) is the general one. The degenerate matrices $\tilde{L}^{\pm}(1)$ become the orthogonal projectors if one substitutes $k_1 = 1/(1+[\mathbf{N}+1])$ and $k_2 = 1/(1+[\mathbf{N}])$.

Now we write the explicit form of the triangle relations

$$\begin{aligned} (-\boldsymbol{a},1)L(x)M(x) &= M'(x)(-\boldsymbol{a},1), \\ L(x)M(x)\begin{pmatrix}1\\ \boldsymbol{a}\end{pmatrix} &= \begin{pmatrix}1\\ \boldsymbol{a}\end{pmatrix}M''(x), \end{aligned} \tag{4.4}$$

$$\begin{aligned} M(x)L(x)\begin{pmatrix}-\boldsymbol{a}^+\\ 1\end{pmatrix} &= \begin{pmatrix}-\boldsymbol{a}^+\\ 1\end{pmatrix}\underline{M}'(x), \\ \left(1,\boldsymbol{a}^+\right)M(x)L(x) &= \underline{M}''(x)\left(1,\boldsymbol{a}^+\right). \end{aligned} \tag{4.5}$$

The pair of equations (4.4) is exactly (3.11), while the pair (4.5) is the dual one. Traces of the monodromies of M' and M'' and of $\underline{M}'$ and $\underline{M}''$ must coincide, they should lead to the same equation (3.14)

$$T\left(x^2\right)Q\left(x^2\right) = \left((-x)^F\prod_f \boldsymbol{u}_f\right)\left(Q'\left(x^2\right)+Q''\left(x^2\right)\right), \tag{4.6}$$

where extra multiplier appears due to our normalization of the transfer matrix (see (2.8)). But locally $M^{\#}$ and $\underline{M}^{\#}$ may be slightly different.

The spectral parameter x in the L-operator (2.1) always stays in the combination $x\boldsymbol{u}$, therefore the shift of the spectral parameter thus may appear as

$$g\left(q^{1/2}x\boldsymbol{u}_f\right) \equiv \boldsymbol{v}_f^{-1}g\left(x\boldsymbol{u}_f\right)\boldsymbol{v}_f. \tag{4.7}$$

Due to this property we can put $x=1$ for the shortness and omit the spectral parameter in our formulae, the x may be restored subsequently in all equations by the shift $\boldsymbol{u}_f \mapsto x\boldsymbol{u}_f$.

The triangle equations (4.4) and (4.5) are equivalent to two systems

$$\begin{aligned} M' &= -\boldsymbol{a}\boldsymbol{v}M, \qquad M'' = q^{-1/2}\lambda(\boldsymbol{a}\boldsymbol{v})^{-1}M, \\ M\boldsymbol{a} &= \boldsymbol{v}^{-1}\left(\boldsymbol{u}^{-1}-\boldsymbol{u}+q^{-1/2}\lambda\boldsymbol{a}^{-1}\boldsymbol{v}^{-1}\right)M, \\ \underline{M}' &= -Mq^{-1/2}\lambda\boldsymbol{v}^{-1}\boldsymbol{a}^+, \qquad \underline{M}'' = M\boldsymbol{v}\left(\boldsymbol{a}^+\right)^{-1}, \\ -q^{-1/2}\lambda\boldsymbol{a}^+M &= M\left(\boldsymbol{u}-\boldsymbol{u}^{-1}-\left(\boldsymbol{a}^+\right)^{-1}\boldsymbol{v}\right)\boldsymbol{v}. \end{aligned} \tag{4.8}$$

In Baxter's equation it is implied that $Q'(x^2) \sim Q(q^{-1}x^2)$ and $Q''(x^2) \sim Q(q\,x^2)$ up to some operator-valued multipliers. These multipliers are to be integrals of motion in a form of pure product over f. There is only one such integral of motion, it is t_0, and hence there must exist a monomial function $\phi(\mathbf{u}) \sim \mathbf{u}^c$ such that

$$M' = \boldsymbol{v} M \phi(\mathbf{u}) \boldsymbol{v}^{-1}, \qquad \underline{M}' = \boldsymbol{v} \phi(\mathbf{u}) M \boldsymbol{v}^{-1}, \tag{4.9}$$

and therefore

$$M'' = -q^{-1/2} \lambda \boldsymbol{v}^{-1} M \boldsymbol{v} \phi^{-1}(\mathbf{u}), \qquad \underline{M}'' = -q^{-1/2} \lambda \phi^{-1}(\mathbf{u}) \boldsymbol{v}^{-1} M \boldsymbol{v}. \tag{4.10}$$

The same $\phi(\mathbf{u})$ is used for M and $\underline{M}$ because Q must be the same. It is important that in (4.9) the multiplier $\phi(\mathbf{u})$ stands from the right of M for M' and from the left of M for $\underline{M}'$. The order of multipliers is governed by Yang-Baxter equation (3.3).

In general, one may put $\phi(\mathbf{u})$ to the other sides, this would give another system for M with another solution. We will not investigate such case separately, because there exists an involutive automorphism τ, defined as

$$\boldsymbol{v}^\tau = \boldsymbol{v}, \qquad \mathbf{u}^\tau = \mathbf{u}^{-1}, \qquad q^\tau = q, \tag{4.11}$$

such that the L-operator is invariant with respect to τ-involution

$$L(1)^\tau = -\sigma_3 L(1) \sigma_3. \tag{4.12}$$

Also it is important that τ does not change q. Therefore $T^\tau(x^2) = (-)^F T(x^2)$, and the other case of positions of ϕ just corresponds to the consideration of M^τ.

With expressions (4.9) for M' and M'', system (4.8) is equivalent to

$$-\mathbf{a} M = M \phi(\mathbf{u}) \boldsymbol{v}^{-1}, \tag{4.13a}$$

$$-q^{-1/2} \lambda M \mathbf{a}^+ = \boldsymbol{v} \phi(\mathbf{u}) M, \tag{4.13b}$$

$$M \mathbf{a} = \boldsymbol{v}^{-1} \left(\mathbf{u}^{-1} - \mathbf{u} + q^{-1/2} \lambda \mathbf{a}^{-1} \boldsymbol{v}^{-1} \right) M, \tag{4.13c}$$

$$-q^{-1/2} \lambda \mathbf{a}^+ M = M \left(\mathbf{u} - \mathbf{u}^{-1} - \left(\mathbf{a}^+ \right)^{-1} \boldsymbol{v} \right) \boldsymbol{v}. \tag{4.13d}$$

It is useful to complement system (4.13) by equations with k_1, k_2 following from (3.3):

$$M \phi^{-1}(\mathbf{u}) k_1(\mathbf{N}) = \phi^{-1}(\mathbf{u}) k_1(\mathbf{N}) M, \tag{4.13e}$$

$$M \phi^{-1}\left(q^{-1/2} \mathbf{u}\right) k_2(\mathbf{N}) = \phi^{-1}\left(q^{-1/2} \mathbf{u}\right) k_2(\mathbf{N}) M. \tag{4.13f}$$

This is the final set of equations that we are going to solve. We will give its solution in two forms. The first one is a formal series solution that admits an interpretation of $\mathbf{a}$ and $\mathbf{a}^+$ as q-oscillator (spectrum of $\mathbf{N}$ is the nonnegative integers, and there exists the vacuum vector for $\mathbf{a}$). Another form implies the Weyl algebra parameterization of $\mathbf{a}$, $\mathbf{a}^+$, when the permutation between h and f spaces plays the significant role. Actually these two forms differ by the notion of the trace in h-space, and q-oscillator trace will give Q_- while the Weyl trace will give Q_+.

4.2. Series solution

First, we test system (4.13) for the formal operator arguments of M. Just considering the expressions $M\boldsymbol{x}M^{-1}\boldsymbol{x}^{-1}$ for several $\boldsymbol{x}$, one may conclude

$$M = M(\mathbf{a}\boldsymbol{v}, \mathbf{u}, \mathbf{N}). \tag{4.14}$$

Hence

$$M \cdot q^{\mathbf{N}}\mathbf{u}^2 = \mathbf{u}^2 q^{\mathbf{N}} \cdot M, \tag{4.15}$$

this trivializes two equivalent relations (4.13e) and (4.13f) of system (4.13).

For further analysis of (4.13) we start from permutation-like (4.13a). Relations like

$$\boldsymbol{x} \cdot M = M \cdot \boldsymbol{y} \tag{4.16}$$

are to be solved as

$$M = \sum_{n\in\mathbb{Z}} \boldsymbol{x}^n \cdot G \cdot \boldsymbol{y}^{-n}, \tag{4.17}$$

and in the case of (4.13a) this gives

$$M = \sum_{n\in\mathbb{Z}} \mathbf{a}^n \cdot G(\mathbf{N}, \mathbf{u}^2) \cdot \left(-\boldsymbol{v}\phi^{-1}(\mathbf{u})\right)^n. \tag{4.18}$$

Note that G does not depend on $\mathbf{a}\boldsymbol{v}$, because any such dependence may be extracted to $\mathbf{a}^n$. Now all other relations from (4.13) must give recursion relations for G. Equation (4.13c) is equivalent to

$$\begin{aligned}(u-u^{-1})G(N,u^2) = &-q^{-1/2}\lambda\phi^{-1}(u)G(N,qu^2) \\ &+\phi(q^{-1/2}u)G(N-1,q^{-1}u^2).\end{aligned} \tag{4.19}$$

Equation (4.13b) gives another permutation-like structure, but with the formal correspondence $\mathbf{a}^+ = [\mathbf{N}]\mathbf{a}^{-1}$ it gives

$$\frac{G(N-1,u^2)}{G(N,qu^2)} = q^{1/2}\lambda\frac{[N]}{\phi^2(u)}. \tag{4.20}$$

Due to (4.20), M may be rewritten in the form of the other permutation-like structure

$$M = \sum_{n\in\mathbb{Z}} \left(-q^{1/2}\lambda^{-1}\boldsymbol{v}\phi(\mathbf{u})\right)^n \cdot G(\mathbf{N}, \mathbf{u}^2) \cdot (\mathbf{a}^+)^{-n}. \tag{4.21}$$

Moreover, this allows one to write M without negative powers of $\mathbf{a}$ or $\mathbf{a}^+$

$$\begin{aligned} M = G(\mathbf{N},\mathbf{u}^2) + \sum_{n=1}^{\infty} \mathbf{a}^n \cdot G(\mathbf{N},\mathbf{u}^2)\cdot\left(-\nu\phi^{-1}(\mathbf{u})\right)^n \\ + \sum_{n=1}^{\infty}\left(-q^{1/2}\lambda^{-1}\nu\phi(\mathbf{u})\right)^{-n}\cdot G(\mathbf{N},\mathbf{u}^2)\cdot(\mathbf{a}^+)^n. \end{aligned} \tag{4.22}$$

Apparently, this form is good for q-oscillator representation.

Equation (4.13d) coincides with (4.19) if one uses the series (4.21). But it is important to note that in general (4.19) and (4.20) are not compatible. Their compatibility condition is the following functional relation for $\phi(\mathbf{u})$ and $[\mathbf{N}]$:

$$\begin{aligned} q^{-1/2}\lambda\left(\frac{[\mathbf{N}]}{\phi(q^{1/2}\mathbf{u})} - \frac{[\mathbf{N}-1]}{\phi(q^{-1/2}\mathbf{u})}\right) \\ = \mathbf{u}^{-1}\left(1 - q^{-1/2}\frac{\phi(\mathbf{u})}{\phi(q^{1/2}\mathbf{u})}\right) - \mathbf{u}\left(1 - q^{1/2}\frac{\phi(\mathbf{u})}{\phi(q^{1/2}\mathbf{u})}\right). \end{aligned} \tag{4.23}$$

Here we used $\phi(\mathbf{u}) \sim \mathbf{u}^c$.

Equation (4.23) has only two solutions for $\phi(\mathbf{u})$ and $[\mathbf{N}]$, corresponding to $|q| < 1$ and $|q| > 1$. In our case $|q| < 1$

$$\phi(\mathbf{u}) = -q^{-1/2}\alpha\mathbf{u}^{-1}, \qquad [\mathbf{N}] = -q^{1/2}\frac{\alpha}{\lambda}\left(1 - q^{-\mathbf{N}}\right), \tag{4.24}$$

where α is a complex parameter, $[\mathbf{N}]$ is normalized so as $[0] = 0$. With these $\phi(\mathbf{u})$ and $[\mathbf{N}]$, equations (4.19) and (4.20) may be solved easily,

$$G_{|q|<1}(N, u^2) = \mathbf{g}_{\alpha,\lambda/\alpha}(N, N-s), \tag{4.25}$$

where $u^2 \equiv q^s$ (see (2.6)), and $\mathbf{g}_{\alpha,\beta}(n,m)$ is defined by (2.24). Parameter α is an avoidable scale of $\mathbf{u}$ and it is convenient to put it to unity, $\alpha \equiv 1$. Note that expressions like $(x;q)_\infty$ in $\mathbf{g}$-function appear as the appropriate solutions of difference relations

$$(x;q)_\infty = (1-x)(qx;q)_\infty, \tag{4.26}$$

and the separation between $|q| < 1$ and $|q| > 1$ is originated from the unavoidable sign of the quadratic exponent $q^{\pm N(N-s)}$. The other solution of zero curvature condition is

$$\phi(u) = \alpha u, \qquad [N] = -\frac{\alpha}{\lambda}\left(1 - q^N\right). \tag{4.27}$$

This gives

$$\begin{aligned} G(N, q^s) = q^{-N(N-s)}\left(q^{-1/2}\frac{\lambda}{\alpha}\right)^{N-s}\left(q^{-1/2}\alpha\right)^N \\ \times \frac{\left(q^{-1-N+s};q^{-1}\right)_\infty\left(q^{-1-N};q^{-1}\right)_\infty}{\left(q^{-1};q^{-1}\right)_\infty}. \end{aligned} \tag{4.28}$$

With $\phi(u)$ defined, the final expressions for M are, in the short form,

$$M = \sum_{n\in\mathbb{Z}} a^n g_{1,\lambda}(N, N-s)(uv)^n \equiv \sum_{n\in\mathbb{Z}} \left(\lambda u v^{-1}\right)^{-n} g_{1,\lambda}(N, N-s)\left(a^+\right)^{-n}, \tag{4.29}$$

and in q-oscillator's form

$$M = g_{1,\lambda}(N, N-s) + \sum_{n=1}^{\infty} a^n g_{1,\lambda}(N, N-s)(uv)^n + \sum_{n=1}^{\infty} \left(\lambda u v^{-1}\right)^n g_{1,\lambda}(N, N-s)\left(a^+\right)^n, \tag{4.30}$$

where, recall (2.6), $u^2 = q^s$.

Substituting $\phi(u) = -q^{-1/2}u^{-1}$ into the expressions for M' and M'', (4.9), and using our definition of the transfer matrix (2.8), we obtain the Baxter equation exactly in the form (2.15).

Existence of the form (4.30) allows one to interpret a, a^+ exactly as q-oscillator generators, such that the spectrum of N is $0, 1, 2, \ldots$ (we have normalized $[N]$ so that $[0] = 0$), and the state $|N = 0\rangle$ is the vacuum, $a\,|N = 0\rangle = 0$. Thus one may define the q-oscillator trace of any operator $F = F(a, a^+, N)$. If

$$F = f_0(N) + \sum_{n\geq 1} f_n(N) a^n + \sum_{n\geq 1} f_n^+(N)\left(a^+\right)^n, \tag{4.31}$$

then taking such trace, one has to take a^0 and $(a^+)^0$th components and then take the sum over $N = 0, 1, 2, \ldots$

$$\mathrm{tr}_{\text{q-osc}}\, F\left(a, a^+, N\right) \stackrel{\text{def}}{=} \sum_{n\geq 0} f_0(n). \tag{4.32}$$

Being applied to the monodromy (3.5) of M, (4.29), this trace definition gives immediately and exactly $Q_-(x^2)$, (2.26).

In general, one may obtain Q_+ at once, considering the τ-involution applied to M and to Q_-

$$M^\tau = \sum_{n\in\mathbb{Z}} \left(a v u^{-1}\right)^n g_{1,\lambda}(N, N+s). \tag{4.33}$$

But there are two objections to consider this case: first, τ-involution changes a little the Baxter equation, and second, M^τ is the degenerate operator,

$$\left(a v u^{-1} - 1\right)\cdot M^\tau = 0, \tag{4.34}$$

and hence we will look for another way to obtain Q_+ operator.

4.3. Extraction of a permutation

Solving (4.13), we mentioned the permutation-like relations. In this section, we suppose that the quantum space f and the auxiliary one h are isomorphic. Our aim is to extract the permutation operator, giving (4.13a) "by hands." As previously, we deal with the case $|q| < 1$, $x = 1$, $\mathbf{u}^2 = q^{\mathbf{s}}$, and

$$\phi(\mathbf{u}) = -q^{-1/2}\mathbf{u}^{-1}, \qquad [\mathbf{N}] = -q^{1/2}\lambda^{-1}\left(1-q^{-\mathbf{N}}\right), \tag{4.35}$$

so that we are looking for another realization of the same operator M. We will search for M in the form

$$M = \mathcal{M}\cdot P_{h,f}, \tag{4.36}$$

where

$$\mathbf{a}P_{h,f} = P_{h,f}(\mathbf{u}\mathbf{v})^{-1}, \qquad \mathbf{N}P_{h,f} = P_{h,f}\mathbf{s}, \qquad P_{h,f}^2 = 1. \tag{4.37}$$

Here the first relation is exactly (4.13a), the second one is the consequence of (4.15), and the last one is the definition of the permutation. System (4.13) for operator $\mathcal{M}$ can be rewritten as follows:

$$\mathbf{a}\cdot\mathcal{M} = \mathcal{M}\cdot\mathbf{a}, \tag{4.38a}$$

$$\mathbf{u}^{-1}\mathbf{v}\cdot\mathcal{M} = \mathcal{M}\cdot\left(1-\mathbf{u}^2\right)\mathbf{u}^{-1}\mathbf{v}, \tag{4.38b}$$

$$\mathcal{M}\cdot(\mathbf{u}\mathbf{v})^{-1} = (\mathbf{u}\mathbf{v})^{-1}\left(1-\mathbf{u}^2+q^{-1}\lambda\mathbf{a}^{-1}\mathbf{v}^{-1}\mathbf{u}\right)\cdot\mathcal{M}, \tag{4.38c}$$

$$q^{-\mathbf{N}}\cdot\mathcal{M} = \mathcal{M}\cdot\left(1+q^{-1}\lambda\left(1-q\mathbf{u}^2\right)^{-1}\mathbf{a}^{-1}\mathbf{v}^{-1}\mathbf{u}\right)q^{-\mathbf{N}}. \tag{4.38d}$$

Solution of it is given by

$$\mathcal{M} = \left(-\lambda\mathbf{a}^{-1}\mathbf{v}^{-1}\mathbf{u};q\right)_\infty\left(q\mathbf{u}^2;q\right)_\infty. \tag{4.39}$$

Operator (4.36) with the definitions (4.37) and (4.39) does solve the system of the relations (4.13). Using the series decomposition for the compact quantum dilogarithms, one may obtain the series representation (there is used $(\mathbf{v}^{-1}\mathbf{u})^n = q^{n(n+1)/2}\mathbf{u}^{2n}(\mathbf{u}\mathbf{v})^{-n}$) for M, (4.36),

$$M = \sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}\lambda^n\mathbf{u}^{2n}\left(q^{1+n}\mathbf{u}^2;q\right)_\infty(\mathbf{u}\mathbf{v})^{-n}P_{h,f}(\mathbf{u}\mathbf{v})^n. \tag{4.40}$$

Note that the function $\mathbf{g}$, (2.24), appears in this decomposition

$$M = \sum_{n\geq 0}\mathbf{g}_{1,\lambda}(n+\mathbf{s},n)(\mathbf{u}\mathbf{v})^{-n}P_{h,f}(\mathbf{u}\mathbf{v})^n. \tag{4.41}$$

In this form all the h-space operators $\mathbf{a}$, $\mathbf{a}^+$, and $\mathbf{N}$ are hidden into the permutation symbol. The permutation operator allows one to calculate the trace in the auxiliary space h in the invariant way via

$$\mathrm{tr}_{\mathrm{inv}}\left(P_{h,1}P_{h,2}\cdots P_{h,F}\right) = P, \tag{4.42}$$

where P is the cyclic shift operator for the chain $f = 1, 2, \ldots, F$, $F+1 \sim 1$

$$\boldsymbol{u}_f P = P \boldsymbol{u}_{f+1}, \qquad \boldsymbol{v}_f P = P \boldsymbol{v}_{f+1}, \qquad f \sim f+F. \tag{4.43}$$

The shift is one of the integrals of motion. Now using (4.41) and the definition of the shift operator, one obtains exactly Q_+, (2.25), for the trace of M-monodromy up to the shift

$$Q_+ P = P Q_+ = \mathrm{tr}_{\mathrm{inv}} \left(M_1 M_2 \cdots M_F \right). \tag{4.44}$$

Now both forms of M-operators have been obtained, (4.22) and (4.41), actually coincide. To show this, we represent $P_{h,f}$ in the following form:

$$P_{h,f} = \sum_{n \in \mathbb{Z}} \delta(\mathbf{N} - \mathbf{s} = n)(\boldsymbol{a u v})^n, \tag{4.45}$$

where $\delta(\mathbf{N} - \mathbf{s} = n)$ is the projector of $\mathbf{N} - \mathbf{s}$ into a state with the eigenvalue n

$$(\mathbf{N} - \mathbf{s})\delta(\mathbf{N} - \mathbf{s} = n) = \delta(\mathbf{N} - \mathbf{s} = n)(\mathbf{N} - \mathbf{s}) = n\delta(\mathbf{N} - \mathbf{s} = n). \tag{4.46}$$

With this form of $P_{h,f}$ (4.41) could be written as follows:

$$M = \sum_{n,k} \boldsymbol{a}^k \boldsymbol{g}_{1,\lambda}(n + \mathbf{s}, n)\delta(\mathbf{N} - \mathbf{s} = n)(\boldsymbol{u v})^k. \tag{4.47}$$

Now one may take the sum over n using the projectors as the delta symbols, and exactly (4.29) appears

$$M = \sum_{k} \boldsymbol{a}^k \boldsymbol{g}_{1,\lambda}(\mathbf{N}, \mathbf{N} - \mathbf{s})(\boldsymbol{u v})^k. \tag{4.48}$$

Such exercises with the projector decomposition of operators are rather formal. One may consider projectors and spectral decompositions of many types, imposing some extra conditions for the spectra of the operators involved. What is actually the difference between both Q operators: the difference is the conjecture about the spectrum of $\mathbf{N}$. Due to the Weyl algebra relations, the spectrum of $\mathbf{N}$ must be equidistant,

$$\mathbf{N} \in \mathbb{Z} + \zeta. \tag{4.49}$$

In the case when $\zeta = 0$ we get q-oscillator representation. In the case when ζ is the same as for $\mathbf{s} \in \mathbb{Z} + \zeta$, we get the isomorphism between h and f spaces and the permutation extracted representation. In general one may generalize both Q_+ and Q_- into Q_ζ, dealing with arbitrary characteristics of $\mathbf{N}$,

$$Q_\zeta = \sum_{\{n_f \in \mathbb{Z} + \zeta\}} \left(\prod_f \boldsymbol{g}_{1,\lambda}\left(n_f, n_f - \mathbf{s}_f\right) \right) \cdot \left(\prod_f (\boldsymbol{u v})_f^{n_f - n_{f-1}} \right). \tag{4.50}$$

A summation over $n \in \mathbb{Z}$ may be restricted in q-hypergeometry by the factor

$$\frac{1}{(q;q)_n} = \frac{\left(q^{1+n};q\right)_\infty}{(q;q)_\infty} = 0 \quad \text{for } n < 0. \tag{4.51}$$

Such restrictions in (4.50) appear when $\zeta = 0$ and when $\mathbf{s} \in \mathbb{Z} + \zeta$, these are exactly the cases of Q_- and Q_+P.

Similar to the spectral decomposition of the permutation operator, one may write down the spectral decomposition of the shift operator

$$P = \sum_{\{n_f \in \mathbb{Z}\}} \left(\prod_f \delta\left(\mathbf{s}_f = n_f + \zeta\right) \right) \left(\prod_f (\mathbf{u}\mathbf{v})_f^{n_f - n_{f-1}} \right). \tag{4.52}$$

In this formula it is implied that ζ is the characteristics of $\mathbf{s}_f$. An example of application of such formula, that is, explicit extraction of the shift operator, is the following summation, where the shift $n_f \mapsto n_f + \mathbf{s}_f$ is done:

$$\begin{aligned} &\sum_{\{n_f \in \zeta + \mathbb{Z}\}} G\left(\left\{n_f, n_f - \mathbf{s}_f\right\}\right) \prod_f (\mathbf{u}\mathbf{v})_f^{n_f - n_{f-1}} \\ &\qquad = \sum_{\{n_f \in \mathbb{Z}\}} G\left(\left\{n_f + \mathbf{s}_f, n_f\right\}\right) \prod_f (\mathbf{u}\mathbf{v})_f^{n_{f+1} - n_f} \cdot P. \end{aligned} \tag{4.53}$$

To obtain it, one has to apply the spectral decomposition of each $\mathbf{s}_f$, and then make the resummation. This trick gives $Q_\zeta = Q_+(x^2)P$ when $\mathbf{s}_f \in \zeta + \mathbb{Z}$.

5. Properties of M operators

5.1. Auxiliary L-operator

Proposition 5.1. *Equation (3.3), provided by (4.13), (4.15), and (4.35), holds for*

$$\tilde{L}(x) = \begin{pmatrix} xq^{\mathbf{N}/2} - x^{-1}q^{-\mathbf{N}/2} & \lambda \mathbf{a}^+ q^{\mathbf{N}/2} \\ \lambda q^{\mathbf{N}/2}\mathbf{a} & -\lambda x^{-1} q^{\mathbf{N}/2} \end{pmatrix}. \tag{5.1}$$

To be exact in our normalization $M = M(1)$, for which (4.13), (4.15), and (4.35) are written down, (3.1) looks like

$$M \cdot L(x) \cdot \tilde{L}(x) = \tilde{L}(x) \cdot L(x) \cdot M, \tag{5.2}$$

and M must intertwine each power of x. Useful relations following from (4.13) are

$$\begin{aligned} M \cdot (\mathbf{u}\mathbf{v})^{-1} &= \mathbf{a} \cdot M, & M \cdot \mathbf{u}\mathbf{v}^{-1} &= q^{\mathbf{N}}\left(\lambda \mathbf{u}\mathbf{v}^{-1} + q^{1/2}\mathbf{a}\right) \cdot M, \\ \mathbf{v}\mathbf{u}^{-1} \cdot M &= M \cdot \lambda \mathbf{a}^+, & \mathbf{u}\mathbf{v} \cdot M &= M \cdot \lambda q^{\mathbf{N}}\left(\mathbf{u}\mathbf{v} + q^{-1/2}\mathbf{a}^+\right). \end{aligned} \tag{5.3}$$

Note, as far the quantum Lax operator (2.1) is called "the quantum relativistic Toda chain L-operator," then operator (5.1) is to be called "the quantum relativistic Dimed self-trapping L-operator" (cf. [21]).

5.2. Intertwining

Now we consider the commutation relations of different Q-operators. Let the operators $Q_1(y)$ and $Q_2(x)$ be constructed with the help of different local $M_{h_1,f}(y)$ and $M_{h_2,f}(x)$ (here we imply different characteristics of h_1 and h_2).

Proposition 5.2. *Two products* $M_{h_1,f}(y)\cdot M_{h_2,f}(x)$ *and* $M_{h_2,f}(x)\cdot M_{h_1,f}(y)$ *are connected by a canonical mapping* $K_{h_1,h_2}(y/x)$ *of the pair of Weyl algebras* h_1 *and* h_2,

$$K_{h_1,h_2}\left(\frac{y}{x}\right)M_{h_1,f}(y)M_{h_2,f}(x)=M_{h_2,f}(x)M_{h_1,f}(y)K_{h_1,h_2}\left(\frac{y}{x}\right), \tag{5.4}$$

where K *acts as follows*:

$$\begin{aligned} K(z)\mathbf{a}_1^+ &= z^{-1}\mathbf{a}_2^+K(z), \qquad K(z)q^{\mathbf{N}_1}=\frac{1+q^{1/2}z\mathbf{a}_1\mathbf{a}_2^+}{1+q^{1/2}z^{-1}\mathbf{a}_1\mathbf{a}_2^+}q^{\mathbf{N}_2}K(z),\\ K(z)\mathbf{a}_2 &= z\mathbf{a}_1K(z), \qquad K(z)q^{\mathbf{N}_2}=q^{\mathbf{N}_1}\frac{1+q^{1/2}z^{-1}\mathbf{a}_1\mathbf{a}_2^+}{1+q^{1/2}z\mathbf{a}_1\mathbf{a}_2^+}K(z). \end{aligned} \tag{5.5}$$

As an example we give the realization of $K(z)$ with the permutation extracted

$$K_{h_1,h_2}(z)=\frac{(-q^{1/2}z\mathbf{a}_1\mathbf{a}_2^+;q)}{(-q^{1/2}z^{-1}\mathbf{a}_1\mathbf{a}_2^+;q)}z^{-\mathbf{N}_1-\mathbf{N}_2}P_{h_1,h_2}, \tag{5.6}$$

where P_{h_1,h_2}—usual external permutation of the spaces h_1 and h_2. This permutation may be canceled from KMM equation, and the following relation for the Q-monodromies appears:

$$\check{K}_{h_1,h_2}\left(\frac{y}{x}\right)\widehat{Q}_{h_1}(y^2)\widehat{Q}_{h_2}(x^2)=\widehat{Q}_{h_1}(x^2)\widehat{Q}_{h_2}(y^2)\check{K}_{h_1,h_2}\left(\frac{y}{x}\right), \tag{5.7}$$

where

$$\begin{aligned} \check{K}_{h_1h_2}\left(\frac{y}{x}\right) &= P_{h_1h_2}K_{h_1h_2}\left(\frac{y}{x}\right)\\ &= \left(\frac{x}{y}\right)^{\mathbf{N}_1+\mathbf{N}_2}\frac{(-q^{1/2}(y/x)\mathbf{a}_1^+\mathbf{a}_2;q)_\infty}{(-q^{1/2}(x/y)\mathbf{a}_1^+\mathbf{a}_2;q)_\infty}, \end{aligned} \tag{5.8}$$

and $\widehat{Q}$—the monodromy of M operators, $Q(x^2)=\mathrm{tr}_h\,\widehat{Q}_h$. Equation (5.7) leads to the pseudo-commutation of the pair of Q matrices with *different* ζ-characteristics and allows one to calculate the Wronskian.

5.3. Wronskian

To calculate the Wronskian, it is necessary to consider (5.7) with $x/y = q^{1/2}$. Then $\check{K}(q^{-1/2}) = q^{(\mathbf{N}_1+\mathbf{N}_2)/2}(1+\mathbf{a}_1^+\mathbf{a}_2)$, and

$$\begin{aligned} q^{(\mathbf{N}_1+\mathbf{N}_2)/2}(1+\mathbf{a}_1^+\mathbf{a}_2)\widehat{Q}_1(q^{-1}x^2)\widehat{Q}_2(x^2) \\ = \widehat{Q}_1(x^2)\widehat{Q}_2(q^{-1}x^2)(1+\mathbf{a}_1^+\mathbf{a}_2)q^{(\mathbf{N}_1+\mathbf{N}_2)/2}. \end{aligned} \tag{5.9}$$

Let δ_W be a projector to the subspace $\mathbf{a}_1^+\mathbf{a}_2 = -1$, that is,

$$\delta_W \cdot (\mathbf{a}_1^+\mathbf{a}_2+1) = (\mathbf{a}_1^+\mathbf{a}_2+1)\cdot\delta_W = 0. \tag{5.10}$$

Then the pseudo-commutation relation provides the following triangle structure:

$$\begin{aligned} \widehat{Q}_1(q^{-1}x^2)\widehat{Q}_2(x^2)\delta_W &= \delta_W\widehat{Q}_1(q^{-1}x^2)\widehat{Q}_2(x^2)\delta_W, \\ \delta_W\widehat{Q}_1(x^2)\widehat{Q}_2(q^{-1}x^2) &= \delta_W\widehat{Q}_1(x^2)\widehat{Q}_2(q^{-1}x^2)\delta_W. \end{aligned} \tag{5.11}$$

Locally we consider the products

$$\begin{aligned} M_1(y)M_2(x)\delta_W &= \sum_{n\in\mathbb{Z}} (\lambda y\mathbf{u}\mathbf{v}^{-1})^n \mathcal{F}_{y,x}(\mathbf{N}_1,\mathbf{N}_2,\mathbf{u}^2)(\mathbf{a}_1^+)^n\delta_W, \\ \delta_W M_1(x)M_2(y) &= \delta_W \sum_{m\in\mathbb{Z}} \mathbf{a}_2^m \tilde{\mathcal{F}}_{x,y}(N_1,N_2,\mathbf{u}^2)(y\mathbf{u}\mathbf{v})^n, \end{aligned} \tag{5.12}$$

where the sum simplified due to

$$\delta_W\cdot(\mathbf{a}_1^+)^n\mathbf{a}_2^{m-n} \equiv \delta_W\cdot(-)^n\mathbf{a}_2^m. \tag{5.13}$$

Triangle structure means that when $y^2 = q^{-1}x^2$, both $\mathcal{F}$ and $\tilde{\mathcal{F}}$ depend actually only on $\mathbf{N}_1+\mathbf{N}_2$ and $\mathbf{u}^2$.

For x and y arbitrary, one has

$$\begin{aligned} \mathcal{F}_{y,x}(\mathbf{N}_1,\mathbf{N}_2,\mathbf{u}^2) &\stackrel{\text{def}}{=} \sum_{m\in\mathbb{Z}} (-\lambda xy\mathbf{u}^2)^m q^{m^2/2}\mathbf{g}_{1,\lambda} \\ &\quad\times(\mathbf{N}_1,\mathbf{N}_1-\mathbf{s}_y-m)\mathbf{g}_{1,\lambda}(\mathbf{N}_2+m,\mathbf{N}_2-\mathbf{s}_x), \\ \tilde{\mathcal{F}}_{x,y}(\mathbf{N}_1,\mathbf{N}_2,\mathbf{u}^2) &\stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} (-\lambda xy\mathbf{u}^2)^n q^{n^2/2}\mathbf{g}_{1,\lambda} \\ &\quad\times(\mathbf{N}_1+n,\mathbf{N}_1-\mathbf{s}_x)\mathbf{g}_{1,\lambda}(\mathbf{N}_2,\mathbf{N}_2-\mathbf{s}_y-n). \end{aligned} \tag{5.14}$$

Here

$$q^{\mathbf{s}_x} = x^2\mathbf{u}^2, \qquad q^{\mathbf{s}_y} = y^2\mathbf{u}^2. \tag{5.15}$$

One may see

$$\tilde{\mathcal{F}}_{x,y}(\mathbf{N}_1,\mathbf{N}_2,\mathbf{u}^2) \equiv \mathcal{F}_{y,x}(\mathbf{N}_2,\mathbf{N}_1,\mathbf{u}^2). \tag{5.16}$$

These sums may be calculated with the help of the Rogers-Ramanujan summation formula. Auxiliary relations for this calculations are

$$\begin{aligned}
&(\lambda x\mathbf{u}\mathbf{v}^{-1})^n(y\mathbf{u}\mathbf{v})^n = q^{n^2/2}(\lambda xy\mathbf{u}^2)^n,\\
&\frac{\mathbf{g}_{1,\lambda}(N+n,N-s)}{\mathbf{g}_{1,\lambda}(N,N-s)} = q^{n(N-s)}\frac{1}{(q^{1+N};q)_n},\\
&\frac{\mathbf{g}_{1,\lambda}(N,N-s-n)}{\mathbf{g}_{1,\lambda}(N,N-s)} = q^{-n^2/2+n/2}(-\lambda q^s)^{-n}(q^{s-N};q)_n,
\end{aligned} \tag{5.17}$$

and the Rogers-Ramanujan celebrated identity is

$$\begin{aligned}
{}_1\Psi_1(x,y;z) &\overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} \frac{(x;q)_n}{(y;q)_n} z^n\\
&= \frac{(q;q)_\infty (y/x;q)_\infty (xz;q)_\infty (q/xz;q)_\infty}{(y;q)_\infty (q/x;q)_\infty (z;q)_\infty (y/xz;q)_\infty},
\end{aligned} \tag{5.18}$$

where the series for ${}_1\Psi_1$ is convergent in

$$\left|\frac{y}{x}\right| < |z| < 1. \tag{5.19}$$

The results of summations are

$$\begin{aligned}
\mathcal{F}_{y,x}(\mathbf{N}_1,\mathbf{N}_2,\mathbf{u}^2) &= \mathbf{g}_{1,\lambda}(\mathbf{N}_1,\mathbf{N}_1-\mathbf{s}_y)\mathbf{g}_{1,\lambda}(\mathbf{N}_2,\mathbf{N}_2-\mathbf{s}_x)\\
&\quad\times {}_1\Psi_1\left(q^{\mathbf{s}_y-\mathbf{N}_1}, q^{1+\mathbf{N}_2}; q^{1/2+\mathbf{N}_2-\mathbf{s}_y}\frac{y}{x}\right),\\
\tilde{\mathcal{F}}_{x,y}(\mathbf{N}_1,\mathbf{N}_2,\mathbf{u}^2) &= \mathbf{g}_{1,\lambda}(\mathbf{N}_1,\mathbf{N}_1-\mathbf{s}_x)\mathbf{g}_{1,\lambda}(\mathbf{N}_2,\mathbf{N}_2-\mathbf{s}_y)\\
&\quad\times {}_1\Psi_1\left(q^{\mathbf{s}_y-\mathbf{N}_2}, q^{1+\mathbf{N}_1}; q^{1/2+\mathbf{N}_1-\mathbf{s}_y}\frac{y}{x}\right).
\end{aligned} \tag{5.20}$$

Put now $y^2 = q^{-1}x^2$, then it appeared

$$\mathcal{F}_{1,2}(\mathbf{N}_1+\mathbf{N}_2,\mathbf{s}_x) \overset{\text{def}}{=} \mathcal{F}_{y,x}(\mathbf{N}_1,\mathrm{N}_2,\mathbf{u}^2) = -\tilde{\mathcal{F}}_{x,y}(\mathbf{N}_1,\mathbf{N}_2,\mathbf{u}^2), \tag{5.21}$$

where $y^2 = q^{-1}x^2$, and

$$\begin{aligned}
\mathcal{F}_{1,2}(\mathbf{N}_1+\mathbf{N}_2,\mathbf{s}_x) &= q^{\mathbf{N}_2(\mathbf{N}_2-\mathbf{s}_x)+\mathbf{N}_1(\mathbf{N}_1-\mathbf{s}_x+1)}\lambda^{\mathbf{N}_1+\mathbf{N}_2-2\mathbf{s}_x+1}\\
&\quad\times\Theta(q^{\mathbf{N}_2-\mathbf{N}_1})\frac{(q^{2+\mathbf{N}_1+\mathbf{N}_2-\mathbf{s}_x};q)_\infty}{(q;q)_\infty^2}.
\end{aligned} \tag{5.22}$$

Here it is used the θ-function notation

$$\Theta(x) = (x;q)_\infty (qx^{-1};q)_\infty (q;q)_\infty = \sum_{n\in\mathbb{Z}} (-x)^n q^{n(n-1)/2}, \tag{5.23}$$

such as

$$\Theta\big(q^k x\big) = (-x)^{-k} q^{-k(k-1)/2}\Theta(x), \qquad \Theta\big(x^{-1}\big) = -x^{-1}\Theta(x). \tag{5.24}$$

Indeed, due to the equidistant of $\mathbf{N}_1$ and $\mathbf{N}_2$, $\mathcal{F}_{1,2}$ depends only on $\mathbf{N}_1+\mathbf{N}_2$.

Now we may calculate the Wronskian. By definition, it is

$$W\big(x^2\big)_{1,2} = Q_1\big(q^{-1}x^2\big)Q_2\big(x^2\big) - Q_1\big(x^2\big)Q_2\big(q^{-1}x^2\big). \tag{5.25}$$

Considering the monodromies of Q_1 and Q_2, standing in the definition of the Wronskian and using (5.7), one may see that the most parts in the subtraction, (5.25), are canceled. Only possible exception is the subspace $\delta_W : \mathbf{a}_1^+ \mathbf{a}_2 = -1$. So to calculate the Wronskian, one has to take a trace only over this subspace. In general, let ζ_1 and ζ_2 be the characteristics of $\mathbf{N}_1$ and $\mathbf{N}_2$, respectively. Then using the definition of $\mathcal{F}$ and $\tilde{\mathcal{F}}$, (5.22) equivalence of $\mathcal{F}$ and $\tilde{\mathcal{F}}$, one may conclude

$$W_{1,2} = \Xi_{1,2} \sum_{n_f \in \zeta_1+\zeta_2+\mathbb{Z}} \bigg(\prod_f \mathcal{F}_{1,2}\big(n_f, \mathbf{s}_f\big)\bigg)\bigg(\prod_f (\mathbf{u}\boldsymbol{v})_f^{n_f - n_{f-1}}\bigg), \tag{5.26}$$

where $\Xi_{1,2}$ is an extra multiplier that may come from the subtraction, δ_W-trace definition and so on. Nevertheless, considering the case $\mathbf{s}_x = \zeta_1$ modulo $\mathbb{Z}$ and $\zeta_2 = 0$, one obtains the following useful form of $\mathcal{F}_{1,2}$:

$$\mathcal{F}_{1,2}(n+\mathbf{s}-, \mathbf{s}) = (-)^n q^{n(n-1)/2} \frac{\lambda^n}{(q;q)_n} \lambda^{-s} \frac{\Theta\big(q^s\big)}{(q;q)_\infty}. \tag{5.27}$$

Extracting now the shift operator as it is described in (4.53), one obtains (2.27). Extra multiplier is equal to unity, this we have checked by a series expansion with respect to λ.

6. Discussion

The technique and results, given in this paper, are rather formal. We have dealt with the single Weyl pair in each site of the lattice, and q is an arbitrary complex number inside the unit circle. It is well known, this regime is absolutely nonphysical, and thus the results presented are to be considered as just an exercise in the field of q-combinatorial analysis. But, nevertheless, some applications of the results and technique presented may be found.

Talking about the Weyl algebra, people usually keep in mind two aspects: the first one is the Faddeev dualization, when $q = \text{ex}\{i\pi e^{i\theta}\}$ with real θ is the universal unitary regime [10, 11, 9], and the second one is the finite state $q = e^{2\pi i/N}$. This paper suggests the third aspect, applied in the backward direction yet: several Toda-chain-type models, physical as well, may be obtained from a model with arbitrary q in the limit $q \mapsto 1+\hbar$, regarded in a special way, such that a rational Weyl algebra mapping is linearized with respect to one of Weyl generators in the first order of $\hbar$. Our experience in the Weyl algebra exercises says that most our results, especially containing the q-dilogarithms and permutations, may be immediately rewritten in the dualized form. In this way the results may be applied to the physical relativistic Toda chain, [12]. It will be done in a separate paper.

The second aspect is also valid, especially in the part of the technique derived. Preliminary considerations show that at the root of unity the model contains the Baxter curve for the Chiral Potts model, the point on Baxter's curve is the spectral parameter of Q-operator, our constant parameter λ is connected with the modulus of Baxter's curve. Remarkable is that in the relativistic Toda chain at the root of unity there appears only one point at Baxter's curve, while in the Chiral Potts model such point lives at each site on the spin chain. This fact makes the relativistic Toda chain much more simple than CPM itself. The investigation of such type models is started in [16].

Acknowledgement

Authors would like to thank A. Isaev, P. Pyatov, S. Pakuliak, and A. Volkov for useful discussions. The work of G. Pronko was supported in part by ESPIRIT project NTCONGS.

This work was supported by RFBR grants No. 98-01-00070 and No. 01-01-00201.

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