ON GENERALIZED DERIVATIVES FOR $C^{1,1}$ VECTOR OPTIMIZATION PROBLEMS

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Received 19 September 2002 and in revised form 16 February 2003

We introduce generalized definitions of Peano and Riemann directional derivatives in order to obtain second-order optimality conditions for vector optimization problems involving $C^{1,1}$ data. We show that these conditions are stronger than those in literature obtained by means of second-order Clarke subdifferential.

1. $C^{1,1}$ vector functions and second-order subdifferentials

A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to be locally lipschitzian (or of class $C^{0,1}$) at $x_0 \in \mathbb{R}^m$ when there exist constants K_{x_0} and δ_{x_0} such that

$$\|f(x_1) - f(x_2)\| \leq K_{x_0}\|x_1 - x_2\|, \tag{1.1}$$

for all $x_1, x_2 \in \mathbb{R}^m$, $\|x_1 - x_0\| \leq \delta_{x_0}$, and $\|x_2 - x_0\| \leq \delta_{x_0}$. For this type of functions, according to Rademacher theorem, f is differentiable almost everywhere (in the sense of Lebesgue measure). Then the first-order Clarke generalized Jacobian of f at x_0, denoted by $\partial f(x_0)$, exists and is given by

$$\partial f(x_0) := \operatorname{cl}\operatorname{conv}\{\lim f'(x_k) : x_k \longrightarrow x_0,\ f'(x_k) \text{ exists}\}, \tag{1.2}$$

where $\operatorname{cl}\operatorname{conv}\{\cdots\}$ stands for the closed convex hull of the set under the parentheses. Now, assume that f is a differentiable vector function from

Journal of Applied Mathematics 2003:7 (2003) 365–376
2000 Mathematics Subject Classification: 90C29, 90C30
URL: http://dx.doi.org/10.1155/S1110757X03209049

$\mathbb{R}^m$ to $\mathbb{R}^n$ whose directional derivative is of class $C^{0,1}$ at $x_0 \in \mathbb{R}^m$. In this case, we say that f is of class $C^{1,1}$ at x_0. The second-order Clarke subdifferential of f at x_0, denoted by $\partial^2 f(x_0)$, is defined as

$$\partial^2 f(x_0) := \operatorname{cl}\operatorname{conv}\{\lim f''(x_k) : x_k \longrightarrow x_0,\ f''(x_k) \text{ exists}\}. \tag{1.3}$$

Thus $\partial^2 f(x_0)$ is a subset of the finite-dimensional space $L(\mathbb{R}^m;L(\mathbb{R}^m;\mathbb{R}^n))$ of linear operators from $\mathbb{R}^m$ to the space $L(\mathbb{R}^m;\mathbb{R}^n)$ of linear operators from $\mathbb{R}^m$ to $\mathbb{R}^n$. The elements of $\partial^2 f(x_0)$ can therefore be viewed as a bilinear function on $\mathbb{R}^m \times \mathbb{R}^m$ with values in $\mathbb{R}^n$. For the case $n = 1$, the term "generalized Hessian matrix" was used in [9] to denote the set $\partial^2 f(x_0)$. By the previous construction, the second-order subdifferential enjoys all properties of the generalized jacobian. For instance, $\partial^2 f(x_0)$ is a nonempty convex compact set of the space $L(\mathbb{R}^m;L(\mathbb{R}^m;\mathbb{R}^n))$ and the set-valued map $x \mapsto \partial^2 f(x)$ is upper semicontinuous. Let $u \in \mathbb{R}^m$; in the following we will denote by Lu the value of a linear operator $L : \mathbb{R}^m \to \mathbb{R}^n$ at the point $u \in \mathbb{R}^m$ and by $H(u,v)$ the value of a bilinear operator $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ at the point $(u,v) \in \mathbb{R}^m \times \mathbb{R}^m$. So we will set

$$\begin{aligned} \partial f(x_0)(u) &= \{Lu : L \in \partial f(x_0)\}, \\ \partial f(x_0)(u,v) &= \{H(u,v) : H \in \partial^2 f(x_0)\}. \end{aligned} \tag{1.4}$$

Some important properties are listed here (see [8]).

(i) Mean value theorem: let f be of class $C^{0,1}$ and $a,b \in \mathbb{R}^m$. Then

$$f(b) - f(a) \in \operatorname{cl}\operatorname{conv}\{\partial f(x)(b-a) : x \in [a,b]\}, \tag{1.5}$$

where $[a,b] = \operatorname{conv}\{a,b\}$.

(ii) Taylor expansion: let f be of class $C^{1,1}$ and $a,b \in \mathbb{R}^m$. Then

$$f(b)-f(a) \in f'(a)(b-a) + \frac{1}{2}\operatorname{cl}\operatorname{conv}\{\partial^2 f(x)(b-a,b-a) : x \in [a,b]\}. \tag{1.6}$$

Several problems of applied mathematics including, for example, portfolio analysis, data classification, and so forth involve functions with no hope of being twice differentiable (see [4, 5, 13] and Example 4.3) but they can be approximated by $C^{1,1}$ objects. Furthermore, the theory of $C^{1,1}$ functions has revealed its importance in some penalty methods, as shown in the following examples.

Example 1.1. Let $g : \mathbb{R}^m \to \mathbb{R}$ be twice continuously differentiable on $\mathbb{R}^m$ and consider $f(x) = [g^+(x)]^2$, where $g^+(x) = \max\{g(x), 0\}$. Then f is $C^{1,1}$ on $\mathbb{R}^m$.

Example 1.2. Consider the minimization problem $\min f_0(x)$ over all $x \in \mathbb{R}^m$ such that $f_1(x) \le 0, \dots, f_k(x) \le 0$. Letting r denote a positive parameter, the augmented Lagrangian L_r [20] is defined on $\mathbb{R}^m \times \mathbb{R}^k$ as

$$L_r(x,y) = f_0(x) + \frac{1}{4r}\sum_{i=1}^{k}\left\{\left[y_i + 2rf_i(x)\right]^+\right\}^2 - y_i^2. \tag{1.7}$$

Upon setting $y = 0$ in the previous expression, we observe that

$$L_r(x,0) = f_0(x) + r\sum_{i=1}^{k}\left[f_i^+(x)\right]^2 \tag{1.8}$$

is the ordinary penalized version of the minimization problem. The augmented Lagrangian L_r is differentiable everywhere on $\mathbb{R}^m \times \mathbb{R}^k$ with

$$\begin{gathered}\nabla_x L_r(x,y) = \nabla f_0(x) + \sum_{j=1}^{k}\left[y_j + 2rf_j(x)\right]^+ \nabla f_j(x) \\ \frac{\partial L_r}{\partial y_i}(x,y) = \max\left\{f_i(x), -\frac{y_i}{2r}\right\}, \quad i = 1,\dots,k,\end{gathered} \tag{1.9}$$

when f_i are C^2 on $\mathbb{R}^m$ and L_r is $C^{1,1}$ on $\mathbb{R}^{m+k}$.

Optimality conditions for $C^{1,1}$ scalar functions have been studied by many authors and numerical methods have been proposed too (see [7, 9, 10, 15, 17, 18, 21, 22, 23]). In [7], Guerraggio and Luc have given necessary and sufficient optimality conditions for vector optimization problems expressed by means of $\partial^2 f(x)$. In this paper, we introduce generalized Peano and Riemann directional derivatives for $C^{1,1}$ vector functions and we study second-order optimality conditions for set-constrained optimization problems.

2. Generalized directional derivatives for $C^{1,1}$ vector functions

Let Ω be an open subset of $\mathbb{R}^m$ and let $f : \Omega \to \mathbb{R}^n$ be a $C^{1,1}$ vector function. For such a function we define

$$\delta_2^d f(x;h) = f(x+2hd) - 2f(x+hd) + f(x) \tag{2.1}$$

with $x \in \Omega$, $h \in \mathbb{R}$, and $d \in \mathbb{R}^m$. The following result characterizes a function of class $C^{1,1}$ in terms of δ_2^d. It has been proved, for the scalar case, in [12, Theorem 2.1].

Proposition 2.1. *Assume that the function $f:\Omega\subset\mathbb{R}^m\to\mathbb{R}^n$ is bounded on a neighborhood of the point $x_0\in\Omega$. Then f is of class $C^{1,1}$ at x_0 if and only if there exist neighborhoods U of x_0 and V of $0\in\mathbb{R}$ such that $\|\delta_2^d f(x;h)\|/h^2$ is bounded on $U\times V\backslash\{0\}$, for all $d\in S^1=\{d\in\mathbb{R}^m:\|d\|=1\}$.*

Proof. The proof follows from recalling that a vector function is of class $C^{1,1}$ at x_0 if and only if each component of f is of class $C^{1,1}$ at x_0. □

The following definition extends to the vector case the well-known definition of the Riemann directional derivative [2].

Definition 2.2. The second Riemann directional derivative of f at $x_0\in\Omega$ in the direction $d\in\mathbb{R}^m$ is defined as

$$f''_R(x_0;d)=\left\{l=\lim_{k\to+\infty}\frac{\delta_2^d f(x_0;t_k)}{h_k^2},\ t_k\downarrow 0\right\},\tag{2.2}$$

that is, the set of all cluster points of sequences $\delta_2^d f(x_0;t_k)/t_k^2$, $t_k\downarrow 0$, $k\to+\infty$.

If $f:\Omega\subset\mathbb{R}^m\to\mathbb{R}$, then $\sup_{l\in f''_R(x_0;d)} l$ $(\inf_{l\in f''_R(x_0;d)} l)$ coincides with the classical definition of the second upper Riemann directional derivative $\overline{f}''_R(x_0;d)$ (second lower Riemann directional derivative $\underline{f}''_R(x_0;d)$). If we define $\Delta_2^d f(x;h)=f(x+hd)-2f(x)+f(x-hd)$, then one can introduce the corresponding Riemann directional derivatives. Riemann [19] introduced (for scalar functions) the notion of second-order directional derivative while he was studying the convergence of trigonometric series. For properties and applications of Riemann directional derivatives, see [1, 2, 8].

Remark 2.3. If we define $\delta_2^{u,v} f(x;s,t)=f(x+su+tv)-f(x+su)-f(x+tv)+f(x)$ with $x\in\Omega$, $s,t\in\mathbb{R}$, and $u,v\in\mathbb{R}^m$, then the set

$$f''_C(x_0;u,v)=\left\{l=\lim_{k\to+\infty}\frac{\delta_2^{u,v} f(x_0;s_k,t_k)}{s_k t_k},\ s_k,t_k\downarrow 0\right\}\tag{2.3}$$

is an extension to the vector case of the definition according to Cominetti and Correa [6]. It is clear that $f''_R(x_0;d)\subset f''_C(x_0;d,d)$.

Definition 2.4. The second Peano directional derivative of f at $x_0 \in \Omega$ in the direction d is defined as

$$f''_P(x_0;d) = \left\{ l = \lim_{k\to+\infty} 2\frac{f(x_0+t_kd) - f(x_0) - t_k\nabla f(x_0)d}{t_k^2},\ t_k \downarrow 0 \right\}, \quad (2.4)$$

that is, the set of all cluster points of sequences $2(f(x_0+t_kd) - f(x_0) - t_k\nabla f(x_0)d)/t_k^2$, $t_k \downarrow 0$, $k \to +\infty$.

If $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}$, then $\sup_{l\in f''_P(x_0;d)} l$ $(\inf_{l\in f''_P(x_0;d)} l)$ coincides with the classical definition of the second upper Peano directional derivative $\overline{f}''_P(x_0;d)$ (second lower Peano directional derivative $\underline{f}''_P(x_0;d)$). Peano [16], studying the Taylor expansion formula for real functions, introduced a concept of a higher-order directional derivative of a function f at a point x_0 known thereafter as the Peano directional derivative. A similar directional derivative (parabolic directional derivative) is also used in [3] in the scalar case. Furthermore, in [14] some optimality conditions for $C^{1,1}$ vector optimization problems involving inequality constraints are given by using this type of generalized directional derivative. In this paper we study optimality conditions for set constraints.

It is trivial to prove that the previous sets are nonempty compact subsets of $\mathbb{R}^n$. The aim of the next sections is to investigate some properties of these generalized directional derivatives for $C^{1,1}$ vector functions and then apply them in order to obtain second-order optimality conditions for vector optimization problems.

3. Preliminary properties

Let $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ be a $C^{1,1}$ scalar function at $x_0 \in \Omega$. The following theorem states an inequality between second upper Peano and Riemann directional derivatives.

Theorem 3.1 [11]. *Let $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ be a $C^{1,1}$ scalar function at $x_0 \in \Omega$. Then $\overline{f}''_P(x_0;d) \le \overline{f}''_R(x_0;d)$.*

Now, let $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ be a $C^{1,1}$ vector function at $x_0 \in \Omega$.

Lemma 3.2. *For all $\xi \in \mathbb{R}^n$, there exists $\tilde{r}_\xi \in f''_R(x_0;d)$ such that $\xi(p - \tilde{r}_\xi) \le 0$, for all $p \in f''_P(x_0;d)$.*

Proof. In fact, for all $\xi \in \mathbb{R}^n$ and eventually by extracting subsequences, we have

$$\overline{\xi f}''_P(x_0;d) \le \overline{\xi f}''_R(x_0;d) = \xi\tilde{r}_\xi, \quad (3.1)$$

where $\tilde{r}_\xi \in f''_R(x_0;d)$. So, for all $p \in f''_P(x_0;d)$, we have $\xi p \leq \overline{\xi f}''_P(x_0;d) \leq \xi \tilde{r}_\xi$ and then the thesis follows. □

Theorem 3.3. *Let $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ be a $C^{1,1}$ vector function at $x_0 \in \Omega$. Then $f''_P(x_0;d) \subset \operatorname{conv} f''_R(x_0;d)$.*

Proof. Ab absurdo, there exists $p^* \in f''_P(x_0;d)$ such that $p^* \notin \operatorname{conv} f''_R(x_0;d)$. Since $\operatorname{conv} f''_R(x_0;d)$ is a compact convex set of $\mathbb{R}^m$, then, by standard separation argument, there exists $\xi \in \mathbb{R}^n$, $\xi \neq 0$, such that $\xi(p^* - r) > 0$, for all $r \in \operatorname{conv} f''_R(x_0;d)$. From Lemma 3.2, there exists $\tilde{r}_\xi \in f''_R(x_0;d)$ such that $\xi(p - \tilde{r}_\xi) \leq 0$, for all $p \in f''_P(x_0;d)$. So $\xi(p^* - \tilde{r}_\xi) \leq 0$ and $\xi(p^* - \tilde{r}_\xi) > 0$. □

Theorem 3.4. *Let $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ be a $C^{1,1}$ vector function at $x_0 \in \Omega$. Then $f''_R(x_0;d) \subset \partial^2 f(x_0)(d,d)$.*

Proof. Let $r \in f''_R(x_0;d)$. Then

$$r = \lim_{k \to +\infty} \frac{f(x_0 + 2t_k d) - 2f(x_0 + t_k d) + f(x_0)}{t_k^2}, \tag{3.2}$$

where $t_k \downarrow 0$. By Taylor formula and the upper semicontinuity of the map $x \mapsto \partial^2 f(x)$, for all $\epsilon > 0$, there exists $k_0(\epsilon) \in \mathbb{N}$ such that, for all $k \geq k_0(\epsilon)$,

$$\frac{f(x_0 + 2t_k d) - 2f(x_0 + t_k d) + f(x_0)}{t_k^2} \in \partial^2 f(x_0)(d,d) + \epsilon B(0,1), \tag{3.3}$$

where $B(0,1)$ is the closed unit ball in $\mathbb{R}^n$. So, taking the limit when $k \to +\infty$ and $\epsilon \to 0$, we have $f''_R(x_0;d) \subset \partial^2 f(x_0)(d,d)$. □

Corollary 3.5. *Let $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ be a $C^{1,1}$ vector function at $x_0 \in \Omega$. Then $f''_P(x_0;d) \subset \partial^2 f(x_0)(d,d)$.*

The following example shows that the inclusion is strict.

Example 3.6. Let $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x^4 \sin(1/x) + x^4, x^4)$. The function f is of class $C^{1,1}$ at $x_0 = 0$ and $f''_P(0;d) = (0,0) \in \partial^2 f(0)(d,d) = [-d^2, d^2] \times \{0\}$.

THEOREM 3.7. *Let $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ be a $C^{1,1}$ vector function at $x_0 \in \Omega$. Then*

$$\begin{aligned} & f''_P(x_0;d) \\ & \quad = \left\{ l = \lim_{k\to+\infty} 2\frac{f(x_0+t_kd_k) - f(x_0) - t_k\nabla f(x_0)d_k}{t_k^2},\ t_k \downarrow 0,\ d_k \longrightarrow d \right\}. \end{aligned} \tag{3.4}$$

Proof. Let

$$\begin{aligned} & \mathcal{A}(x_0;d) \\ & \quad = \left\{ l = \lim_{k\to+\infty} 2\frac{f(x_0+t_kd_k) - f(x_0) - t_k\nabla f(x_0)d_k}{t_k^2},\ t_k \downarrow 0,\ d_k \longrightarrow d \right\}. \end{aligned} \tag{3.5}$$

Then the inclusion $f''_P(x_0;d) \subset \mathcal{A}$ follows immediately considering $d_k = d$. Vice versa, let $l \in \mathcal{A}$, there exist $t_k \downarrow 0$, $d_k \to d$ such that

$$\begin{aligned} \frac{l}{2} &= \lim_{k\to+\infty} \frac{f(x_0+t_kd_k) - f(x_0) - t_k\nabla f(x_0)d_k}{t_k^2} \\ &= \lim_{k\to+\infty} \frac{f(x_0+t_kd_k) - f(x_0+t_kd)}{t_k^2} \\ &\quad + \frac{f(x_0+t_kd) - f(x_0) - t_k\nabla f(x_0)d - t_k\nabla f(x_0)(d_k-d)}{t_k^2}. \end{aligned} \tag{3.6}$$

Taking eventually a subsequence

$$\begin{aligned} & \lim_{k\to+\infty} \frac{f(x_0+t_kd) - f(x_0) - t_k\nabla f(x_0)d}{t_k^2} = \frac{p}{2}, \quad p \in f''_P(x_0;d), \\ & \lim_{k\to+\infty} \left| \frac{f_i(x_0+t_kd_k) - f_i(x_0+t_kd) - t_k\nabla f_i(x_0)(d_k-d)}{t_k^2} \right| \\ & \qquad = \lim_{k\to+\infty} \left| \frac{t_k\nabla f_i(\xi_k)(d_k-d) - t_k\nabla f_i(x_0)(d_k-d)}{t_k^2} \right| \\ & \qquad \leq \lim_{k\to+\infty} \frac{K\|d_k-d\|\|\xi_k-x_0\|}{t_k} = 0, \end{aligned} \tag{3.7}$$

where K is the Lipschitz modulus of the derivative ∇f around x_0, $\xi_k \in [x_0 + t_k d_k, x_0 + t_k d]$ and $(\xi_k - x_0)/t_k \to d$. Therefore

$$\lim_{k \to +\infty} \frac{f(x_0 + t_k d_k) - f(x_0 + t_k d) - t_k \nabla f(x_0)(d_k - d)}{t_k^2} = 0 \tag{3.8}$$

and then $p = l$. □

4. Set-constrained optimization problems

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a $C^{1,1}$ vector function and consider the following set-constrained optimization problem:

$$\min_{x \in X} f(x), \tag{4.1}$$

where X is a subset of $\mathbb{R}^m$. For such a problem, the following definition states the classical notion of local minimum point and local weak minimum point.

Definition 4.1. Let C be a pointed closed convex cone of $\mathbb{R}^n$. Then $x_0 \in X$ is called a local minimum point of (4.1) if there exists a neighborhood $N \cap X$ of x_0 such that no $x \in N \cap X$ satisfies $f(x_0) - f(x) \in C \backslash \{0\}$.

Definition 4.2. Let C be a pointed closed convex cone of $\mathbb{R}^n$. Then $x_0 \in X$ is called a local weak minimum point of (4.1) if there exists a neighborhood $N \cap X$ of x_0 such that no $x \in N \cap X$ satisfies $f(x_0) - f(x) \in \operatorname{int} C$.

The following example shows the importance of the class of $C^{1,1}$ vector functions in order to obtain smooth approximations of strongly nondifferentiable optimization problems.

Example 4.3. In many applications in portfolio analysis, data classification, and approximation theory [4, 5, 13], one can be interested in solving optimization problems as

$$\min_{x \in X} \big(f(x), \operatorname{supp}(x)\big), \tag{4.2}$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a smooth function, X is a subset of $\mathbb{R}^m$, and $\operatorname{supp}(x) = |\{i : x_i \neq 0\}|$ ($|A|$ is the cardinality of the set $A \subset \mathbb{N}$). For example, in portfolio optimization, the variables correspond to commodities to be bought, the function $\operatorname{supp}(x)$ specifies that not too many different types of commodities can be chosen, the function $f(x)$ is a "measure of risk," and the constraints X prescribe levels of "performance." In data classification context, the function f is a measure of the margin between the separating planes, and the minimization of $\mathrm{supp}(x)$ concerns the number of features of x, that is, the number of nonzero components of the vector, in order to keep the complexity of the classification rule low (this improves the understandability of the model results).

This type of optimization problems involves strongly nondifferentiable functions; one way to solve it is to replace the function $\mathrm{supp}(x)$ with an approximation $\mathrm{supp}^*_\alpha(x)$ as

$$\mathrm{supp}^*_\alpha(x) = \sum_{i=1}^{m} \left[\max\left\{e^{-\alpha x_i}, e^{\alpha x_i}\right\}\right]^2. \tag{4.3}$$

Then the function $x \to (f(x), \mathrm{supp}^*_\alpha(x))$ is a $C^{1,1}$ vector function, for all $\alpha > 0$.

Definition 4.4. Let A be a given subset of $\mathbb{R}^n$ and $x_0 \in \mathrm{cl}\, A$. The sets

$$\begin{aligned} WF(A, x_0) &= \left\{d \in \mathbb{R}^n : \exists t_k \downarrow 0,\ x_0 + t_k d \in X\right\}, \\ T(A, x_0) &= \left\{d \in \mathbb{R}^n : \exists t_k \downarrow 0,\ \exists d_k \to d,\ x_0 + t_k d_k \in X\right\} \end{aligned} \tag{4.4}$$

are called cone of weak feasible directions and tangent cone to A at x_0.

Theorem 4.5 (necessary optimality condition). *If x_0 is a local weak minimum point, then, for all $d \in \mathbb{R}^m$ such that $-\nabla f(x_0)d \in C$ and $d \in WF(X, x_0)$,*

$$f''_P(x_0; d) \not\subset -\mathrm{int}\, C. \tag{4.5}$$

Proof. Ab absurdo, there exist $d \in \mathbb{R}^n$, $-\nabla f(x)d \in C$, and $d \in WF(X, x_0)$ such that $f''_P(x_0; d) \subset -\mathrm{int}\, C$. Now, for all $l \in f''_P(x_0; d)$, we have

$$2\frac{f(x_0 + t_k d) - f(x_0) - t_k \nabla f(x_0) d}{t_k^2} = l + o(t_k^2), \tag{4.6}$$

where $l \in f''_P(x_0; d) \subset -\mathrm{int}\, C$ and there exists k_0 such that for all $k \geq k_0$ we have

$$2\frac{f(x_0 + t_k d) - f(x_0) - t_k \nabla f(x_0) d}{t_k^2} \in -\mathrm{int}\, C \tag{4.7}$$

and then

$$f(x_0+t_kd)-f(x_0)\in t_k\nabla f(x_0)d-\operatorname{int}C\subset -\operatorname{int}C. \tag{4.8}$$

□

Corollary 4.6. *If x_0 is a local weak minimum point, then, for all $d\in\mathbb{R}^m$ such that $-\nabla f(x_0)d\in C$ and $d\in WF(X,x_0)$, we have*

$$\partial^2 f(x_0)(d,d)\not\subset -\operatorname{int}C. \tag{4.9}$$

Example 4.7. Let $f:\mathbb{R}\to\mathbb{R}^2$, $f(x)=(x^4\sin(1/x)-x^2/4,-x^2)$, and $C=\mathbb{R}^2_+$. The point $x_0=0$ is not a local weak minimum point. We have $f''_P(0;d)=(-d^2/2,-2d^2)\in -\operatorname{int}\mathbb{R}^2_+$ (the necessary condition is not satisfied) but $\partial^2 f(0)(d,d)=[-3d^2/2,d^2/2]\times\{-2d^2\}$ (the necessary condition is satisfied).

Theorem 4.8 (sufficient optimality condition). *Let $x_0\in X$ and suppose that $\nabla f(x_0)d\notin -\operatorname{int}C$, for all $d\in T(X,x_0)$. If for all $d\neq 0$, $d\in T(X,x_0)$, and $\nabla f(x_0)d\in -(C\setminus\operatorname{int}C)$ there exists a neighborhood U of d such that, for all $v\in U(d)$, $\nabla f(x_0)v\notin -\operatorname{int}C$, and $f''_P(x_0;d)\subset\operatorname{int}C$, then x_0 is a local minimum point.*

Proof. Ab absurdo, there exists $x_k\in X$, $x_k\to x_0$ such that $f(x_k)-f(x_0)\in -C\setminus\{0\}$. If $d_k=(x_k-x_0)/\|x_k-x_0\|$, then, eventually by extracting subsequence, $x_k=x_0+t_kd_k$ and $d_k\to d$, $\|d\|=1$. So $d\in T(X,x_0)$. Furthermore,

$$f(x_k)-f(x_0)=t_k\nabla f(x_0)d_k+o(t_k) \tag{4.10}$$

and then $\nabla f(x_0)d\in -C$ since C is closed. So $\nabla f(x_0)d\in -(C\setminus\operatorname{int}C)$ and then there exists $U(d)$ such that $\nabla f(x_0)v\notin -\operatorname{int}C$, for all $v\in U(d)$.

Let $p=\lim_{k\to+\infty}2((f(x_k)-f(x_0)-t_k\nabla f(x_0)d_k)/t_k^2)\in\operatorname{int}C$, then $p\in f''_P(x_0;d)$ and, since $\operatorname{int}C$ is open, there exists k_0 such that, for all $k\geq k_0$,

$$2\frac{f(x_k)-f(x_0)-t_k\nabla f(x_0)d_k}{t_k^2}\in\operatorname{int}C \tag{4.11}$$

and then

$$f(x_k)-f(x_0)\in t_k\nabla f(x_0)d_k+\operatorname{int}C\subset(-\operatorname{int}C)^c+\operatorname{int}C\subset(-C)^c. \tag{4.12}$$

□

COROLLARY 4.9. *Let $x_0 \in X$ and suppose that $\nabla f(x_0)d \notin -\operatorname{int} C$, for all $d \in T(X,x_0)$. If for all $d \neq 0$, $d \in T(X,x_0)$, and $\nabla f(x_0)d \in -(C \setminus \operatorname{int} C)$, there exists a neighborhood U of d such that, for all $v \in U$, $\nabla f(x_0)v \notin -\operatorname{int} C$, and $\partial^2 f(x_0)(d,d) \subset \operatorname{int} C$, x_0 is a local minimum point.*

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