FINITE-DIFFERENCE METHOD FOR PARAMETERIZED SINGULARLY PERTURBED PROBLEM

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We study uniform finite-difference method for solving first-order singularly perturbed boundary value problem (BVP) depending on a parameter. Uniform error estimates in the discrete maximum norm are obtained for the numerical solution. Numerical results support the theoretical analysis.

1. Introduction

In this paper, we are going to devise a finite-difference method for the following parameter-dependant singularly perturbed boundary value problem (BVP):

$$Lu := \varepsilon u'(x) + a(x)u(x) = f(x,\lambda), \quad x \in \Omega = (0,l], \tag{1.1}$$

$$u(0) = A, \qquad u(l) = B, \tag{1.2}$$

where A, B are given constants and $a(x)$, $f(x,\lambda)$ are sufficiently smooth functions such that

$$\begin{gathered} a(x) \geq \alpha > 0 \quad \text{in } \bar{\Omega} = [0,l], \\ 0 < m_1 \leq \left|\frac{\partial f}{\partial \lambda}\right| \leq M_1 < \infty \quad \text{in } \bar{\Omega} \times \mathbb{R}. \end{gathered} \tag{1.3}$$

$\varepsilon > 0$ is a small parameter and $\{u(x),\lambda\}$ is a solution.

For $\varepsilon \ll 1$, the function $u(x)$ has a boundary layer of thickness $O(\varepsilon)$ near $x = 0$.

Under the above conditions, there exists a unique solution to problem (1.1), (1.2) (see [7, 12]). An overview of some existence and uniqueness results and applications of parameterized equations may be obtained, for example, in [6, 7, 8, 9, 12, 13, 15, 16]. In [7, 9, 12], have also been considered some approximating aspects of this kind of problems. But designed in the above-mentioned papers, algorithms are only concerned with the regular cases (i.e., when the boundary layers are absent).

The numerical analysis of singular perturbation cases has always been far from trivial because of the boundary layer behavior of the solution. Such problems undergo rapid

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changes within very thin layers near the boundary or inside the problem domain [4, 10, 11]. It is well known that standard numerical methods on uniform meshes for solving such problems are unstable and fail to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods to these problems, whose accuracy does not depend on the parameter value ε, that is, methods that are convergent ε-uniformly. For the various approaches on the numerical solution of differential equations with steep, continuous solutions, we may refer to the monographs [4, 5, 14].

Here we analyze a fitted difference scheme on a uniform mesh for the numerical solution of the problem (1.1), (1.2). In Section 2, we state some important properties of the exact solution. In Section 3, we present the difference scheme and obtain uniform error estimates for the truncation term and appropriate solution on a uniform mesh. Uniform convergence is proved in the discrete maximum norm. In Section 4, we formulate the iterative algorithm for solving the discrete problem and give the illustrative numerical results. The approach to construct discrete problem and error analysis for approximate solution is similar to those ones from [1, 2, 3].

Henceforth, C and c denote the generic positive constants independent of ε and of the mesh parameter. A subscripted such constant is also independent of ε and mesh parameter, but whose value is fixed.

2. The continuous problem

In this section, we give uniform bounds of the solution of the BVP (1.1), (1.2), which will be used to analyze properties of the appropriate difference problem.

Lemma 2.1. *For the solution $\{u(x),\lambda\}$ of the problem (1.1), (1.2),*

$$|\lambda| \le c_0, \tag{2.1}$$

$$\|u\|_\infty \le c_1, \tag{2.2}$$

where

$$c_0 = \frac{\|a\|_\infty}{m_1\left(1-\exp\left(-\|a\|_\infty l\right)\right)}\left(|A|+|B|\right)+m_1^{-1}\|F\|_\infty,$$

$$c_1 = |A| + \alpha^{-1}\|F\|_\infty + c_0\alpha^{-1}M_1, \tag{2.3}$$

$$\left(F(x) = f(x,0),\ \|a\|_\infty \equiv \|a\|_{\infty,\bar\Omega} := \max_{\bar\Omega}|a(x)|\right).$$

Proof. We rewrite (1.1) as

$$\varepsilon u'(x) + a(x)u(x) = f(x,0) + \frac{\partial \tilde f}{\partial\lambda}\lambda, \tag{2.4}$$

where

$$\frac{\partial \tilde f}{\partial\lambda} = \frac{\partial f}{\partial\lambda}(x,\lambda^*), \quad \lambda^* = \gamma\lambda,\ 0<\gamma<1. \tag{2.5}$$

Integrating (2.4), we get

$$u(x) = A\exp\left(-\frac{1}{\varepsilon}\int_0^x a(t)dt\right) + \frac{1}{\varepsilon}\int_0^x F(\tau)\exp\left(-\frac{1}{\varepsilon}\int_\tau^x a(t)dt\right)d\tau + \frac{1}{\varepsilon}\lambda\int_0^x \frac{\partial f}{\partial\lambda}(\tau,\lambda^*)\exp\left(-\frac{1}{\varepsilon}\int_\tau^x a(t)dt\right)d\tau, \tag{2.6}$$

from which, by setting the boundary condition $u(l) = B$, we have

$$\lambda = \frac{B - A\exp\left(-(1/\varepsilon)\int_0^l a(t)dt\right) - (1/\varepsilon)\int_0^l F(\tau)\exp\left(-(1/\varepsilon)\int_\tau^l a(t)dt\right)d\tau}{(1/\varepsilon)\int_0^l(\partial f/\partial\lambda)(\tau,\lambda^*)\exp\left(-(1/\varepsilon)\int_\tau^l a(t)dt\right)d\tau}. \tag{2.7}$$

Applying the mean-value theorem for integrals, we deduce that

$$\begin{aligned}&\frac{\left|(1/\varepsilon)\int_0^l F(\tau)\exp\left(-(1/\varepsilon)\int_\tau^l a(t)dt\right)d\tau\right|}{\left|(1/\varepsilon)\int_0^l(\partial f/\partial\lambda)(\tau,\lambda^*)\exp\left(-(1/\varepsilon)\int_\tau^l a(t)dt\right)d\tau\right|}\\ &\quad= \frac{\left|(1/\varepsilon)\int_0^l F(\tau)\exp\left(-(1/\varepsilon)\int_\tau^l a(t)dt\right)d\tau\right|}{(1/\varepsilon)\left|(\partial f/\partial\lambda)(x^*,\lambda^*)\right|\int_0^l\exp\left(-(1/\varepsilon)\int_\tau^l a(t)dt\right)d\tau}\\ &\quad\le m_1^{-1}\|F\|_\infty, \quad 0 < x^* < l.\end{aligned} \tag{2.8}$$

It then follows from (2.7) that

$$|\lambda| \le \frac{\left|B - A\exp\left(-(1/\varepsilon)\int_0^l a(t)dt\right)\right|}{m_1(1/\varepsilon)\int_0^l\exp\left(-(1/\varepsilon)\int_\tau^l a(t)dt\right)d\tau} + m_1^{-1}\|F\|_\infty, \tag{2.9}$$

which, for $\varepsilon \le 1$, immediately leads to (2.1).

Next, from (2.6), we see that

$$|u(x)| \le |A|\exp\left(-\frac{\alpha x}{\varepsilon}\right) + \alpha^{-1}\left[1 - \exp\left(-\frac{\alpha x}{\varepsilon}\right)\right]\left(\|F\|_\infty + |\lambda|M_1\right) \tag{2.10}$$

and using the estimate (2.1), we obtain (2.2). □

3. Discrete problem and convergence

3.1. Derivation of the difference scheme. In what follows, we denote by ω_h the uniform mesh on Ω:

$$\omega_h = \{x_i = ih,\ i = 1,\dots,N;\ Nh = l\}, \qquad \bar{\omega}_h = \omega_h \cup \{x = 0\}. \tag{3.1}$$

To simplify the notation, we set $g_i = g(x_i)$ for any function $g(x)$, while g_i^h denotes an approximation of $g(x)$ at x_i.

For any mesh function $\{w_i\}$ defined on $\bar{\omega}_h$, we use the discrete maximum norm

$$\|w\|_\infty \equiv \|w\|_{\infty,\bar{\omega}_h} := \max_{0\le i\le N} |w_i|. \tag{3.2}$$

The approach of generating difference method is through the integral identity

$$\chi_i h^{-1}\int_{x_{i-1}}^{x_i} Lu\varphi_i(x)dx = \chi_i h^{-1}\int_{x_{i-1}}^{x_i} f(x,\lambda)\varphi_i(x)dx, \quad 1\le i\le N, \tag{3.3}$$

with the exponential basis functions

$$\varphi_i(x) = \exp\left(-\frac{a_i(x_i - x)}{\varepsilon}\right), \quad x_{i-1}\le x\le x_i, \tag{3.4}$$

where

$$\chi_i = \left(h^{-1}\int_{x_{i-1}}^{x_i}\varphi_i(x)dx\right)^{-1} = \frac{a_i\rho}{1-\exp(-a_i\rho)}, \quad \rho = \frac{h}{\varepsilon}. \tag{3.5}$$

We note that function $\varphi_i(x)$ is the solution of the problem

$$-\varepsilon\varphi_i'(x) + a_i\varphi_i(x) = 0, \quad x_{i-1}\le x < x_i,\ \varphi_i(x_i) = 1. \tag{3.6}$$

The relation (3.3) is rewritten as

$$\chi_i h^{-1}\varepsilon\int_{x_{i-1}}^{x_i} u'(x)\varphi_i(x)dx + a_i\chi_i h^{-1}\int_{x_{i-1}}^{x_i} u(x)\varphi_i(x)dx + R_i = f(x_i,\lambda) \tag{3.7}$$

with the remainder term

$$\begin{aligned} R_i = {} & \chi_i h^{-1}\int_{x_{i-1}}^{x_i}[a(x) - a(x_i)]\varphi_i(x)dx \\ & + \chi_i h^{-1}\int_{x_{i-1}}^{x_i}[f(x_i,\lambda) - f(x,\lambda)]\varphi_i(x)dx. \end{aligned} \tag{3.8}$$

To be consistent with [1, 2, 3], we then obtain

$$\varepsilon\theta_i u_{\bar{x},i} + a_i u_i + R_i = f(x_i,\lambda), \quad 1\le i\le N, \tag{3.9}$$

where

$$\theta_i = 1 + \chi_i h^{-1} a_i \varepsilon^{-1} \int_{x_{i-1}}^{x_i} (x - x_i)\varphi_i(x)dx, \qquad u_{\bar{x},i} = \frac{u_i - u_{i-1}}{h}, \tag{3.10}$$

and a simple calculation gives us

$$\theta_i = \frac{a_i\rho}{1 - \exp(-a_i\rho)} \exp(-a_i\rho). \tag{3.11}$$

As a consequence of (3.9), we propose the following difference scheme for approximating (1.1), (1.2):

$$L^h u_i^h := \varepsilon\theta_i u_{\bar{x},i}^h + a_i u_i^h = f(x_i, \lambda^h), \quad 1 \le i \le N, \qquad u_0^h = A, \quad u_N^h = B, \tag{3.12}$$

where θ_i is defined by (3.11).

3.2. Uniform error estimates. To investigate the convergence of the method, note that the error functions $z_i^h = u_i^h - u_i$, $0 \le i \le N$, $\mu^h = \lambda^h - \lambda$ are the solution of the discrete problem

$$\varepsilon\theta_i z_{\bar{x},i}^h + a_i z_i^h = f(x_i, \mu^h + \lambda) - f(x_i, \lambda) + R_i, \quad 1 \le i \le N, \tag{3.13}$$

$$z_0^h = 0, \qquad z_N^h = 0, \tag{3.14}$$

where θ_i and R_i are given by (3.11) and (3.8), respectively.

Lemma 3.1. *For the error function R_i,*

$$\|R\|_{\infty,\omega_h} \le Ch, \tag{3.15}$$

provided $a \in C^1(\bar{\Omega})$ and $|\partial f/\partial x| \le C$ for $x \in \bar{\Omega}$ and λ satisfying (2.1).

The proof easily follows from the explicit expression of R_i defined by (3.8).

Lemma 3.2. *The solution $\{z_i^h, \mu^h\}$ of the problem (3.13), (3.14) satisfies*

$$|\mu^h| \le m_1^{-1} \|R\|_{\infty,\omega_h}, \tag{3.16}$$

$$\|z^h\|_{\infty,\bar{\omega}_h} \le \alpha^{-1}(1 + m_1^{-1} M_1) \|R\|_{\infty,\omega_h}. \tag{3.17}$$

Proof. From (3.13), we obtain

$$z_i^h = \frac{\varepsilon\theta_i}{\varepsilon\theta_i + ha_i} z_{i-1}^h + \frac{h(\partial\tilde{f}/\partial\lambda)_i}{\varepsilon\theta_i + ha_i}\mu^h + \frac{hR_i}{\varepsilon\theta_i + ha_i}, \tag{3.18}$$

where

$$\left(\frac{\partial\tilde{f}}{\partial\lambda}\right)_i = \frac{\partial f}{\partial\lambda}(x_i, \lambda + \gamma\mu^h), \quad 0 < \gamma < 1. \tag{3.19}$$

Solving the first-order difference equation with respect to z_i^h and setting the boundary condition $z_0^h = 0$, we get

$$z_i^h = \mu^h h \sum_{k=1}^{i} \frac{(\partial\tilde{f}/\partial\lambda)_k}{\varepsilon\theta_k + ha_k} Q_{ik} + h\sum_{k=1}^{i} \frac{R_k}{\varepsilon\theta_k + ha_k} Q_{ik}, \tag{3.20}$$

where

$$Q_{ik} = \begin{cases} 1, & k = i, \\ \displaystyle\prod_{j=k+1}^{i} \frac{\varepsilon\theta_j}{\varepsilon\theta_j + ha_j}, & 1 \le k \le i-1. \end{cases} \tag{3.21}$$

For $i = N$, taking into consideration that $z_N^h = 0$, we have

$$\mu^h = -\frac{\sum_{k=1}^{N} \left(R_k/(\varepsilon\theta_k + ha_k)\right) Q_{N,k}}{\sum_{k=1}^{N} \left((\partial\tilde{f}/\partial\lambda)_k/(\varepsilon\theta_k + ha_k)\right) Q_{N,k}}, \tag{3.22}$$

from which, since $\varepsilon\theta_i + ha_i > 0$ $(1 \le i \le N)$, the required result (3.16) easily follows.

Finally, applying the maximum principle for difference operator $L^h z_i^h := \varepsilon\theta_i z_{\bar{x},i}^h + a_i z_i^h$, $1 \le i \le N$, to (3.13) yields

$$\|z^h\|_{\infty,\bar{\omega}_h} \le \alpha^{-1}\left(M_1 |\mu^h| + \|R\|_{\infty,\omega_h}\right), \tag{3.23}$$

which, along with (3.16), leads to (3.17). □

Combining the two previous lemmas gives us the following convergence result.

Theorem 3.3. *Let $\{u(x), \lambda\}$ be the solution of (1.1), (1.2) and $\{u_i^h, \lambda^h\}$ the solution of (3.13), (3.14). Then*

$$|\lambda - \lambda^h| \le Ch, \qquad \|u - u^h\|_{\infty,\bar{\omega}_h} \le Ch. \tag{3.24}$$

4. Numerical results

In this section, we present some numerical experiments in order to illustrate the present method.

(a) We solve the nonlinear problem (3.12) using the following quasilinearization technique:

$$\begin{gathered}\varepsilon\theta_i u_{\bar{x},i}^{(n)} + a_i u_i^{(n)} = f(x_i,\lambda^{(n-1)}), \quad 1 \le i < N,\\ u_0^{(n)} = A,\\ \lambda^{(n)} = \lambda^{(n-1)} - \frac{f(l,\lambda^{(n-1)}) - \theta_N\rho^{-1}(B - u_{N-1}^{(n)}) - a_N B}{(\partial f/\partial\lambda)(l,\lambda^{(n-1)})},\end{gathered} \tag{4.1}$$

$n = 1,2,\ldots$; $\lambda^{(0)}$ given. (For simplicity, the h on u_i is omitted.) The initial guess $\lambda^{(0)}$ is being chosen by condition (2.1).

(b) Consider the test problem

$$\begin{gathered}\varepsilon u'(x) + \frac{1}{1+x^2}u(x) = 2\lambda + \sin\frac{\lambda x}{2}, \quad 0 \le x \le 1,\\ u(0) = 1, \qquad u(1) = 0.\end{gathered} \tag{4.2}$$

The initial guess in (4.2) is taken as $\lambda^{(0)} = 0.00039$ and stopping criterion is

$$\max_i |u^{(n)} - u^{(n-1)}| \le 10^{-5}; \qquad |\lambda^{(n)} - \lambda^{(n-1)}| \le 10^{-5}. \tag{4.3}$$

We use a double-mesh method (see, e.g., [5]) to compute the experimental rates of convergence:

$$p_u^{\varepsilon,h} = \frac{\ln(e_u^{\varepsilon,h}/e_u^{\varepsilon,h/2})}{\ln 2}, \qquad e_u^{\varepsilon,h} = \max_{0\le i\le N} |u_i^h - u_{2i}^{h/2}| \tag{4.4}$$

for u_i^h, and

$$p_\lambda^{\varepsilon,h} = \frac{\ln(e_\lambda^{\varepsilon,h}/e_\lambda^{\varepsilon,h/2})}{\ln 2}, \qquad e_\lambda^{\varepsilon,h} = |\lambda^h - \lambda^{h/2}| \tag{4.5}$$

for λ^h.

Tables 4.1 and 4.2 contain some numerical results for different values of ε and h, based on the double-mesh principle. The result established here is that the discrete solution is uniformly convergent with respect to the perturbation parameter, and also clearly, that we obtain first-order convergence, so Theorem 3.3 is sharp.

Table 4.1. Errors $\{e_u^{\varepsilon,h}, e_\lambda^{\varepsilon,h}\}$ and convergence rates $\{p_u^{\varepsilon,h}, p_\lambda^{\varepsilon,h}\}$ on ω_h for $h = 1/8$ and $h = 1/16$.

ε	$h = 1/8$		$h = 1/16$		$p_u^{\varepsilon,h}$	$p_\lambda^{\varepsilon,h}$
	$e_u^{\varepsilon,h}$	$e_\lambda^{\varepsilon,h}$	$e_u^{\varepsilon,h/2}$	$e_\lambda^{\varepsilon,h/2}$		
10^{-3}	0.00002235	0.00000506	0.00001069	0.00000248	1.064	1.028
10^{-4}	0.00002235	0.00000506	0.00001069	0.00000248	1.064	1.027
10^{-5}	0.00002235	0.00000506	0.00001069	0.00000248	1.064	1.027
10^{-6}	0.00002235	0.00000506	0.00001069	0.00000248	1.064	1.027
10^{-7}	0.00002235	0.00000506	0.00001069	0.00000248	1.064	1.027
10^{-8}	0.00002235	0.00000506	0.00001069	0.00000248	1.064	1.027
10^{-9}	0.00002235	0.00000506	0.00001069	0.00000248	1.064	1.027
10^{-10}	0.00002235	0.00000506	0.00001069	0.00000248	1.064	1.027
10^{-11}	0.00002235	0.00000506	0.00001069	0.00000248	1.064	1.027
10^{-12}	0.00002235	0.00000506	0.00001069	0.00000248	1.064	1.027

Table 4.2. Errors $\{e_u^{\varepsilon,h}, e_\lambda^{\varepsilon,h}\}$ and convergence rates $\{p_u^{\varepsilon,h}, p_\lambda^{\varepsilon,h}\}$ on ω_h for $h = 1/16$ and $h = 1/32$.

ε	$h = 1/16$		$h = 1/32$		$p_u^{\varepsilon,h}$	$p_\lambda^{\varepsilon,h}$
	$e_u^{\varepsilon,h}$	$e_\lambda^{\varepsilon,h}$	$e_u^{\varepsilon,h/2}$	$e_\lambda^{\varepsilon,h/2}$		
10^{-3}	0.00001152	0.00000248	0.00000566	0.00000122	1.028	1.028
10^{-4}	0.00001152	0.00000248	0.00000571	0.00000123	1.014	1.014
10^{-5}	0.00001152	0.00000248	0.00000571	0.00000123	1.014	1.014
10^{-6}	0.00001152	0.00000248	0.00000571	0.00000123	1.014	1.014
10^{-7}	0.00001152	0.00000248	0.00000571	0.00000123	1.014	1.014
10^{-8}	0.00001152	0.00000248	0.00000571	0.00000123	1.014	1.014
10^{-9}	0.00001152	0.00000248	0.00000571	0.00000123	1.014	1.014
10^{-10}	0.00001152	0.00000248	0.00000571	0.00000123	1.014	1.014
10^{-11}	0.00001152	0.00000248	0.00000571	0.00000123	1.014	1.014
10^{-12}	0.00001152	0.00000248	0.00000571	0.00000123	1.014	1.014

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