GENERALIZED RESOLVENTS AND SPECTRUM FOR A CERTAIN CLASS OF PERTURBED SYMMETRIC OPERATORS

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The generalized resolvents for a certain class of perturbed symmetric operators with equal and finite deficiency indices are investigated. Using the Weinstein-Aronszajn formula, we give a classification of the spectrum.

1. Introduction

The present paper is concerned with the study of spectral properties for a certain class of linear symmetric operator T, defined in the Hilbert space H of the form $T = A + B$, where A is a closed linear symmetric operator, with nondensely defined domain in general, $D(A) \subset H$, and B is a finite-rank operator of the form

$$Bf = \sum_{k=1}^{n} a_k (f, y_k) y_k, \tag{1.1}$$

where $y_1, y_2, \dots, y_n$ is a linearly independant system in H, $a_1, a_2, \dots, a_n \in \mathbb{R}$. We remark that the operator T can be considered as a perturbation of the operator A by the finite-rank operator B.

The case when A is a first-order or second-order differential operator in the spaces $L^2(0, 2\pi)$, $L^2(0, \infty)$ or in the Hilbert space of vector-valued functions, and B is a one-dimensional perturbation ($n = 1$), has been studied by many authors (see, e.g., [9, 20, 24]).

In particular, certain integrodifferential equations of the above type occur in quantum mechanical scattering theory [8].

In this paper, the generalized resolvents of perturbed symmetric operator T with equal and finite deficiency indices are investigated. Using the Weinstein-Aronszajn formula (see, e.g., [18]), we give a classification of the spectrum. Finally, the obtained results are applied to the study of two classes of first-order and second-order differential operators.

We note that the spectral theory of perturbed symmetric and selfadjoint operators have been investigated using various methods by many authors [3, 4, 5, 6, 11, 12, 13, 14, 15, 16, 17, 21, 22].

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2. Preliminaries

Let A be a closed symmetric operator with nondensely defined domain in a separable Hilbert space H with equal deficiency indices (m,m), and $m<\infty$. We denote by $\rho(A)$ the resolvent set of the operator A, the resolvent operator $R_\lambda(A)$ of A is defined as $R_\lambda(A)=(A-\lambda I)^{-1}$. The complement of $\rho(A)$ in the complex plane is called the spectrum of A and denoted by $\sigma(A)$. There is a decomposition of the spectrum $\sigma(A)$ into three disjoint subsets, at least one of which is not empty [1, 2, 10]:

$$\sigma(A)=P\sigma(A)\cup C\sigma(A)\cup PC\sigma(A), \tag{2.1}$$

$P\sigma(A)$ is called the point spectrum, $C\sigma(A)$ the continuous spectrum, and $PC\sigma(A)$ the point-continuous spectrum. We denote the essential spectrum of the operator A by $\sigma_e(A)=C\sigma(A)\cup PC\sigma(A)$.

For arbitrary $\lambda\in\mathbb{C}$, we denote $P_\lambda=N_\lambda\cap(D(A)\oplus N_{\overline{\lambda}})$, where $N_\lambda=H\Theta(A-\lambda I)D(A)$ is the deficiency subspace of the operator A [1, 2].

It is known [23] that $P_\lambda=\{0\}$ if and only if $\overline{D(A)}=H$, and if $\overline{D(A)}\neq H$, then the subset

$$G_\lambda=\{[\varphi,\psi]\in N_\lambda\times N_{\overline{\lambda}}:\varphi-\psi\in D(A)\} \tag{2.2}$$

is a graph of the isometric operator X_λ with domain P_λ and values in $P_{\overline{\lambda}}$.

We denote by $\mathfrak{I}$ the set of linear operators F defined from N_i to N_{-i}, such that $\|F\|\leq 1$. For each analytic operator-valued function $F(\lambda)$ in $\mathbb{C}^+$, with $\mathbb{C}^+=\{\lambda:\operatorname{Im}\lambda>0\}$, and values in $\mathfrak{I}$, we introduce the set $\Omega_F(\infty)$ consisting of elements $h\in N_i$ such that

$$\lim_{\lambda\to\infty,\lambda\in\mathbb{C}_\varepsilon^+}|\lambda|\left[\|h\|-\|F(\lambda)h\|\right]<\infty, \tag{2.3}$$

where $C_\varepsilon^+=\{\lambda\in\mathbb{C}^+:\varepsilon<\arg\lambda<\pi-\varepsilon\}$, $0<\varepsilon<\pi/2$.

It is known [27] that $\Omega_F(\infty)$ is a vector space and for each $h\in\Omega_F(\infty)$,

$$\lim_{\lambda\to\infty,\lambda\in\mathbb{C}_\varepsilon^+}F(\lambda)h=F_0(\infty)h \tag{2.4}$$

exists in the sense of the strong topology, and $F_0(\infty)$ is an isometric operator.

According to the theory of Štraus [28], the generalized resolvents of A are given by the formula

$$R_\lambda(A)=R_\lambda=\left(A_{F(\lambda)}-\lambda I\right)^{-1},\quad R_{\overline{\lambda}}=R_\lambda^*,\quad \lambda\in\mathbb{C}^+, \tag{2.5}$$

where $A_{F(\lambda)}$ is an extension of A which is determined by the function $F(\lambda)$, whose values are operators from the deficiency subspace N_i to the deficiency subspace N_{-i} such that $\|F(\lambda)\|\leq 1$ and $F(\lambda)$ satisfy the condition

$$F_0(\infty)\psi=X_i\psi,\quad \text{for } \psi=0 \text{ only}, \tag{2.6}$$

then $A_{F(\lambda)}$ is a restriction on H of a selfadjoint operator defined in a certain extended Hilbert space and is called quasiselfadjoint extension of the operator A [28] defined on $D(A_{F(\lambda)}) = D(A) + (F(\lambda) - I)N_i$ by

$$A_{F(\lambda)}(f + F(\lambda)\varphi - \varphi) = Af + iF(\lambda)\varphi + i\varphi, \quad f \in D(A),\ \varphi \in N_i. \tag{2.7}$$

For selfadjoint extensions with exit in the space in which acts the considered operators, see, for example, [12, 21] and the references therein.

We denote by $\aleph$ the set of analytic operator functions $F(\lambda)$ in $\mathbb{C}^+$ with values in $\mathfrak{I}$ satisfying the condition (2.6).

Remark 2.1. To each selfadjoint extension of the operator A corresponds a certain constant operator function $F(\lambda) = V$, where V is an isometric operator defined from N_i over N_{-i} satisfying the condition $V\psi = X_i\psi$ for $\psi = 0$ only, and reciprocally.

We denote by $\mathring{A}$ a selfadjoint extension of A and we introduce the operator

$$\mathring{U}_{\lambda\lambda_0} = (\mathring{A} - \lambda_0 I)(\mathring{A} - \lambda I)^{-1}, \quad \operatorname{Im}\lambda > 0. \tag{2.8}$$

We note that (see [19])

$$\mathring{U}_{\lambda\lambda_0} N_{\overline{\lambda_0}} = N_{\overline{\lambda}}, \quad (\operatorname{Im}\lambda)(\operatorname{Im}\lambda_0) \neq 0. \tag{2.9}$$

We denote by

$$\varphi_i^{(1)}, \varphi_i^{(2)}, \dots, \varphi_i^{(m)} \tag{2.10}$$

a basis of N_{-i}. From (2.9), $\varphi_\lambda^{(k)} = \mathring{U}_{\lambda i}\varphi_i^{(k)}$, $k = 1, 2, \dots, m$ form a basis for $N_{\overline{\lambda}}$. In particular, the vectors

$$\varphi_{-i}^{(k)} = \mathring{U}\varphi_i^{(k)}, \quad k = 1, 2, \dots, m, \tag{2.11}$$

where $\mathring{U} = \mathring{U}_{-ii}$ is the Cayley transform [1, 2] of $\mathring{A}$, form an orthogonal basis of N_i.

To get a convenient formula of the generalized resolvents of A, we will need the following notation:

$$\Phi_{\lambda\mu} = (\lambda - \overline{\mu})\left[\left(\varphi_\lambda^{(k)}, \varphi_\mu^{(s)}\right)\right]_{k,s=1}^{m}, \qquad C(\lambda) = \Phi_{\lambda i}^{-1}\Phi_{\lambda(-i)}, \tag{2.12}$$

where E is the identity matrix of order m, $\Omega(\lambda)$ is an analytic matrix function in $\mathbb{C}^+$ corresponding, in the bases (2.10) and (2.11), to the operator function $F(\lambda) \in \aleph$ and $\varphi_\lambda = (\varphi_\lambda^{(1)}, \dots, \varphi_\lambda^{(m)})^t$, $(f, \varphi_{\overline{\lambda}})^t = ((f, \varphi_{\overline{\lambda}}^{(1)}), \dots, (f, \varphi_{\overline{\lambda}}^{(m)}))$, t denotes the transpose, and (φ_λ, g) is defined analogously.

In what follows, we denote by Φ the set of matrices $\Omega(\lambda)$, $\lambda \in \mathbb{C}^+$, associated in the bases (2.10) and (2.11) to the operator functions $F(\lambda) \in \aleph$.

According to the notation used in [7], the generalized resolvents of A are given by

$$R_\lambda(A)f = R_\lambda f = \mathring{R}_\lambda f + (f,\varphi_{\overline{\lambda}})^t [E - \Omega(\lambda)][C(\lambda)\Omega(\lambda) - E]^{-1}\Phi_{\overline{\lambda}i}^{-1}\varphi_\lambda, \\ R_{\overline{\lambda}} = R_\lambda^*, \quad \lambda \in \mathbb{C}^+, \tag{2.13}$$

where $\mathring{R}_\lambda$ is the resolvent of $\mathring{A}$ and $\Omega(\lambda) \in \Phi$.

Remark 2.2. The formula (2.13) defines a resolvent of a selfadjoint extension of A if and only if $\Omega(\lambda)$ is a unitary constant matrix.

3. Resolvent and spectrum of a symmetric perturbed operator

Let $T = A + B$ be defined on $D(T) = D(A)$, where A is a linear closed symmetric operator in H and B is a finite-rank operator.

Lemma 3.1. *For $\lambda \in \rho(A) \cap \rho(T)$, the resolvent $R_\lambda(T)$ of the operator T is given by*

$$R_\lambda(T) = R_\lambda(A) - R_\lambda(A)[I + BR_\lambda(A)]^{-1}BR_\lambda(A). \tag{3.1}$$

Proof. For $\lambda \in \rho(A) \cap \rho(T)$, the operator

$$R_\lambda(A)[I + BR_\lambda(A)]^{-1} = R_\lambda(T) \tag{3.2}$$

exists and is bounded. Then, we get

$$\begin{aligned}(T - \lambda I)&[R_\lambda(A) - R_\lambda(A)(I + BR_\lambda(A))^{-1}BR_\lambda(A)] \\ &= (A - \lambda I + B)[R_\lambda(A) - R_\lambda(A)(I + BR_\lambda(A))^{-1}BR_\lambda(A)] \\ &= I + BR_\lambda(A) - (I + BR_\lambda(A))(I + BR_\lambda(A))^{-1}BR_\lambda(A) = I\end{aligned} \tag{3.3}$$

as required. □

Remark 3.2. If $\|BR_\lambda(A)\| < 1$, then from (3.1), we obtain

$$R_\lambda(T) = R_\lambda(A)(I + BR_\lambda(A))^{-1} = R_\lambda(A)\sum_{k=0}^{\infty}(-1)^k[BR_\lambda(A)]^k. \tag{3.4}$$

Now, the aim is to give a convenient expression of $(I + BR_\lambda(A))^{-1}$ in a more specific case.

So, we study in detail the case when B is a finite-rank operator. Then,

$$Bf = \sum_{k=1}^{n} a_k(f, y_k)y_k, \quad f \in H, \tag{3.5}$$

where $a_1, a_2, \dots, a_n \in \mathbb{R}$; $\{y_1, y_2, \dots, y_n\}$ is a linearly independent system in H. If we put

$$(I + BR_\lambda(A))^{-1}BR_\lambda(A)f = y, \tag{3.6}$$

we have

$$y = BR_\lambda(A)f - BR_\lambda(A)y, \tag{3.7}$$

then, $y \in \operatorname{Im} B$, so that

$$y = \sum_{k=1}^{n} c_k y_k. \tag{3.8}$$

From (3.7) and (3.8), we get

$$\sum_{k=1}^{n} c_k y_k = BR_\lambda(A)f - \sum_{k=1}^{n} c_k BR_\lambda(A)y_k, \tag{3.9}$$

with

$$c_k + a_k \sum_{j=1}^{n} c_j \left(R_\lambda(A)y_j, y_k\right) = a_k \left(R_\lambda(A)f, y_k\right). \tag{3.10}$$

The determinant $\Delta(\lambda)$ of the system (3.10) is given by

$$\Delta(\lambda) = \det\left\{\left[\delta_{kj} + a_k \left(R_\lambda(A)y_j, y_k\right)\right]_{k,j=1}^{n}\right\}, \tag{3.11}$$

where δ_{kj} is the Kronecker symbol. If we suppose that $\Delta(\lambda) \neq 0$, the solution of (3.10) is given by

$$c_k = c_k(\lambda; f) = \frac{\left(f, \Delta_k(\lambda)\right)}{\Delta(\lambda)}, \quad k = 1, 2, \ldots, n, \tag{3.12}$$

where $\Delta_k(\lambda)$ is the determinant obtained from $\overline{\Delta(\lambda)}$ by replacing the kth column by $[a_j R_{\overline{\lambda}}(A) y_j]_{j=1}^{n}$. So, from (3.1), we have

$$R_\lambda(T)f = R_\lambda(A)f - \sum_{k=1}^{n} \frac{\left(f, \Delta_k(\lambda)\right)}{\Delta(\lambda)} R_\lambda(A) y_k. \tag{3.13}$$

This completes the proof of the following theorem.

THEOREM 3.3. *Let $\lambda \in \rho(A)$ such that $\Delta(\lambda) \neq 0$. Then, $\lambda \in \rho(T)$ and the resolvent of the operator T is given by (3.13).*

Remark 3.4. From (3.13), we note that the resolvent $R_\lambda(T)$ is a perturbation of $R_\lambda(A)$ by a finite-rank operator.

Remark 3.5. For the particular case $n = 1$ and $a_1 = 1$, the formula (3.13) was established in [9].

Remark 3.6. If $\lambda \in \rho(A)$ such that $\Delta(\lambda) = 0$, then λ is an eigenvalue of the operator T.

Proof. We can show that there exists an element

$$\psi = \sum_{k=1}^{n} \alpha_k y_k \tag{3.14}$$

such that $R_\lambda(A)\psi$ is an eigenvector of the operator T, corresponding to the eigenvalue λ. Consequently, we have

$$a_k \sum_{j=1}^{n} \alpha_j \big(R_\lambda(A) y_j, y_k\big) + \alpha_k = 0, \quad k = \overline{1,n}. \tag{3.15}$$

Since the determinant of this system $\Delta(\lambda) = 0$, it admits a nontrivial solution, which gives the desired result. □

Theorem 3.7. *Let μ be a fixed complex number. Then, the following holds.*

(a) *If $\mu \in \rho(A)$ and $\Delta(\mu) \neq 0$, then $\mu \in \rho(T)$.*
(b) *If $\mu \in \rho(A)$ and $\Delta(\mu) = 0$, then $\mu \in P\sigma(T)$ and the multiplicity of μ as an eigenvalue of T is equal to the order of the zero of $\Delta(\lambda)$ at μ.*
(c) *If $\mu \in P\sigma(A)$ and μ of multiplicity $k > 0$ and if μ is a pole of $\Delta(\lambda)$ of multiplicity p $(k \geq p)$, then*
 (1) *for $k > p$, it holds that $\mu \in P\sigma(T)$ of multiplicity $(k - p)$,*
 (2) *for $k = p$, it holds that $\mu \in \rho(T)$.*
(d) *If $\mu \in P\sigma(A)$ is neither a zero, nor a pole of $\Delta(\lambda)$, then $\mu \in P\sigma(T)$.*
(e) *If $\mu \in P\sigma(A)$ of multiplicity k and μ is a root of the function $\Delta(\lambda)$ of order p, then $\mu \in P\sigma(T)$ of order $(k + p)$.*
(f) *The essential spectra $\sigma_e(A)$ and $\sigma_e(T)$, respectively of the operators A and T, coincide.*

Proof. It is sufficient to evaluate the function

$$C(\lambda) = \det\{I + BR_\lambda(A)\}. \tag{3.16}$$

To this end, let $y \in \operatorname{Im} B$. Then,

$$BR_\lambda(A)y = \sum_{k=1}^{n} a_k \big(y, R_\lambda^*(A) y_k\big) y_k, \tag{3.17}$$

it is clear that $C(\lambda) = \Delta(\lambda)$, and the function $\Delta(\lambda)$ is meromorphic in $\rho(A) \cup P\sigma(A)$. From the formula of Weinstein and Aronszajn [18], we have

$$\overline{\vartheta}(\lambda; T) = \overline{\vartheta}(\lambda; A) + \vartheta(\lambda; \Delta), \tag{3.18}$$

where

$$\overline{\vartheta}(\lambda;A)=\begin{cases}0 & \text{if } \lambda\in\rho(A),\\ k & \text{if } \lambda\in P\sigma(A) \text{ and of multiplicity } k,\\ +\infty & \text{otherwise},\end{cases}$$
$$\vartheta(\lambda;\Delta)=\begin{cases}k & \text{if } \lambda \text{ is a zero of } \Delta(\lambda) \text{ of order } k,\\ -k & \text{if } \lambda \text{ is a pole of } \Delta(\lambda) \text{ of order } k,\\ 0 & \text{for other } \lambda\in\Omega,\end{cases} \tag{3.19}$$

which gives the desired result. □

4. Generalized resolvents

Now, we suppose that A is a symmetric operator with deficiency indices (m,m), $m<\infty$.

Lemma 4.1. *Let $\lambda\in\mathbb{C}$ such that $\operatorname{Im}\lambda>0$ and $\varphi_\lambda(A)\in N_{\overline{\lambda}}(A)$. Then, the element $\varphi_\lambda(T)$, defined by the formula*

$$\varphi_\lambda(T)=D(\lambda)\varphi_\lambda(A)=\varphi_\lambda(A)-\sum_{k=1}^{n}\frac{(\varphi_\lambda(A),\mathring{g}_k(\lambda))}{\mathring{\Delta}(\lambda)}R_\lambda(\mathring{A})y_k, \tag{4.1}$$

is an element of the deficiency subspace $N_{\overline{\lambda}}(T)$, where

$$D(\lambda)=I-R_\lambda(\mathring{A})\left[I+BR_\lambda(\mathring{A})\right]^{-1}B=I-R_\lambda(\mathring{T})B,\qquad \mathring{g}_k(\lambda)=(\mathring{A}-\overline{\lambda}I)\mathring{\Delta}_k(\lambda), \tag{4.2}$$

$\mathring{\Delta}(\lambda)$ and $\mathring{\Delta}_k(\lambda)$ are defined similarly as $\Delta(\lambda)$ and $\Delta_k(\lambda)$ in the formula (3.13) by putting the operator $\mathring{A}$ instead of the operator A.

Proof. Since the operators $\mathring{A}$ and $\mathring{T}=\mathring{A}+B$ are selfadjoint and λ is nonreal, then $\lambda\in\rho(\mathring{A})\cap\rho(\mathring{T})$. In addition, from Theorem 3.3 we have $\mathring{\Delta}(\lambda)\neq 0$. Furthermore, for each $f\in D(A)=D(T)$, we have

$$\begin{aligned}([\mathring{T}-\overline{\lambda}I]f,D(\lambda)\varphi_\lambda(A)) &= (D^*(\lambda)[\mathring{T}-\overline{\lambda}I]f,\varphi_\lambda(A))\\ &= ([I-BR_{\overline{\lambda}}(\mathring{T})](\mathring{T}-\overline{\lambda}I)f,\varphi_\lambda(A))\\ &= ((\mathring{A}-\overline{\lambda}I)f,\varphi_\lambda(A))\\ &= 0,\end{aligned} \tag{4.3}$$

and the equality

$$\varphi_\lambda(T)=\varphi_\lambda(A)-\sum_{k=1}^{n}\frac{(\varphi_\lambda(A),\mathring{g}_k(\lambda))}{\mathring{\Delta}(\lambda)}R_\lambda(\mathring{A})y_k \tag{4.4}$$

results from (3.13). □

Remark 4.2. We note that if $\varphi_\lambda(A)\neq 0$, then $\varphi_\lambda(T)\neq 0$.

Proof. If we suppose the contrary, we obtain $R_\lambda(\overset{\circ}{T})B\varphi_\lambda(A) = \varphi_\lambda(A)$, which gives $\overset{\circ}{A}\varphi_\lambda(A) = \lambda\varphi_\lambda(A)$. This leads to a contradiction, since a selfadjoint operator can not have nonreal eigenvalues. □

Remark 4.3. If $D(A)$ is dense in H, then $\varphi_\lambda(A)$ and $\varphi_\lambda(T)$ are, respectively, eigenfunctions of the operators A^* and T^*, corresponding to the eigenvalues $\overline{\lambda}$.

Let $\varphi_i^{(k)}(T) = D(i)\varphi_\lambda^{(k)}(A)$, $k = 1,2,\dots,m$, defined by the formula (4.1). If $\varphi_i^{(1)}(A)$, $\varphi_i^{(2)}(A),\dots,\varphi_i^{(m)}(A)$ is a basis of the deficiency subspace $N_i(A)$ of the operator A, then $\varphi_i^{(1)}(T),\varphi_i^{(2)}(T),\dots,\varphi_i^{(m)}(T)$ is a basis of the deficiency subspace $N_i(T)$ of the operator T. Putting

$$\begin{aligned} \overset{\circ}{U}_{\lambda\lambda_0}(\overset{\circ}{T}) &= (\overset{\circ}{T} - \lambda_0 I)R_\lambda(\overset{\circ}{T}), \qquad & \varphi_\lambda^{(k)}(T) &= \overset{\circ}{U}_{\lambda i}(\overset{\circ}{T})\varphi_i^{(k)}(T), \quad k = 1,2,\dots,m, \\ \varphi_\lambda(T) &= \left(\varphi_\lambda^{(1)}(T),\dots,\varphi_\lambda^{(m)}(T)\right)^t, \qquad & \Phi_{\lambda\mu}(T) &= (\lambda - \overline{\mu})\left[\left(\varphi_\lambda^{(k)}(T),\varphi_\mu^{(j)}(T)\right)\right]_{k,j=1}^m, \end{aligned} \tag{4.5}$$

$C(\lambda) = \Phi_{\overline{\lambda} i}^{-1}(T)\Phi_{\lambda(-i)}(T)$ denotes the characteristic matrix of the operator T, and $\omega(\lambda)$ the corresponding matrix of order $m \times m$, in the bases $\varphi_i^{(1)}(T)$, $\varphi_i^{(2)}(T),\dots,\varphi_i^{(m)}(T)$ and $\varphi_{-i}^{(1)}(T)$, $\varphi_{-i}^{(2)}(T),\dots,\varphi_{-i}^{(m)}(T)$.

Theorem 4.4. *The set of all generalized resolvents of the operator T is given by*

$$R_\lambda(T)f = R_\lambda(\overset{\circ}{T})f + \left(f,\varphi_{\overline{\lambda}}(T)\right)^t\left[E - \omega(\lambda)\right]\left[C(\lambda)\omega(\lambda) - E\right]^{-1}\Phi_{\overline{\lambda} i}^{-1}(T)\varphi_\lambda(T), \quad \forall f \in H, \tag{4.6}$$

where

$$R_\lambda(\overset{\circ}{T})f = R_\lambda(\overset{\circ}{A})f - \sum_{k=1}^n \frac{\left(f,\overset{\circ}{\Delta}_k(\lambda)\right)}{\overset{\circ}{\Delta}(\lambda)}R_\lambda(\overset{\circ}{A})y_k. \tag{4.7}$$

Proof. The proof results from Lemma 4.1 and formula (2.13). □

We denote, respectively, by A_ω and T_ω the quasiselfadjoint extensions of operators A and T corresponding to the operator function $F(\lambda) \in \mathfrak{I}$, defined by the matrix $\omega(\lambda)$.

Remark 4.5. To selfadjoint extensions of these operators correspond the constant unitary matrices $\omega = [\omega_{ij}]$.

Theorem 4.6. *Suppose that $y_1,y_2,\dots,y_n \in \operatorname{Im} A$, μ is an eigenvalue of the quasiselfadjoint extension A_ω of the operator A, $\mu \in P\sigma(A_\omega)$. If $\mu \in \rho(\overset{\circ}{A})$ and $\overset{\circ}{\Delta}(\mu) \neq 0$, then μ is an eigenvalue of the operator $T_\omega = A_\omega + B$ and the corresponding eigenfunction $\varphi_\mu(T_\omega)$ is given by*

$$\varphi_\mu(T_\omega) = D(\mu)\varphi_\mu(A_\omega) = \varphi_\mu(A_\omega) - \sum_{k=1}^n \frac{\left(\varphi_\mu(A_\omega),\overset{\circ}{g}_k(\mu)\right)}{\overset{\circ}{\Delta}(\mu)}R_\mu(\overset{\circ}{A})y_k, \tag{4.8}$$

where $\varphi_\mu(A_\omega)$ is the eigenfunction of the operator A_ω, corresponding to the eigenvalue μ.

Proof. Since $y_1, y_2, \dots, y_n \in \operatorname{Im} A$, then $B\varphi_\mu(A) \in \operatorname{Im} A$. We also have

$$\varphi_\mu(T_\omega) = D(\mu)\varphi_\mu(A_\omega) = \varphi_\mu(A_\omega) - R_\mu(\mathring{T})B\varphi_\mu(A) = \varphi_\mu(A_\omega) - \psi_\mu, \tag{4.9}$$

where

$$\psi_\mu = R_\mu(\mathring{T})B\varphi_\mu(A) \in D(A). \tag{4.10}$$

Then,

$$\begin{aligned} T_\omega\varphi_\mu(T_\omega) &= T_\omega(\varphi_\mu(A_\omega) - \psi_\mu) \\ &= (A_\omega + B)\varphi_\mu(A_\omega) - T_\omega R_\mu(\mathring{T})B\varphi_\mu(A) \\ &= \mu\varphi_\mu(A_\omega) + B\varphi_\mu(A_\omega) - B\varphi_\mu(A_\omega) + \mu R_\mu(\mathring{T})B\varphi_\mu(A) \\ &= \mu\varphi_\mu(T_\omega). \end{aligned} \tag{4.11}$$

□

5. Applications

5.1. Perturbed first-order differential operator. Consider in $L^2(0,2\pi)$ the operator $T = A + B$, where A is defined by $Ay = iy'$ with domain $D(A) = H_0^1(0,2\pi)$ and B is given by

$$(By)(x) = \sum_{k=1}^{n} a_k(y, y_k)y_k(x), \tag{5.1}$$

where $y_1, y_2, \dots, y_n \in L^2(0,2\pi)$ and $a_k \in \mathbb{R}$, for all $k = \overline{1,n}$. From [1, 2], the operator A is regular symmetric of deficiency indices $(1,1)$ and each selfadjoint extension of A has a discrete spectrum.

Theorem 5.1. *The generalized resolvent $R_\lambda(T_\theta)$ of T, corresponding to the function $\omega(\lambda) = \theta(\lambda)$, is an integral operator with kernel*

$$K(x,t) = \left[1_{[x,2\pi]}(x) + \frac{1}{\theta(\lambda)e^{2\pi\lambda i} + 1}\right]e^{i\lambda(t-x)} + \sum_{k=1}^{n}\theta_k(\lambda,x)\phi_k(\lambda,t), \tag{5.2}$$

where $1_{[x,2\pi]}(x)$ is the characteristic function of the interval $[x,2\pi]$,

$$\phi_k(\lambda,t) = (\Delta_k^\theta(\lambda))(t), \qquad \theta_k(\lambda,x) = \frac{(R_\lambda(A_\theta)y_k)(x)}{\Delta^\theta(\lambda)}, \tag{5.3}$$

where $R_\lambda(A_\theta)$, associated to the function $\theta(\lambda)$, is given by

$$(R_\lambda(A_\theta)y)(x) = \int_0^x y(t)e^{i\lambda(t-x)}dt - \frac{1}{\theta(\lambda)e^{2\pi ti} + 1}\int_0^{2\pi} y(t)e^{\lambda i(t-x)}dt \tag{5.4}$$

with

$$\Delta^\theta(\lambda) = \{\delta_{k_j} + a_k(R_\lambda(A_\theta)y_j, y_k)\}, \tag{5.5}$$

and Δ_k^θ is the determinant obtained from $\overline{\Delta^\theta(\lambda)}$ replacing the kth column by $[a_k R_{\overline{\lambda}}(A_\theta)y_k]_1^n$.

Proof. The proof results from [26] and Theorem 3.3. □

Corollary 5.2. *Let T_θ be a selfadjoint extension of T corresponding to the function θ, $|\theta| = 1$.*

(1) *The spectrum of T_θ is simple if and only if the roots of $\Delta^\theta(\lambda)$ are simple and for $k = 0, \pm 1, \pm 2, \dots, \Delta^\theta(1/2 + k - \varphi_0/2\pi) \neq 0$, where $\{1/2 + k - \varphi_0/2\pi\}$ is the spectrum of A_θ, and $\varphi_0 = \arg\theta$.*

(2) $\sigma(T_\theta) = P\sigma(T_\theta) = E_1 \cup E_2$, *where E_1 is the set of points of $\sigma(A_\theta) = \{1/2 + k - \varphi_0/2\pi, k = 0, \pm 1, \pm 2, \dots\}$ in which $\Delta^\theta(\lambda)$ is analytic, E_2 is the set of roots of $\Delta^\theta(\lambda)$.*

Proof. The proof results from (5.4), Theorem 3.7, and Lemma 4.1. □

5.2. Perturbed second-order differential operator. Consider in $L^2(0,\infty)$ the operator $T = A + B$, where A is defined by

$$Ay = -y'' + x^2 y \tag{5.6}$$

with domain $D(A)$ consisting of all variables y which satisfy

(i) $y \in L^2(0,\infty)$,
(ii) y' is absolutely continuous on all compact subintervals of $[0,\infty[$,
(iii) $Ay \in L^2(0,\infty)$,
(IV) $y(0) = y(\infty) = \lim_{x\to\infty} y(x) = 0$, $y'(0) = y'(\infty) = 0$,

and B is given by

$$(By)(x) = \sum_{k=1}^{n} a_k (y, y_k) y_k(x), \tag{5.7}$$

where $y_1, y_2, \dots, y_n \in L^2(0,2\pi)$ and $a_k \in IR$, for all $k = \overline{1,n}$.

From [1, 2], the operator A is symmetric of deficiency indices $(1,1)$. Let u_1, u_2 be two solutions of (5.6), satisfying the initial conditions

$$\begin{aligned} u_1(0,\lambda) &= 1, \qquad u_1'(x,\lambda)\big|_{x=0} = 0, \\ u_2(0,\lambda) &= 0, \qquad u_2'(x,\lambda)\big|_{x=0} = -1. \end{aligned} \tag{5.8}$$

There exists a function $m(\lambda)$ [29] analytic in $\mathbb{C}\backslash\mathbb{R}$ such that

$$\psi(x,\lambda) = u_2(x,\lambda) + m(\lambda) u_1(x,\lambda) \in L^2(0,\infty). \tag{5.9}$$

Theorem 5.3. *The generalized resolvents $R_\lambda(T_\theta)$ of the operator T are defined by*

$$R_\lambda(T_\theta) y = R_\lambda(A_\theta) y - \sum_{k=1}^{n} \frac{(y, \Delta_k^\theta(\lambda))}{\Delta^\theta(\lambda)} R_\lambda(A_\theta) y_k, \quad \operatorname{Im}\lambda > 0, \tag{5.10}$$

where

$$R_\lambda(A_\theta)y = \psi(x,\lambda)\int_0^x y(s)u_1(s,\lambda)ds + u_1(x,\lambda)\int_x^\infty y(s)\psi(s,\lambda)ds - \frac{\psi(x,\lambda)}{\theta(\lambda)+m(\lambda)}\int_0^\infty y(s)\psi(s,\lambda)ds, \tag{5.11}$$

$$\Delta^\theta(\lambda) = \det\{\sigma_{j_k} + a_k(R_\lambda(A_\theta)y_j, y_k)\}, \quad \lambda \in \mathbb{C}^+, \tag{5.12}$$

with $\theta(\lambda)$ *an arbitrary function analytic in* $\mathbb{C}^+$ *and such that* $\operatorname{Im}\theta(\lambda) \geq 0$ *or* $\theta(\lambda)$ *is an infinite constant.*

Proof. First, we show that for $\lambda \in \mathbb{C}^+$, $\Delta^\theta(\lambda) \neq 0$ (then, $\Delta^\theta \neq 0$). We know (see [1, 2]) that for each quasiselfadjoint extension of a symmetric operator, $\mathbb{C}^+$ is contained in the set of regular points of this operator. Then, if $\lambda \in \mathbb{C}^+$, we have $\lambda \in \rho(A_\theta)$ and $\lambda \in \rho(T_\theta)$. If we suppose that $\lambda \in \mathbb{C}^+$ and $\Delta^\theta(\lambda) = 0$, from Theorem 3.7, we obtain $\lambda \in P\sigma(T_\theta)$, which is a contradiction. The formula (5.11) results from [25]. Using Theorem 3.3, we end the proof. □

COROLLARY 5.4. *Let* T_θ *be a selfadjoint extension associated to* $\theta \in \overline{IR}$, *let* $\lambda_1, \lambda_2, \ldots$ *be the roots of* $\Delta^\theta(\lambda)$ *in* $\rho(A_\theta)$ *and let* $z_1, z_2, \ldots$ *be the poles of* $\Delta^\theta(\lambda)$. *Then,*

$$P\sigma(T_\theta) = \left(P\sigma(A_\theta)\setminus\{z_i\}_1^\infty\right) \cup \{\lambda_j\}_1^\infty. \tag{5.13}$$

Proof. The proof results from (b) and (c) of Theorem 3.7. □

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