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Research Article

Some New Fixed-Point Theorems for a (ψ, ϕ)-Pair Meir-Keeler-Type Set-Valued Contraction Map in Complete Metric Spaces

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We obtain some new fixed point theorems for a (ψ, ϕ)-pair Meir-Keeler-type set-valued contraction map in metric spaces. Our main results generalize and improve the results of Klim and Wardowski, (2007).

1. Introduction and Preliminaries

Let (X, d) be a metric space, Y a subset of X, and $f : Y \to X$ a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that, for all $x, y \in Y$,

$$d(fx, fy) \leq \alpha \cdot d(x, y). \tag{1.1}$$

The well-known Banach's fixed-point theorem asserts that if $Y = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, a mapping $f : X \to X$ is called a quasi-contraction if there exists $k < 1$ such that

$$d(fx, fy) \leq k\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}, \tag{1.2}$$

for any $x, y \in X$. In 1974, Ćirić [2] introduced these maps and proved an existence and uniqueness fixed-point theorem.

Throughout this paper, by $\mathbb{R}$ we denote the set of all real numbers, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space. Let $C(X)$ denote a collection of all nonempty closed subsets of X and $CB(X)$ a collection of all nonempty closed and bounded subsets of X.

The existence of fixed points for various multivalued contractive mappings had been studied by many authors under different conditions. In 1969, Nadler Jr. [3] extended the famous Banach contraction principle from single-valued mapping to multivalued mapping and proved the below fixed-point theorem for multivalued contraction.

Theorem 1.1 (see [3]). *Let (X, d) be a complete metric space, and let T be a mapping from X into $CB(X)$. Assume that there exists $c \in [0, 1)$ such that*

$$\mathscr{H}(Tx, Ty) \leq cd(x, y) \quad \forall x, y \in X, \tag{1.3}$$

where $\mathscr{H}$ denotes the Hausdorff metric on $CB(X)$ induced by d; that is, $H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}$, for all $A, B \in CB(X)$ and $D(x, B) = \inf_{z\in B} d(x, z)$. Then T has a fixed point in X.

In 1989, Mizoguchi-Takahashi [4] proved the following fixed-point theorem.

Theorem 1.2 (see [4]). *Let (X, d) be a complete metric space, and let T be a map from X into $CB(X)$. Assume that*

$$\mathscr{H}(Tx, Ty) \leq \xi(d(x, y)) \cdot d(x, y), \tag{1.4}$$

for all $x, y \in X$, where $\xi : [0, \infty) \to [0, 1)$ satisfies $\limsup_{s\to t^+} \xi(s) < 1$ for all $t \in [0, \infty)$. Then T has a fixed point in X.

In 2006, Feng and Liu [5] gave the following theorem.

Theorem 1.3 (see [5]). *Let (X, d) be a complete metric space, and let $T : X \to C(X)$ be a multivalued map. If there exist $b, c \in (0, 1)$, $c < b$ such that for any $x \in X$, there is $y \in T(x)$ satisfying the following two conditions:*

(i) $b \cdot d(x, y) \leq D(x, Tx)$,

(ii) $D(x, Ty) \leq c \cdot d(x, y)$.

Then T has a fixed point in X provided that the mapping $f : X \to \mathbb{R}$ defined by $f(x) = D(x, Tx)$, $x \in X$, is lower semicontinuous; that is, if for any $\{x_n\} \subset X$ and $x \in X$, $x_n \to x$, then $f(x) \leq \lim_{n\to\infty} \inf f(x_n)$.

In 2007, Klim and Wardowski [6] proved the following fixed point theorem.

Theorem 1.4 (see [6]). *Let (X, d) be a complete metric space, and let $T : X \to C(X)$ be a multivalued map. Assume that the following conditions hold:*

(i) *the mapping $f : X \to \mathbb{R}$ defined by $f(x) = D(x, Tx)$, $x \in X$, is lower semicontinuous;*

(ii) *there exist* $b \in (0,1)$ *and* $\varphi : [0,\infty) \to [0,b)$ *such that*

$$\forall_{t\in[0,\infty)} \quad \left\{\limsup_{r\to t^+} \varphi(r) < b\right\},$$
$$\forall_{x\in X} \quad \exists_{y\in Tx} \quad \{bd(x,y) \le D(x,Tx) \land D(y,Ty) \le \varphi(d(x,y)) \cdot d(x,y)\}. \tag{1.5}$$

Then T has a fixed point in X.

Recently, Pathak and Shahzad [7] introduced a new class of mapping $\Theta[0,A)$ and generalized the results of Klim and Wardowski [6]. Suppose that $A \in (0,\infty]$, $\Theta[0,A)$ denote the class of functions $\theta : [0,A) \to \mathbb{R}$ satisfying the following conditions:

(1) θ is nondecreasing on $[0,A)$;

(2) $\theta(t) > 0$ for all $t \in (0,A)$;

(3) θ is subadditive in $(0,A)$; that is, $\theta(t_1 + t_2) \le \theta(t_1) + \theta(t_2)$.

The following theorem was introduced in Pathak and Shahzad [7].

Theorem 1.5 (see [7]). *Let (X,d) be a complete metric space and suppose that $T : X \to C(X)$. Assume that the following conditions hold:*

(i) *the mapping $f : X \to \mathbb{R}$ defined by $f(x) = D(x,Tx)$, $x \in X$, is lower semicontinuous,*

(ii) *there exists $\alpha : (0,\infty) \to (0,1)$ such that*

$$\forall_{t\in[0,\infty)} \quad \left\{\limsup_{r\to t^+} \alpha(r) < 1\right\}, \tag{1.6}$$

(iii) *there exists $\theta \in \Theta[0,A)$ satisfying the following condition:*

$$\forall_{x\in X} \quad \exists_{y\in Tx} \quad \{\theta(d(x,y)) \le \theta(D(x,Tx))\},$$
$$\forall_{x\in X} \quad \exists_{y\in Tx} \quad \{\theta(D(y,Ty)) \le \alpha(d(x,y)) \cdot \theta(d(x,y))\}. \tag{1.7}$$

Then T has a fixed point in X.

Later, Kamran and Kiran [8] improved some results of Pathak and Shahzad [7] by allowing T to have values in closed subsets of X. They proved that the function $\theta \in \Theta[0,A)$ is positive homogenous in $[0,A)$, that is,

(4) $\theta(at) \le a\theta(t)$ for all $a > 0$, $t \in [0,A)$,

and denote by $\Theta_h[0,A)$ the class of functions $\theta \in \Theta[0,A)$ satisfying condition (4). They proved the following theorem.

Theorem 1.6 (see [8]). *Let (X,d) be a complete metric space and suppose that α is a function from $(0,\infty)$ to $[0,1)$ such that*

$$\forall_{t\in[0,\infty)} \quad \left\{\limsup_{r\to t^+} \alpha(r) < 1\right\}. \tag{1.8}$$

Suppose that $T : X \to C(X)$. Assume that the following condition holds:

$$\theta(D(y, Ty)) \leq \alpha(d(x, y)) \cdot \theta(d(x, y)), \quad \text{for each } x \in X, \ y \in Tx, \tag{1.9}$$

where $\theta \in \Theta_h[0, A)$. Then

(i) *for each $x_0 \in X$, there exists an orbit $\{x_n\}$ of T and $\xi \in X$ such that $\lim_{n\to\infty} x_n = \xi$;*

(ii) *ξ is a fixed point of T if and only if the function $f(x) = D(x, Tx)$ is T-orbitally lower semicontinuous at ξ.*

2. Main Results

In this section, we first recall the notion of the Meir-Keeler-type function (see [9]). A function $\psi : [0, \infty) \to [0, \infty)$ is said to be a Meir-Keeler-type function, if, for each $\eta \in [0, \infty)$, there exists $\delta > 0$ such that for $t \in [0, \infty)$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$. We now define a new stronger Meir-Keeler-type function, as follows.

Definition 2.1. One calls $\psi : [0, \infty) \to [0, 1)$ the stronger Meir-Keeler-type function, if, for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0, \infty)$ with $\eta \leq t < \delta + \eta$, there exists $\gamma_\eta \in [0, 1)$ such that $\psi(t) < \gamma_\eta$.

Remark 2.2. It is clear that, if the function $\xi : [0, \infty) \to [0, 1)$ satisfies

$$\limsup_{s \to t^+} \xi(s) < 1 \tag{2.1}$$

for all $t \in [0, \infty)$, then ξ is also a stronger Meir-Keeler-type function.

Example 2.3. (1) If $\psi : [0, \infty) \to [0, 1), \psi(t) = k$ with $k \in (0, 1)$, then ψ is a stronger Meir-Keeler-type function.

(2) If $\psi : [0, \infty) \to [0, 1), \psi(t) = t/(t+1)$, then ψ is a stronger Meir-Keeler-type function.

Definition 2.4. Let $\psi : [0, \infty) \to [0, 1)$, $\phi : [0, \infty) \to [b, 1)$ be two functions where $0 < b < 1$. Then the mappings ψ, ϕ are called a (ψ, ϕ)-pair Meir-Keeler-type function, if, for each $\eta > 0$, there exists $\delta > 0$ such that, for $t \in [0, \infty)$ with $\eta \leq t < \delta + \eta$, there exists $\gamma_\eta \in [0, 1)$ such that $\psi(t)/\phi(t) < \gamma_\eta$.

Remark 2.5. It is clear that if the functions $\psi : [0, \infty) \to [0, 1)$, $\phi : [0, \infty) \to [b, 1)$ satisfy

$$\limsup_{s \to t^+} \frac{\psi(s)}{\phi(s)} < 1, \tag{2.2}$$

for all $t \in [0, \infty)$, then ϕ, ψ are also a (ψ, ϕ)-pair Meir-Keeler-type function.

Example 2.6. If $\psi : [0, \infty) \to [0, 1)$, $\psi(t) = t/(4t + 1)$ and $\phi : [0, \infty) \to [0, 1), \phi(t) = t/(3t + 1)$, then ϕ, ψ are a (ψ, ϕ)-pair Meir-Keeler-type function.

Definition 2.7. Let (X, d) be a metric space, let $\psi : [0, \infty) \to [0, 1)$, $\phi : [0, \infty) \to [b, 1)$ be two functions where $0 < b < 1$, and let $T : X \to 2^X$ be a set-valued map. Then T is called a (ψ, ϕ)-pair Meir-Keeler-type set-valued contraction map, if the following conditions hold:

(C1) for each $\eta > 0$, there exists $\delta > 0$ such that for $x \in X$ with $\eta \le D(x, Tx) < \delta + \eta$, there exists $\gamma_\eta \in [0, 1)$ such that

$$\frac{\psi(D(x, Tx))}{\phi(D(x, Tx))} < \gamma_\eta, \tag{2.3}$$

(C2) for all $x \in X$, there exists $y \in Tx$ such that

$$\begin{aligned} &\phi(D(x, Tx)) \cdot d(x, y) \le D(x, Tx), \\ &D(y, Ty) \le \psi(D(x, Tx)) \cdot d(x, y). \end{aligned} \tag{2.4}$$

In this paper, we obtain some new fixed-point theorems for a (ψ, ϕ)-pair Meir-Keeler-type set-valued contraction map in metric spaces. Our main results generalize and improve the results of Klim and Wardowski [6]. We now state our main theorem as follows.

Theorem 2.8. *Let (X, d) be a complete metric space, and let $T : X \to C(X)$ be a (ψ, ϕ)-pair Meir-Keeler-type set-valued contraction map. Then T has a fixed point in X provided the mapping $f : X \to \mathbb{R}$ defined by $f(x) = D(x, Tx)$, $x \in X$, is lower semicontinuous.*

Proof. Given $x_0 \in X$ and by (C2), there exists $x_1 \in X$ such that $x_1 \in Tx_0$. Since T is a (ψ, ϕ)-pair Meir-Keeler type set-valued contraction map, there exists $x_1 \in Tx_0$ such that

$$\begin{aligned} &\phi(D(x_0, Tx_0)) \cdot d(x_0, x_1) \le D(x_0, Tx_0), \\ &D(x_1, Tx_1) \le \psi(D(x_0, Tx_0)) \cdot d(x_0, x_1). \end{aligned} \tag{2.5}$$

Continuing this process, we can choose a sequence $\{x_n\} \subset X$ with $x_{n+1} \in Tx_n$ such that, for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} &\phi(D(x_n, Tx_n)) \cdot d(x_n, x_{n+1}) \le D(x_n, Tx_n), \\ &D(x_{n+1}, Tx_{n+1}) \le \psi(D(x_n, Tx_n)) \cdot d(x_n, x_{n+1}). \end{aligned} \tag{2.6}$$

Therefore, we can deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned} D(x_{n+1}, Tx_{n+1}) &\le \frac{\psi(D(x_n, Tx_n))}{\phi(D(x_n, Tx_n))} \cdot D(x_n, Tx_n) \\ &< D(x_n, Tx_n). \end{aligned} \tag{2.7}$$

Thus, the sequence $\{D(x_n, Tx_n)\}_{n=0}^{\infty}$ is decreasing and bounded below. Then there exists $\eta \geq 0$ such that

$$\lim_{n\to\infty} D(x_n, Tx_n) = \eta. \tag{2.8}$$

Hence, there exists $\kappa_0 \in \mathbb{N}$ and $\delta > 0$ such that, for all $n \geq \kappa_0$,

$$\eta \leq D(x_n, Tx_n) < \eta + \delta. \tag{2.9}$$

By the condition (C1), we have that there exists $\gamma_\eta \in [0, 1)$ such that

$$\frac{\varphi(D(x_n, Tx_n))}{\phi(D(x_n, Tx_n))} < \gamma_\eta, \quad \forall n \geq \kappa_0. \tag{2.10}$$

So for each $n \in \mathbb{N}$ with $n \geq \kappa_0$, by (2.6), we can deduce that

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \frac{\varphi(D(x_n, Tx_n))}{\phi(D(x_n, Tx_n))} \\
&\leq \frac{\varphi(D(x_{n-1}, Tx_{n-1}))}{\phi(D(x_n, Tx_n))} \cdot d(x_{n-1}, x_n) \\
&\leq \frac{\varphi(D(x_{n-1}, Tx_{n-1}))}{\phi(D(x_n, Tx_n))} \cdot \frac{D(x_{n-1}, Tx_{n-1})}{\phi(D(x_{n-1}, Tx_{n-1}))} \\
&\leq \frac{1}{b} \cdot \frac{\varphi(D(x_{n-1}, Tx_{n-1}))}{\phi(D(x_{n-1}, Tx_{n-1}))} \cdot D(x_{n-1}, Tx_{n-1}) \\
&\leq \frac{1}{b} \cdot \gamma_\eta \cdot D(x_{n-1}, Tx_{n-1}) \\
&\leq \frac{1}{b} \cdot \gamma_\eta^2 \cdot D(x_{n-2}, Tx_{n-2}) \\
&\;\;\vdots \\
&\leq \frac{1}{b} \cdot \gamma_\eta^{n-\kappa_0} \cdot D(x_{\kappa_0}, Tx_{\kappa_0}).
\end{aligned} \tag{2.11}$$

Take $m, n \in \mathbb{N}$ with $m > n > \kappa_0$. Then we get

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \frac{1}{b} \cdot \frac{\gamma_\eta^{n-\kappa_0} \cdot D(x_{\kappa_0}, Tx_{\kappa_0})}{1 - \gamma_\eta}, \tag{2.12}$$

and so we conclude that

$$d(x_n, x_m) \longrightarrow 0, \quad \text{as } m, n \longrightarrow \infty, \tag{2.13}$$

since $0 \leq \gamma_\eta < 1$. Thus, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since X is complete, there exists $\mu \in X$ such that $x_n \to \mu$ as $n \to \infty$.

Since $f : X \to \mathbb{R}$, $f(x) = d(x, Tx)$, $x \in X$, is lower semicontinuous, we have

$$0 \leq d(\mu, T\mu) = f(\mu) \leq \liminf_{\infty} d(x_n, Tx_n) = 0. \tag{2.14}$$

The closeness of $T\mu$ implies $\mu \in T\mu$. □

The following is a simple example for Theorem 2.8, and it generalize the result of Klim and Wardowski [6].

Example 2.9. Let $X = [0,1]$ be a metric space with the standard metric d. Let $T : X \to C(X)$ be defined by

$$T(x) = \left\{\frac{1}{3}x^2\right\}, \quad \forall x \in X. \tag{2.15}$$

Let $\varphi : [0,\infty) \to [0,1)$, $\phi : [0,\infty) \to [2/3, 1)$ be defined by

$$\varphi(t) = \frac{4}{9} + \frac{1}{9(t+1)}, \quad \phi(t) = \frac{2}{3} + \frac{1}{9(t+1)}, \quad \forall t \in [0,\infty). \tag{2.16}$$

Then T is a (φ, ϕ)-pair Meir-Keeler-type set-valued contraction map, and $0 \in X$ is a fixed point of T.

In particular, if we let $\phi(t) = 2/3$, then this example satisfies all of the conditions of Theorem 1.4 (that was introduced in Klim and Wardowski [6]).

Using Example 3.1 in [6] and Example 1 in [10], we get the following another example for Theorem 2.8.

Example 2.10. Let $X = [0,1]$ be a metric space with the standard metric d. Let $T : X \to C(X)$ be defined as in Example 3.1 of Klim and Wardowski [6]:

$$T(x) = \begin{cases} \left\{\frac{1}{2}x^2\right\}, & \text{if } x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ \left\{\frac{17}{96}, \frac{1}{4}\right\}, & \text{if } x = \frac{15}{32}. \end{cases} \tag{2.17}$$

Let $\varphi : [0,\infty) \to [0,1)$ be defined as in Example 1 of Ćirić [10]:

$$\varphi(t) = \begin{cases} \max\left\{\frac{1}{12}, \frac{23}{12}t\right\}, & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \left\{\frac{23}{24}\right\}, & \text{if } t \in \left(\frac{1}{2}, \infty\right), \end{cases} \tag{2.18}$$

and let $\phi : [0,\infty) \to [1/12, 1)$ be defined by

$$\phi(t) = \sqrt{\varphi(t)} = \begin{cases} \max\left\{\sqrt{\dfrac{1}{12}}, \sqrt{\dfrac{23t}{12}}\right\}, & \text{if } t \in \left[0, \dfrac{1}{2}\right], \\ \left\{\sqrt{\dfrac{23}{24}}\right\}, & \text{if } t \in \left(\dfrac{1}{2}, \infty\right). \end{cases} \tag{2.19}$$

Clearly, a function

$$f(x) = D(x, Tx) = \begin{cases} x - \dfrac{1}{2}x^2, & \text{if } x \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right], \\ \dfrac{7}{32}, & \text{if } x = \dfrac{15}{32}, \end{cases} \tag{2.20}$$

is lower semicontinuous. Then ϕ, φ are a (φ, ϕ)-pair Meir-Keeler-type function, and T is a (φ, ϕ)-pair Meir-Keeler-type set-valued contraction map. Moreover, by Theorem 2.8, we have that $0 \in X$ is a fixed point of T.

If we let $T : X \to C(X)$ be closed, then we also have the following fixed result.

Theorem 2.11. *Let (X, d) be a complete metric space, and let $T : X \to C(X)$ be a (φ, ϕ)-pair Meir-Keeler-type set-valued contraction map and closed. Then T has a fixed point in X.*

Proof. Following the proof of Theorem 2.8, we get that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since X is complete, there exists $\mu \in X$ such that $x_n \to \mu$ as $n \to \infty$. Since T is closed and $x_{n+1} \in Tx_n$, we have that $\mu \in T\mu$. □

The following is a simple example for Theorem 2.11.

Example 2.12. Let $X = [0, 1]$ be a metric space with the metric $d(x, y) := x$ for all $(x, y) \in X \times X$. Let $T : X \to C(X)$ be defined by

$$T(x) = \left\{\frac{1}{4}x^2\right\}, \quad \forall x \in X. \tag{2.21}$$

Let $\varphi : [0,\infty) \to [0, 1)$, $\phi : [0,\infty) \to [1/4, 1)$ be defined by

$$\varphi(t) = \frac{1}{4}, \qquad \phi(t) = \frac{1}{2}, \quad \forall t \in [0,\infty). \tag{2.22}$$

Then T is a (φ, ϕ)-pair Meir-Keeler-type set-valued contraction map and closed, and $0 \in X$ is a fixed point of T.

Applying Theorem 2.8 and Remark 2.5, we are easy to get the following result.

Theorem 2.13. *Let (X, d) be a complete metric space, let $\varphi : [0,\infty) \to [0,1)$, $\phi : [0,\infty) \to [b,1)$ be two functions where $0 < b < 1$, and let $T : X \to C(X)$ be a set-valued contraction map. Suppose the following conditions hold:*

(1) *for each $t \in [0,\infty)$,*

$$\limsup_{s \to t^+} \frac{\varphi(s)}{\phi(s)} < 1, \tag{2.23}$$

(2) *for all $x \in X$, there exists $y \in Tx$ such that*

$$\begin{gathered} \phi(D(x,Tx)) \cdot d(x,y) \leq D(x,Tx), \\ D(y,Ty) \leq \varphi(D(x,Tx)) \cdot d(x,y). \end{gathered} \tag{2.24}$$

Then T has a fixed point in X provided the mapping $f : X \to \mathbb{R}$ defined by $f(x) = D(x,Tx), x \in X$, is lower semicontinuous.

The following is a simple example for Theorem 2.13.

Example 2.14. Let $X = [0,1]$ be a metric space with the metric $d, d(x,y) := x$ for all $(x,y) \in X \times Y$. Let $T : X \to C(X)$ be defined as in Example 3.1 of Klim and Wardowski [6]:

$$T(x) = \begin{cases} \left\{ \dfrac{1}{2}x^2 \right\}, & \text{if } x \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right], \\ \left\{ \dfrac{17}{96}, \dfrac{1}{4} \right\}, & \text{if } x = \dfrac{15}{32}. \end{cases} \tag{2.25}$$

Let $\varphi : [0,\infty) \to [0,1)$ be defined as in Example 1 of Ćirić [10]:

$$\varphi(t) = \begin{cases} \max\left\{ \dfrac{1}{12}, \dfrac{23}{12}t \right\}, & \text{if } t \in \left[0, \dfrac{1}{2}\right], \\ \left\{ \dfrac{23}{24} \right\}, & \text{if } t \in \left(\dfrac{1}{2}, \infty\right), \end{cases} \tag{2.26}$$

and let $\phi : [0,\infty) \to [1/12, 1)$ be defined by

$$\phi(t) = \sqrt{\varphi(t)}. \tag{2.27}$$

Clearly, a function

$$f(x) = D(x,Tx) = x \tag{2.28}$$

is lower semicontinuous. Clearly, $\limsup_{s \to t^+}(\varphi(s)/\phi(s)) < 1$. We also conclude the following.

Case 1. If $x \in [0, 15/32) \cup (15/32, 1]$, then $y = T(x) = (1/2)x^2$, and ψ, ϕ satisfy the condition (2) of Theorem 2.13.

Case 2. If $x = 15/32$, then $y = T(x) = 1/4$ (resp., $y = T(x) = 17/96$), and ψ, ϕ also satisfy the condition (2) of Theorem 2.13.

Thus, by Theorem 2.13, we have that $0 \in X$ is a fixed point of T.

Using Example 2.10, we also get the following example for Theorem 2.13.

Example 2.15. Let $X = [0, 1]$ be a metric space with the standard metric d. Let $T : X \to C(X)$ be defined as

$$T(x) = \begin{cases} \left\{ \dfrac{1}{2}x^2 \right\}, & \text{if } x \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right], \\ \left\{ \dfrac{17}{96}, \dfrac{1}{4} \right\}, & \text{if } x = \dfrac{15}{32}. \end{cases} \tag{2.29}$$

Let $\psi : [0, \infty) \to [0, 1)$ be defined as

$$\psi(t) = \begin{cases} \max\left\{ \dfrac{1}{12}, \dfrac{23}{12}t \right\}, & \text{if } t \in \left[0, \dfrac{1}{2}\right], \\ \left\{ \dfrac{23}{24} \right\}, & \text{if } t \in \left(\dfrac{1}{2}, \infty\right), \end{cases} \tag{2.30}$$

and $\phi : [0, \infty) \to [1/12, 1)$ be defined by

$$\phi(t) = \sqrt{\psi(t)}. \tag{2.31}$$

Clearly, $\limsup_{s \to t^+}(\psi(s)/\phi(s)) < 1$, and ϕ, ψ satisfies all of the conditions of Theorem 2.13. So, we have that $0 \in X$ is a fixed point of T.

If we let the function $\phi : [0, \infty) \to [b, 1)$ be $\phi(t) = b$ for all $t \in [0, \infty)$ and let the function $\psi : [0, \infty) \to [0, b)$, $b \in (0, 1)$, be a stronger Meir-Keeler-type function; that is for if, for each $\eta > 0$, there exists $\delta > 0$ such that, for $t \in [0, \infty)$ with $\eta \le t < \delta + \eta$, there exists $\gamma_\eta \in [0, b)$ such that $\psi(t) < \gamma_\eta$, then, by Theorem 2.8, it is easy to get the following theorem.

Theorem 2.16. *Let (X, d) be a complete metric space, let $\psi : [0, \infty) \to [0, b)$, $b \in (0, 1)$ be a stronger Meir-Keeler-type function, and let $T : X \to C(X)$ be a set-valued contraction map. Suppose that, for all $x \in X$, there exists $y \in Tx$ such that*

$$\begin{gathered} b \cdot d(x, y) \le D(x, Tx), \\ D(y, Ty) \le \psi(D(x, Tx)) \cdot d(x, y). \end{gathered} \tag{2.32}$$

Then T has a fixed point in X provided that the mapping $f : X \to \mathbb{R}$ defined by $f(x) = D(x, Tx)$, $x \in X$, is lower semicontinuous.

The following is a simple example for Theorem 2.16.

Example 2.17. Let $X = [0,1]$ be a metric space with the standard metric d. Let $T : X \to C(X)$ be defined by

$$T(x) = \left\{ \frac{1}{3}x^2 \right\}, \quad \forall x \in \mathrm{X}. \tag{2.33}$$

Let $\psi : [0,\infty) \to [0,2/3)$ be defined by

$$\psi(t) = \frac{4}{9} + \frac{1}{9(t+1)}, \quad \forall t \in [0,\infty). \tag{2.34}$$

Then ψ a stronger Meir-Keeler-type function, and $0 \in X$ is a fixed point of T.

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