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Research Article

Semicompatibility and Fixed Point Theorems for Reciprocally Continuous Maps in a Fuzzy Metric Space

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The aim of this paper is to prove a common fixed point theorem for six mappings on fuzzy metric space using notion of semicompatibility and reciprocal continuity of maps satisfying an implicit relation. We proposed to reanalysis the theorems of Imdad et al. (2002), Popa (2001), Popa (2002) and Singh and Jain (2005).

1. Introduction

The fuzzy theory has become an area of active research for the last forty years. It has a wide range of applications in the field of science and engineering, for example, population dynamics, computer programming, nonlinear dynamical systems, medicine and so forth. The concept of fuzzy sets was introduced initially by Zadeh [1] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and application. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [2], and George and Veeramani [3] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Later, many authors, for example, [4–6] proved common fixed point theorems in fuzzy metric spaces. Grabiec's [7] followed Kramosil and Michalek and obtained the fuzzy version of Banach contraction principle. Vasuki [8] obtained the fuzzy version of common fixed point theorem which had extra conditions. Pant [9] introduced the notion of reciprocal continuity of mappings in metric spaces. Pant and Jha [10] proved an analogue of the result given by Balasubramaniam et al. [11]. Popa [12] proved theorem for weakly compatible noncontinuous mapping using implicit relation. Recently Singh and Jain [13], and Singh and Chauhan [14] have introduced semicompatible, compatible, and weak compatible maps in fuzzy metric space.

The purpose of this paper is to prove a common fixed point theorem in fuzzy metric spaces using weak compatibility, semicompatibility, an implicit relation, and reciprocal continuity. Here, we generalize the result of [15] by

(1) increasing the number of self maps from four to six,

(2) using the notion of reciprocal continuity.

2. Preliminaries

Definition 2.1. A binary operation $*: [0,1]^2 \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all a, b, c and $d \in [0,1]$.

Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:

(FMS-1) $M(x, y, 0) = 0$,

(FMS-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,

(FMS-3) $M(x, y, t) = M(y, x, t)$,

(FMS-4) $M(x, y, t) * M(y, z, s) \geq M(x, z, t + s)$,

(FMS-5) $M(x, y, \cdot) : [0, \infty) \to [0,1]$ is left continuous.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Remark 2.3. Every metric space (X, d) induces a fuzzy metric space $(X, M, *)$, where $a * b = \min\{a, b\}$ and for all $a, b \in X$, $M(x, y, t) = t/(t + d(x, y))$, for all $t > 0$, $M(x, y, 0) = 0$, which is called the fuzzy metric space induced by the metric d.

Lemma 2.4. *For all $x, y \in X$, let $M(x, y, \cdot)$ be a nondecreasing function.*

Definition 2.5. Let $(X, M, *)$ be a fuzzy metric space.

(a) The sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if

$$\lim_{n \to \infty} M(x_n, x, t) = 1 \quad \forall t > 0. \tag{2.1}$$

(b) The sequence $\{x_n\}$ in X is said to be a Cauchy sequence in X if

$$\lim_{n \to \infty} M(x_n, x_{n+p}, t) = 1 \quad \forall t > 0,\ p > 0. \tag{2.2}$$

(c) The space is said to be complete if every Cauchy sequence in it converges to a point of it.

Remark 2.6. Since $*$ is continuous, it follows from (FMS-4) that the limit of a sequence in a fuzzy metric space is unique.

In this paper, $(X, M, *)$ is considered to be the fuzzy metric space with condition

(FMS-6)

$$\lim_{n\to\infty} M(x, y, t) = 1 \quad \forall x, y \in X. \tag{2.3}$$

Lemma 2.7. *Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with the condition (FMS-6). If there exists a number $k \in (0, 1)$ such that*

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t) \tag{2.4}$$

for all $t > 0$, then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.8. *Let A and B be two self-maps on a complete fuzzy metric space $(X, M, *)$ such that for some $k \in (0, 1)$, for all $x, y \in X$ and $t > 0$,*

$$M(Ax, By, kt) \geq \min\{M(x, y, t), M(Ax, x, t)\}. \tag{2.5}$$

Then, A and B have a unique common fixed point in X.

Definition 2.9. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then, the mappings are said to be weak compatible if they commute at their coincidence points, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 2.10. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then, the mappings are said to be compatible if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1, \quad \forall t > 0 \tag{2.6}$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X. \tag{2.7}$$

Remark 2.11. Compatibility implies weak compatibility. The converse is not true.

Definition 2.12. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be semicompatible if

$$\lim_{n\to\infty} M(ASx_n, Sx, t) = 1, \quad \forall t > 0. \tag{2.8}$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X. \tag{2.9}$$

Remark 2.13. Semicompatibility implies weak compatibility. The converse is not true.

Proposition 2.14. *Let A and S be self-maps on a fuzzy metric space $(X, M, *)$. If S is continuous, then (A, S) is semicompatible if and only if (A, S) is compatible.*

Remark 2.15. Semicompatibility of the pair (A, S) does not imply the semicompatibility of (S, A) as seen in Example 2.16.

Example 2.16. Let $X = [0, 1]$ and let $(X, M, *)$ be the fuzzy metric space with $M(x, y, t) = [\exp|x - y|/t]^{-1}$, for all $x, y \in X$, $t > 0$. Define self-map S as follows:

$$\mathbf{Sx} = \begin{cases} x & \text{if } 0 \le x < \dfrac{1}{3}, \\ 1 & \text{if } x \ge \dfrac{1}{3}. \end{cases} \tag{2.10}$$

Let I be the identity map on X and $x_n = 1/3 - 1/n$. Then, $\{Ix_n\} = \{x_n\} \to 1/3$ and$\{Sx_n\} = \{x_n\} \to 1/3$. Thus, $\{ISx_n\} = \{Sx_n\} \to 1/3 \neq S(1/3)$. Hence, (I, S) is not semicompatible. Again, as (I, S) is commuting, it is compatible. Further, for any sequence $\{x_n\}$ in X such that $\{x_n\} \to x$ and $\{S\, x_n\} \to x$, we have $\{SIx_n\} = \{Sx_n\} \to x = Ix$. Hence, (S, I) is always semicompatible.

Definition 2.17. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then, the mappings are said to be reciprocally continuous if

$$\lim_{n\to\infty} ASx_n = Ax, \qquad \lim_{n\to\infty} SAx_n = Sx \tag{2.11}$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X. \tag{2.12}$$

If A and S are both continuous, then they are obviously reciprocally continuous, but the converse is not true.

2.1. A Class of Implicit Relations

Let K_4 be the set of all real continuous functions
$F : R_+^4 \to R$, nondecreasing in first argument and satisfying the following conditions:

(i) for $u, v \ge 0$, $F(u, v, v, u) \ge 0$ or $F(u, v, u, v) \ge 0$ implies that $u \ge v$;

(ii) $F(u, u, 1, 1) \ge 0$ implies that $u \ge 1$.

3. Main Result

Theorem 3.1. *Let A, B, S, T, I, and J be self-mappings of a complete fuzzy metric space $(X, M, *)$ such that*

(a) $AB(X) \subset J(X)$ *and* $ST(X) \subset I(X)$;

(b) *the pair* (AB, I) *is semicompatible and* (ST, J) *is weak compatible;*

(c) *the pair* (AB, I) *is reciprocally continuous.*

For some $F \in K_4$, there exists $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$,

$$F(M(ABx, STy, kt), M(Ix, Jy, t), M(ABx, Ix, t), M(STy, Jy, kt)) \geq 0, \tag{3.1}$$

$$F(M(ABx, STy, kt), M(Ix, Jy, t), M(ABx, Ix, kt), M(STy, Jy, t)) \geq 0. \tag{3.2}$$

Then AB, ST, I, and J have a unique common fixed point. Furthermore, if the pairs $(A, B), (A, I), (B, I), (S, T), (S, J)$, and (T, J) are commuting mapping then, A, B, S, T, I, and J have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $AB(X) \subset J(X)$ and $ST(X) \subset I(X)$, there exist $x_1, x_2 \in X$ such that $ABx_0 = Jx_1$ and $STx_1 = Ix_2$. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n+1} = ABx_{2n} = Jx_{2n+1}, \qquad y_{2n+2} = STx_{2n+1} = Ix_{2n+2} \tag{3.3}$$

for $n = 0, 1, 2, \ldots$. Now putting in (3.1) $x = x_{2n}$, $y = x_{2n+1}$, we obtain

$$F(M(ABx_{2n}, STx_{2n+1}, kt), M(Ix_{2n}, Jx_{2n+1}, t), M(ABx_{2n}, Ix_{2n}, t), M(STx_{2n+1}, Jx_{2n+1}, kt)) \geq 0 \tag{3.4}$$

that is,

$$F(M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+2}, y_{2n+1}, kt)) \geq 0. \tag{3.5}$$

Using (i), we get

$$M(y_{2n+2}, y_{2n+1}, kt) \geq M(y_{2n+1}, y_{2n}, t). \tag{3.6}$$

Analogously, putting $x = x_{2n+2}$, $y = x_{2n+1}$ in (3.2), we have

$$F(M(y_{2n+3}, y_{2n+2}, kt), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+3}, y_{2n+2}, kt), M(y_{2n+1}, y_{2n+2}, t)) \geq 0. \tag{3.7}$$

Using (i), we get

$$M(y_{2n+3}, y_{2n+2}, kt) \geq M(y_{2n+1}, y_{2n+2}, t). \tag{3.8}$$

Thus, from (3.6) and (3.8), for n and t, we have

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t). \tag{3.9}$$

Hence, by Lemma 2.7, $\{y_n\}$ is a Cauchy sequence in X, which is complete. Therefore, $\{y_n\}$ converges to $p \in X$. The sequences $\{ABx_{2n}\}$, $\{STx_{2n+1}\}$, $\{Ix_{2n}\}$, and $\{Jx_{2n+1}\}$, being subsequences of $\{y_n\}$, also converge to p, that is,

$$\begin{gathered}\{ABx_{2n}\} \longrightarrow p, \qquad \{STx_{2n+1}\} \longrightarrow p, \\ \{Ix_{2n}\} \longrightarrow p, \qquad \{Jx_{2n+1}\} \longrightarrow p.\end{gathered} \tag{3.10}$$

The reciprocal continuity of the pair (AB, I) gives

$$ABIx_{2n} \longrightarrow ABp, \qquad IABx_{2n} \longrightarrow Ip. \tag{3.11}$$

The semicompatibility of the pair (AB, I) gives

$$\lim_{n\to\infty} ABIx_{2n} = Ip. \tag{3.12}$$

From the uniqueness of the limit in a fuzzy metric space, we obtain that

$$ABp = Ip. \tag{3.13}$$

Step 1. By putting $x = p$, $y = x_{2n+1}$ in (3.1), we obtain

$$F(M(ABp, STx_{2n+1}, kt), M(Ip, Jx_{2n+1}, t), M(ABp, Ip, t), M(STx_{2n+1}, Jx_{2n+1}, kt)) \geq 0. \tag{3.14}$$

Letting n tends to infinity and using (3.10) and (3.13), we get

$$F(M(Ip, p, kt), M(Ip, p, t), M(Ip, Ip, t), M(p, p, kt)) \geq 0. \tag{3.15}$$

As F is nondecreasing in first argument, we have

$$F(M(Ip, p, t), M(Ip, p, t), 1, 1) \geq 0. \tag{3.16}$$

Using (ii), we have $M(Ip, p, t) \geq 1$ for all $t > 0$, which gives $M(Ip, p, t) = 1$, that is,

$$Ip = p = ABp. \tag{3.17}$$

Step 2. As $AB(X) \subset J(X)$, there exists $u \in X$ such that $ABp = Ip = p = Ju$.

Putting $x = x_{2n}$, $y = u$ in (3.1), we obtain that

$$F(M(ABx_{2n}, STu, kt), M(Ix_{2n}, Ju, t), M(ABx_{2n}, Ix_{2n}, t), M(STu, Ju, kt)) \geq 0. \tag{3.18}$$

Letting n tends to infinity and using (3.10), we get

$$F(M(p, STu, kt), 1, 1, M(STu, p, kt)) \geq 0. \tag{3.19}$$

Using (i), we have $M(p, STu, kt) \geq 1$ for all $t > 0$, which gives $M(p, STu, kt) = 1$.

Thus, $p = STu$. Therefore, $STu = Ju = p$. Since (ST, J) is weak compatible, we get $JSTu = STJu$, that is,

$$STp = Jp. \tag{3.20}$$

Step 3. By putting $x = p$, $y = p$ in (3.1) and using (3.17) and (3.20), we obtain

$$F(M(ABp, STp, kt), M(Ip, Jp, t), M(ABp, Ip, t), M(STp, Jp, kt)) \geq 0 \tag{3.21}$$

that is, $F(M(ABp, STp, kt), M(ABp, STp, t), 1, 1) \geq 0$.

As F is nondecreasing in first argument, we have

$$F(M(ABp, STp, t), M(ABp, STp, t), 1, 1) \geq 0. \tag{3.22}$$

Using (ii), we have $M(ABp, STp, t) \geq 1$ for all $t > 0$, which gives $M(ABp, STp, t) = 1$.

Thus,

$$ABp = STp. \tag{3.23}$$

Therefore, $p = ABp = STp = Ip = Jp$, that is, p is a common fixed point of AB, ST, I, and J.

Uniqueness. Let q be another common fixed point of AB, ST, I, and J. Then, $q = ABq = STq = Iq = Jq$.

By putting $x = p$ and $y = q$ in (3.1), we get

$$F(M(ABp, STq, kt), M(Ip, Jq, t), M(ABp, Ip, t), M(STq, Jq, kt)) \geq 0 \tag{3.24}$$

that is, $F(M(p, q, kt), M(p, q, t), 1, 1) \geq 0$.

As F is nondecreasing in first argument, we have

$$F(M(p, q, t), M(p, q, t), 1, 1) \geq 0. \tag{3.25}$$

Using (ii), we have $M(p, q, t) \geq 1$ for all $t > 0$, which gives $M(p, q, t) = 1$, that is, $p = q$. Therefore, p is the unique common fixed point of the self-maps AB, ST, I, and J.

Finally, we need to show that p is also a common fixed point of A, B, S, T, I, and J. For this, let p be the unique common fixed point of both the pairs (AB, I) and (ST, J). Then, by using commutativity of the pair (A, B), (A, I), and (B, I), we obtain

$$\begin{gathered} Ap = A(ABp) = A(BAp) = AB(Ap), \qquad Ap = A(Ip) = I(Ap), \\ Bp = B(ABp) = B(A(Bp)) = BA(Bp) = AB(Bp), \qquad Bp = B(Ip) = I(Bp), \end{gathered} \tag{3.26}$$

which shows that Ap and Bp are common fixed point of (AB, I), yielding thereby

$$Ap = p = Bp = Ip = ABp \tag{3.27}$$

in the view of uniqueness of the common fixed point of the pair (AB, I). Similarly, using the commutativity of (S, T), (S, J), (T, J), it can be shown that

$$Sp = Tp = Jp = STp = p. \tag{3.28}$$

Now, we need to show that $Ap = Sp$ $(Bp = Tp)$ also remains a common fixed point of both the pairs (AB, I) and (ST, J). For this, put $x = p$ and $y = p$ in (3.1), and using (3.27) and (3.28), we get

$$F(M(ABp, STp, kt), M(Ip, Jp, t), M(ABp, Ip, t), M(STp, Jp, kt)) \geq 0, \tag{3.29}$$

that is,

$$F(M(Ap, Sp, kt), M(Ap, Sp, t), M(Ap, Ap, t), M(Sp, Sp, kt)) \geq 0. \tag{3.30}$$

As F is nondecreasing in first argument, we have

$$F(M(Ap, Sp, t), M(Ap, Sp, t), 1, 1) \geq 0. \tag{3.31}$$

Using (ii), we obtain

$$M(Ap, Sp, t) \geq 1 \quad \forall t > 0, \tag{3.32}$$

which gives $M(Ap, Sp, t) = 1$, that is, $Ap = Sp$. Similarly, it can be shown that $Bp = Tp$. Thus, p is the unique common fixed point of A, B, S, T, I, and J. This completes the proof of our theorem.

□

Since semicompatibility implies weak compatibility, we have the following.

Corollary 3.2. *Let A, B, S, T, I, and J be self-maps of a complete fuzzy metric space $(X, M, *)$ satisfying the conditions (a),* (3.1)*, and* (3.2) *of the above theorem, and the pairs (AB, I) and (ST, J) are semicompatible and one of the pair, (AB, I) or (ST, J), is reciprocally continuous. Then, AB, ST, I,* *and J have a unique common fixed point. Furthermore, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J), and (T,J) are commuting mapping, then A,B,S,T,I, and J have a unique common fixed point.*

If we take $B = T =$ identity map in Theorem 3.1, then we have the following.

Corollary 3.3. *Let A,S,I, and J be self-mappings of a complete fuzzy metric space $(X,M,*)$ such that*

(a) *$A(X) \subset J(X)$ and $S(X) \subset I(X)$;*

(b) *(A,I), (S,J) are semicompatible commuting pair of maps;*

(c) *the pair (A,I) or (S,J) is reciprocally continuous.*

For some $F \in K_4$, there exists $k \in (0,1)$ such that for all $x,y \in X$ and $t > 0$,

$$\begin{aligned} &F(M(Ax,Sy,kt),M(Ix,Jy,t),M(Ax,Ix,t),M(Sy,Jy,kt)) \geq 0, \\ &F(M(Ax,Sy,kt),M(Ix,Jy,t),M(Ax,Ix,kt),M(Sy,Jy,t)) \geq 0. \end{aligned} \tag{3.33}$$

Then, A,S,I, and J have a unique common fixed point.

This theorem proves that the theorem of Singh holds even when the pairs are reciprocally continuous.

On taking $I =$ identity mapping in Theorem 3.1, we have the following result for 5 self-maps.

Corollary 3.4. *Let A,B,S,T, and J be self-mappings of a complete fuzzy metric space $(X,M,*)$ such that*

(a) *$AB(X) \cap ST(X) \subset J(X)$,*

(b) *(ST,J) is weak compatible.*

For some $F \in K_4$, there exists $k \in (0,1)$ such that for all $x,y \in X$ and $t > 0$,

$$\begin{aligned} &F(M(ABx,STy,kt),M(x,Jy,t),M(ABx,x,t),M(STy,Jy,kt)) \geq 0 \\ &F(M(ABx,STy,kt),M(x,Jy,t),M(ABx,x,kt),M(STy,Jy,t)) \geq 0. \end{aligned} \tag{3.34}$$

Then, AB, ST, and J have a unique common fixed point. Furthermore, if the pairs (A,B), (S,T), (S,J), and (T,J) are commuting mapping then A,B,S,T, and J have a unique common fixed point.

If we take $B = T = I = J =$ identity mapping in Theorem 3.1, then the conditions (a), (b), and (c) are satisfied trivially, and we get the following result.

Corollary 3.5. *Let A and S be self-mappings of a complete fuzzy metric space $(X,M,*)$ such that for some $F \in K_4$, there exists $k \in (0,1)$ such that for all $x,y \in X$ and $t > 0$,*

$$\begin{aligned} &F(M(Ax,Sy,kt),M(x,y,t),M(Ax,x,t),M(Sy,y,kt)) \geq 0, \\ &F(M(Ax,Sy,kt),M(x,y,t),M(Ax,x,kt),M(Sy,y,t)) \geq 0. \end{aligned} \tag{3.35}$$

Then, A and S have a unique common fixed point in X.

Theorem 3.6. *Let A, B, S, T, I, and J be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying conditions (a), (c),* (3.1), *and* (3.2) *of Theorem 3.1, and (AB, I) is compatible, and (ST, J) is weak compatible. Then, AB, ST, I, and J have a unique common fixed point. Furthermore, if the pairs (A, B), (A, I), (B, I), (S, T), (S, J), and (T, J) are commuting mapping then A, B, S, T, I, and J have a unique common fixed point.*

Proof. As in the proof of Theorem 3.1, the sequence $\{y_n\}$ converges to $p \in X$, and (3.10) is satisfied.

The reciprocal continuity of the pair (AB, I) gives

$$ABIx_{2n} \longrightarrow ABp, \qquad IABx_{2n} \longrightarrow Ip. \tag{3.36}$$

The compatibility of the pair (AB, I) gives

$$\lim_{n\to\infty} ABIx_{2n} = ABp = \lim_{n\to\infty} IABx_2n. \tag{3.37}$$

From the uniqueness of the limit in a fuzzy metric space, we obtain that $ABp = Ip$.

Step 1. By putting $x = ABx_{2n}$, $y = x_{2n+1}$ in (3.1), we obtain

$$\begin{gathered} F(M(AB\,ABx_{2n}, STx_{2n+1}, kt), M(IABx_{2n}, Jx_{2n+1}, t), \\ M(AB\,ABx_{2n}, IABx_{2n}, t), M(STx_{2n+1}, Jx_{2n+1}, kt)) \geq 0. \end{gathered} \tag{3.38}$$

Letting n tends to infinity and using (3.10) and (3.13), we get

$$F(M(ABp, p, kt), M(ABp, p, t), M(ABp, ABp, t), M(p, p, kt)) \geq 0. \tag{3.39}$$

As F is nondecreasing in first argument, we have

$$F(M(ABp, p, t), M(ABp, p, t), 1, 1) \geq 0. \tag{3.40}$$

Using (ii), we have $M(ABp, p, t) \geq 1$ for all $t > 0$, which gives $M(ABp, p, t) = 1$, that is,

$$Ip = p = ABp. \tag{3.41}$$

Step 2. As $AB(X) \subset J(X)$, there exists $u \in X$ such that $ABp = Ip = p = Ju$.

Putting $x = x_{2n}$, $y = u$ in (3.1), we obtain that

$$F(M(ABx_{2n}, STu, kt), M(Ix_{2n}, Ju, t), M(AB\ x_{2n}, Ix_{2n}, t), M(STu, Ju, kt)) \geq 0. \tag{3.42}$$

Letting n tends to infinity and using (3.10), we get

$$F(M(p, STu, kt), 1, 1, M(STu, p, kt)) \geq 0. \tag{3.43}$$

Using (i), we have $M(p, STu, kt) \geq 1$ for all $t > 0$, which gives $M(p, STu, kt) = 1$. Thus, $p = STu$. Therefore, $STu = Ju = p$. Since (ST, J) is weak compatible, we get $JSTu = STJu$, that is,

$$STp = Jp. \tag{3.44}$$

Step 3. By putting $x = p$, $y = p$ in (3.1) and using (3.17) and (3.20), we obtain

$$F(M(ABp, STp, kt), M(Ip, Jp, t), M(ABp, Ip, t), M(STp, Jp, kt)) \geq 0, \tag{3.45}$$

that is, $F(M(ABp, STp, kt), M(ABp, STp, t), 1, 1) \geq 0$.

As F is nondecreasing in first argument, we have

$$F(M(ABp, STp, t), M(ABp, STp, t), 1, 1) \geq 0. \tag{3.46}$$

Using (ii), we have $M(ABp, STp, t) \geq 1$ for all $t > 0$, which gives $M(ABp, STp, t) = 1$. Thus,

$$ABp = STp. \tag{3.47}$$

Therefore, $p = ABp = STp = Ip = Jp$, that is, p is a common fixed point of AB, ST, I, and J.

The rest of the proof is the same as in Theorem 3.1.

□

Since compatibility implies weak compatibility, we have the following.

Corollary 3.7. *Let A, B, S, T, I, and J be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying conditions (a),* (3.1), *and* (3.2) *of Theorem 3.1, and the pairs (AB, I) and (ST, J) are compatible, and one of the pair, (AB, I) or (ST, J), is reciprocally continuous. Then, AB, ST, I, and J have a unique common fixed point in X. Furthermore, if the pairs (A, B), (A, I), (B, I), (S, T), (S, J), and (T, J) are commuting mapping, then A, B, S, T, I, and J have a unique common fixed point.*

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.
[2] O. Kramosil and J. Michalek, "Fuzzy metric and statistical metric spaces," *Kybernetika*, vol. 11, pp. 326–334, 1975.
[3] A. George and P. Veeramani, "On some results in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 395–399, 1994.
[4] V. Gregori and A. Sapena, "On fixed-point theorems in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 125, no. 2, pp. 245–252, 2002.
[5] D. Miheţ, "A Banach contraction theorem in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 144, no. 3, pp. 431–439, 2004.
[6] B. Schweizer, H. Sherwood, and R. M. Tardiff, "Contractions on probabilistic metric spaces: examples and counterexamples," *Stochastica*, vol. 12, no. 1, pp. 5–17, 1988.
[7] M. Grabiec, "Fixed points in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 27, no. 3, pp. 385–389, 1988.
[8] R. Vasuki, "Common fixed points for *R*-weakly commuting maps in fuzzy metric spaces," *Indian Journal of Pure and Applied Mathematics*, vol. 30, no. 4, pp. 419–423, 1999.

[9] R. P. Pant, "Common fixed points of four mappings," *Bulletin of the Calcutta Mathematical Society*, vol. 90, no. 4, pp. 281–286, 1998.

[10] R. P. Pant and K. Jha, "A remark on common fixed points of four mappings in a fuzzy metric space," *Journal of Fuzzy Mathematics*, vol. 12, no. 2, pp. 433–437, 2004.

[11] P. Balasubramaniam, S. Muralisankar, and R. P. Pant, "Common fixed points of four mappings in a fuzzy metric space," *Journal of Fuzzy Mathematics*, vol. 10, no. 2, pp. 379–384, 2002.

[12] V. Popa, "Some fixed point theorems for weakly compatible mappings," *Radovi Matematički*, vol. 10, no. 2, pp. 245–252, 2001.

[13] B. Singh and S. Jain, "Semi-compatibility, compatibility and fixed point theorem in fuzzy metric space," *Journal of the Chuncheong Mathematical Society*, pp. 1–22, 2005.

[14] B. Singh and M. S. Chauhan, "Common fixed points of compatible maps in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 115, no. 3, pp. 471–475, 2000.

[15] B. Singh and S. Jain, "Semicompatibility and fixed point theorems in fuzzy metric space using implicit relation," *International Journal of Mathematics and Mathematical Sciences*, no. 16, pp. 2617–2629, 2005.

[16] M. Imdad, S. Kumar, and M. S. Khan, "Remarks on some fixed point theorems satisfying implicit relations," *Radovi Matematički*, vol. 11, no. 1, pp. 135–143, 2002.

[17] V. Popa, "Fixed points for non-surjective expansion mappings satisfying an implicit relation," *Buletinul Ştiinţific al Universităţii Baia Mare. Seria B*, vol. 18, no. 1, pp. 105–108, 2002.