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Research Article

Analysis of a System for Linear Fractional Differential Equations

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The main purpose of this paper is to obtain the unique solution of the constant coefficient homogeneous linear fractional differential equations $D_{t_0}^q X(t) = PX(t), X(a) = B$ and the constant coefficient nonhomogeneous linear fractional differential equations $D_{t_0}^q X(t) = PX(t) + D, X(a) = B$ if P is a diagonal matrix and $X(t) \in C_{1-q}[t_0,T] \times C_{1-q}[t_0,T] \times \cdots \times C_{1-q}[t_0,T]$ and prove the existence and uniqueness of these two kinds of equations for any $P \in L(R^m)$ and $X(t) \in C_{1-q}[t_0,T] \times C_{1-q}[t_0,T] \times \cdots \times C_{1-q}[t_0,T]$. Then we give two examples to demonstrate the main results.

1. Introduction

System of fractional differential equations has gained a lot of interest because of the challenges it offers compared to the study of system of ordinary differential equations. Numerous applications of this system in different areas of physics, engineering, and biological sciences have been presented in [1–3]. The differential equations involving the Riemman-Liouville differential operators of fractional order $0 < q < 1$ appear to be more important in modeling several physical phenomena and therefore seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations. The existence and uniqueness of solution for fractional differential equations with any $X(t) \in C[t_0,T] \times C[t_0,T] \times \cdots \times C[t_0,T]$ have been studied in many papers, see [4–28]. In [4] Daftradar-Gejji and Babakhani have studied the existence and uniqueness of

$$D_0^q(X(t) - X_0) = PX(t), \tag{1.1}$$

where D_0^q denotes the standard Riemman-Liouville fractional derivative, $0 < q < 1, X(t) = (x_1(t), x_2(t), \ldots x_m(t))^T$, $X(0) = X_0 = (x_{10}, x_{20}, \ldots, x_{m0})^T$, $P \in L(R^m)$ which is an m dimensional linear space. They have obtained that the system (1.1) has a unique solution defined on [0,T] if $P \in L(R^m)$ and $X(t) \in C[t_0,T] \times C[t_0,T] \times \cdots \times C[t_0,T]$. In [17] Belmekki et al. have studied the existence of periodic solution for some linear fractional differential equation in $C_{1-q}[0,1]$. In [21] Ahmad and Nieto have studied the Riemann-Liouwille fractional differential equations with fractional boundary conditions. In comparison with the earlier results of this type we get more general assumptions. We assume $X(t) \in C_{1-q}[t_0,T] \times C_{1-q}[t_0,T] \times \cdots \times C_{1-q}[t_0,T]$ instead of $X(t) \in C[t_0,T] \times C[t_0,T] \times \cdots \times C[t_0,T]$ and consider the following system of fractional differential equations:

$$\begin{gathered} D_{t_0}^q X(t) = PX(t), \quad X(a) = B, \\ D_{t_0}^q X(t) = PX(t) + D, \quad X(a) = B, \end{gathered} \tag{1.2}$$

where D_0^q denotes the standard Riemman-Liouville fractional derivative, $0 < q < 1$, $P \in L(R^m)$,

$$X(t) \in C_{1-q}[t_0,T] \times C_{1-q}[t_0,T] \times \cdots \times C_{1-q}[t_0,T], \tag{1.3}$$

$(a,B) \in (t_0,T] \times R^m$ and D is a constant vector. We completely generalize the results in [4] and obtain the new results if $X(t) \in C_{1-q}[t_0,T] \times C_{1-q}[t_0,T] \times \cdots \times C_{1-q}[t_0,T]$. Furthermore, we also obtain some results of the unique solution of the homogeneous and nonhomogeneous initial value problems with the classical Mittag-Leffler special function [5] which is similar to the ordinary differential equations. Now we introduce the first Mittag-Leffler function $e_q(t-t_0)$ defined by

$$e_q(t-t_0) = \sum_{k=1}^{+\infty} \frac{(t-t_0)^{kq-1}}{\Gamma(kq)}. \tag{1.4}$$

The function $e_q(t-t_0)$ belongs to $C_{1-q}[t_0,T]$. Indeed, taking the norm in $C_{1-q}[t_0,T]$, we have

$$\left\| e_q(t-t_0) \right\|_{1-q} \le \sum_{k=1}^{+\infty} \frac{(T-t_0)^{(k-1)q}}{\Gamma(kq)} < +\infty. \tag{1.5}$$

The formula remains valid for $q \to 1^-$. In this case, $e_1(t-t_0) = \exp(t-t_0)$. Then we introduce the second Mittag-Leffler function $E_q(t-t_0)$ defined by

$$E_q(t-t_0) = \sum_{k=1}^{+\infty} \frac{(t-t_0)^{kq}}{\Gamma(kq+1)}. \tag{1.6}$$

The formula remains also valid for $q \to 1^-$. In this case, $E_1(t-t_0) = \exp(t-t_0) - 1$.

The paper is organized as follows. In Section 2 we recall the definitions of fractional integral and derivative and related basic properties and preliminary results used in the text. In Section 3 we obtain the unique solution of the constant coefficient homogeneous and nonhomogeneous linear fractional differential equations for P being the diagonal matrix. In Section 4 we prove the existence and uniqueness of these two kinds of equations for any $P \in L(R^m)$. In Section 5 we give some specific examples to illustrate the results.

2. Definitions and Preliminary Results

Let us denote by $C[t_0, T]$ the space of all continuous real functions defined on $[t_0, T]$, which turns out to be a Banach space with the norm

$$\|x\| = \max_{t \in [t_0, T]} |x(t)|. \tag{2.1}$$

We define similarly another Banach space $C_{1-q}[t_0, T]$, in which function $x(t)$ is continuous on $(t_0, T]$ and $(t-t_0)^{1-q}x(t)$ is continuous on $[t_0, T]$ with the norm:

$$\|x\|_{1-q} = \max_{t \in [t_0, T]} (t-t_0)^{1-q}|x(t)|. \tag{2.2}$$

$L[t_0, T]$ is the space of real functions defined on $[t_0, T]$ which are Lebesgue integrable on $[t_0, T]$.

Obviously $C_{1-q}[t_0, T] \subset L(t_0, T)$.

The definitions and results of the fractional calculus reported below are not exhaustive but rather oriented to the subject of this paper. For the proofs, which are omitted, we refer the reader to [6] or other texts on basic fractional calculus.

Definition 2.1 (see [6]). The fractional primitive of order $q > 0$ of function $x(t) \in C_{1-q}[t_0, T]$ is given by

$$I_{t_0}^q x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} x(s)\,ds. \tag{2.3}$$

From [17] we know $I_{t_0}^q x(t)$ exists for all $q > 0$, when $x \in C_{1-q}[t_0, T]$; consider also that when $x \in C[t_0, T]$ then $I_{t_0}^q x(t) \in C[t_0, T]$ and moreover

$$I_{t_0}^q x(t_0) = 0. \tag{2.4}$$

Definition 2.2 (see [6]). The fractional derivative of order $0 < q < 1$ of a function $x(t) \in C_{1-q}[t_0, T]$ is given by

$$D_{t_0}^q x(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_{t_0}^t (t-s)^{-q} x(s) ds. \tag{2.5}$$

We have $D_{t_0}^q I_{t_0}^q x(t) = x(t)$ for all $x(t) \in C_{1-q}[t_0, T]$.

Lemma 2.3 (see [6]). *Let $0 < q < 1$. If one assumes $x(t) \in C_{1-q}[t_0, T]$, then the fractional differential equation*

$$D_{t_0}^q x(t) = 0 \tag{2.6}$$

has $x(t) = c(t-t_0)^{q-1}, c \in R$, as solutions.

From this lemma we can obtain the following law of composition.

Lemma 2.4 (see [6]). *Assume that $x(t) \in C_{1-q}[t_0, T]$ with a fractional derivative of order $0 < q < 1$ that belongs to $C_{1-q}[t_0, T]$. Then*

$$I_{t_0}^q D_{t_0}^q x(t) = x(t) + c(t-t_0)^{q-1}, \tag{2.7}$$

for some $c \in R$. When the function x is in $C[t_0, T]$, then $c = 0$.

Lemma 2.5 (see [6]). *Let U be a nonempty closed subset of a Banach space E, and let $\alpha_n \geq 0$ for every n and such $\sum_{n=0}^{\infty} \alpha_n$ converges. Moreover, let the mapping $A : U \rightarrow U$ satisfy the inequality*

$$\|A^n u - A^n v\| \leq \alpha_n \|u - v\|, \tag{2.8}$$

for every $n \in N$ and any $u, v \in U$. Then, A has a uniquely defined fixed point u^*. Furthermore, for any $u_0 \in U$, the sequence $(A^n u_0)_{n=1}^{\infty}$ converges to this fixed point u^*.

Lemma 2.6 (see [12]). *Let $P \in L(R^m)$ and have real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$. Then there exists a basis of R^m in which the matrix representation of P assumes Jordan form, that is, the matrix of P is made of diagonal blocks of the form $\operatorname{diag}(J_1, J_2, \ldots, J_r)$, where each J_i consists of diagonal blocks of the form*

$$\begin{pmatrix} \lambda_i & 0 & \cdots & 0 & 0 \\ 1 & \lambda_i & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda_i \end{pmatrix}. \tag{2.9}$$

Lemma 2.7 (see [12]). *Let $P \in L(R^m)$ and have complex eigenvalues $\mu_j = \alpha_j + i\beta_j$, $j = 1, 2, \ldots, r$, with multiplicity. Then there exists a basis of R^m, where P has matrix form* $\operatorname{diag}(\widehat{J}_1, \widehat{J}_2, \ldots, \widehat{J}_r)$*, where each $\widehat{J}_i$ consists of diagonal blocks of the type*

$$\begin{pmatrix} D & 0 & \cdots & 0 & 0 \\ I_2 & D & \cdots & 0 & 0 \\ 0 & I_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_2 & D \end{pmatrix}, \quad D = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}, \; I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.10}$$

Lemma 2.8 (see [12]). *Let $P \in L(R^m)$. Then R^m has a basis giving P a matrix representation composed of diagonal blocks of type J_i and/or matrices $\widehat{J}_i$, where J_i and $\widehat{J}_i$ are as defined in the preceding lemmas.*

Now, we will introduce Lemma 2.9 to prove the following Theorem 4.4 in Section 4.

Lemma 2.9. *Let $0 < q < 1$. Assume that $x(t)$ and $f(t)$ belong to $C_{1-q}[t_0, T]$. Then For the initial value problem*

$$D_{t_0}^q x(t) = \lambda x(t) + f(t), \quad x(a) = b \tag{2.11}$$

has a unique solution $x(t) \in C_{1-q}[t_0, T]$ provided $t_0 < a < a_0$, where a_0 is a suitable constant depending on t_0, q, and λ.

Proof. The initial value problem (2.11) will be solved in two steps.

(1) Local existence.

Our problem is equivalent to the problem of determination of fixed points of the following operator:

$$Ax(t) = c(t - t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} (\lambda x(s) + f(s)) ds, \tag{2.12}$$

with

$$c = \left(b - \frac{1}{\Gamma(q)} \int_{t_0}^{a} (a - s)^{q-1} (\lambda x(s) + f(s)) ds \right) (a - t_0)^{1-q}. \tag{2.13}$$

It is immediate to verify that $A : C_{1-q}[t_0,T] \to C_{1-q}[t_0,T]$ is also well defined. Indeed,

$$\begin{aligned}
&\left|(t-t_0)^{1-q}Ax(t)\right| \\
&\quad\le |\lambda|\left|\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(s-t_0)^{q-1}(s-t_0)^{1-q}x(s)ds\right| \\
&\qquad+\left|\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(s-t_0)^{q-1}(s-t_0)^{1-q}f(s)ds\right| \\
&\quad\le |c|+|\lambda|\|x\|_{1-q}\left|I^{q}(t-t_0)^{q-1}\right|+\|f\|_{1-q}\left|I^{q}(t-t_0)^{q-1}\right| \\
&\quad\le |c|+|\lambda|\|x\|_{1-q}\frac{\Gamma(q)}{\Gamma(2q)}(t-t_0)^{q}+\|f\|_{1-q}\frac{\Gamma(q)}{\Gamma(2q)}(t-t_0)^{q},
\end{aligned} \tag{2.14}$$

for $x(t)$ and $f(t)$ belong to $\in C_{1-q}[t_0,T]$.

Then we can also prove A is a contraction operator. Indeed,

$$\begin{aligned}
(t-t_0)^{1-q}\left|Ax(t)-Ay(t)\right| &\le |\lambda|\frac{\Gamma(q)}{\Gamma(2q)}(a-t_0)^{q}\|x-y\|_{1-q}+|\lambda|\frac{\Gamma(q)}{\Gamma(2q)}(t-t_0)^{q}\|x-y\|_{1-q} \\
&\le |\lambda|\frac{\Gamma(q)}{\Gamma(2q)}(a-t_0)^{q}\|x-y\|_{1-q}+|\lambda|\frac{\Gamma(q)}{\Gamma(2q)}(T-t_0)^{q}\|x-y\|_{1-q},
\end{aligned} \tag{2.15}$$

for all $x(t), y(t) \in C_{1-q}[t_0,T]$. Let us assume

$$|\lambda|\frac{\Gamma(q)}{\Gamma(2q)}(a-t_0)^{q} < \frac{1}{2}, \tag{2.16}$$

that is,

$$a < a_0 = \left(\frac{\Gamma(2q)}{2|\lambda|\Gamma(q)}\right)^{1/q} + t_0. \tag{2.17}$$

Taking $T-a>0$ sufficiently small, we also have

$$|\lambda|\frac{\Gamma(q)}{\Gamma(2q)}(T-t_0)^{q} < \frac{1}{2}, \tag{2.18}$$

and then

$$\left\|Ax(t)-Ay(t)\right\|_{1-q} \le L\left\|x-y\right\|_{1-q}, \tag{2.19}$$

with $L < 1$. Therefore A is a contraction operator. This shows that initial problem (2.11) has a unique solution.

(2) Continuation of solution.

Since we know the value of $x(t)$ on $(t_0, a]$, then we can compute

$$c_* = \left(b - \frac{1}{\Gamma(q)} \int_{t_0}^{a} (a-s)^{q-1} (\lambda x(s) + f(s)) ds \right) (a-t_0)^{1-q}. \tag{2.20}$$

We can solve the integral problem

$$y(t) = c_*(t-t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} (\lambda y(s) + f(s)) ds, \tag{2.21}$$

obtaining a unique solution $y(t) \in C_{1-q}[t_0, T]$ for all $T > t_0$. Now $x(t)$ and $y(t)$ agree on $(t_0, a]$. Thus the solution admits $y(t)$ as its continuation. Hence the proof of Lemma 2.9 is complete . □

3. Initial Value Problem: Continuous Solutions on $(t_0, T]$

We open this section with some basic examples, concerning the case when the solutions in $C_{1-q}[t_0, T] \times C_{1-q}[t_0, T] \times \cdots \times C_{1-q}[t_0, T]$ are submitted to an initial condition.

Theorem 3.1. *Let $0 < q < 1$. For all $(a, B) \in (t_0, T] \times R^m$ the initial value problem*

$$D_{t_0}^{q} X(t) = 0, \quad X(a) = B \tag{3.1}$$

admits

$$X(t) = B(a-t_0)^{1-q}(t-t_0)^{q-1}, \tag{3.2}$$

as unique solution in $C_{1-q}[t_0, T] \times C_{1-q}[t_0, T] \times \cdots \times C_{1-q}[t_0, T]$.

Proof. According to Lemma 2.4, the initial value problem (3.1) is equivalent to the following equations:

$$X(t) = C(t-t_0)^{q-1}, \qquad C = B(a-t_0)^{1-q}. \tag{3.3}$$

Hence the proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let $0 < q < 1$. Assume*

$$F(t) = (f_1(t), f_2(t), \ldots, f_m(t))^T \in C_{1-q}[t_0, T] \times C_{1-q}[t_0, T] \times \cdots \times C_{1-q}[t_0, T]. \tag{3.4}$$

Then for all $(a, B) \in (t_0, T] \times R^m$ *the initial value problem*

$$D_{t_0}^q X(t) = F(t), \quad X(a) = B \tag{3.5}$$

has a unique solution in

$$C_{1-q}[t_0, T] \times C_{1-q}[t_0, T] \times \cdots \times C_{1-q}[t_0, T], \tag{3.6}$$

given by

$$X(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T, \tag{3.7}$$

with

$$x_i(t) = \left(b_i - \frac{1}{\Gamma(q)} \int_{t_0}^{a} (a-s)^{q-1} f_i(s) ds \right)(a-t_0)^{1-q}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f_i(s) ds, \quad (i = 1, 2, \ldots, m). \tag{3.8}$$

Proof. According to Lemma 2.4, the initial value problem (3.5) is equivalent to the following equations:

$$\begin{aligned} X(t) &= C(t-t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} F(s) ds, \\ C &= \left(B - \frac{1}{\Gamma(q)} \int_{t_0}^{a} (a-s)^{q-1} F(s) ds \right)(a-t_0)^{1-q}. \end{aligned} \tag{3.9}$$

Hence the proof of Theorem 3.2 is complete. □

The result remains true even if $q \to 1^-$. In this case, (3.5) is reduced to the ordinary differential equations

$$X'(t) = F(t), \quad X(a) = B, \tag{3.10}$$

which have a unique solution in

$$C[t_0, T] \times C[t_0, T] \times \cdots \times C[t_0, T], \tag{3.11}$$

given by

$$X(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T, \tag{3.12}$$

with

$$x_i(t) = b_i - \int_{t_0}^{a} f_i(s)ds + \int_{t_0}^{t} f_i(s)ds, \quad (i = 1, 2, \ldots, m). \tag{3.13}$$

Theorem 3.3. *Let $0 < q < 1$. For all $(a, B) \in (t_0, T] \times R^m$ the initial value problem*

$$D_{t_0}^q X(t) = PX(t), \quad X(a) = B, \tag{3.14}$$

where

$$P = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_m \end{pmatrix} \tag{3.15}$$

has a unique solution in

$$C_{1-q}[t_0, T] \times C_{1-q}[t_0, T] \times \cdots \times C_{1-q}[t_0, T], \tag{3.16}$$

given by

$$X(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T, \tag{3.17}$$

with

$$x_i(t) = b_i e_q^{-1}\left(\lambda_i^{1/q}(a - t_0)\right) e_q\left(\lambda_i^{1/q}(t - t_0)\right), \quad (i = 1, 2, \ldots, m). \tag{3.18}$$

Proof. We can write (3.27) in the following form:

$$\begin{gathered} D_{t_0}^q x_1(t) = \lambda_1 x_1(t), \\ D_{t_0}^q x_2(t) = \lambda_2 x_2(t), \\ \vdots \\ D_{t_0}^q x_m(t) = \lambda_m x_m(t), \\ x_1(a) = b_1, \ldots, x_m(a) = b_m. \end{gathered} \tag{3.19}$$

According to Lemma 2.4,

$$D_{t_0}^q x_i(t) = \lambda_i x_i(t) \tag{3.20}$$

is equivalent to the following equations:

$$x_i(t) = c_i(t-t_0)^{q-1} + I_{t_0}^{q}(\lambda_i x_i(t)), \tag{3.21}$$

for some $c_i \in R$. From (3.21) we obtain, by iteration,

$$x_i(t) = c_i\Gamma(q)\left(\frac{(t-t_0)^{q-1}}{\Gamma(q)} + \frac{\lambda_i(t-t_0)^{2q-1}}{\Gamma(2q)} + \cdots + \frac{\lambda_i^{n-1}(t-t_0)^{nq-1}}{\Gamma(nq)}\right) + \lambda_i^n I_{t_0}^{nq} x_i(t). \tag{3.22}$$

Letting $n \to +\infty$, $\|\lambda_i^n I_{t_0}^{nq} x_i(t)\|_{1-q} \to 0$ if $x_i(t) \in C_{1-q}[t_0, T]$. Indeed,

$$\left\|\lambda_i^n I_{t_0}^{nq} x_i(t)\right\|_{1-q} \le \left|\lambda_i^n\right| \|x_i(t)\|_{1-q} \frac{\Gamma(q)}{\Gamma((n+1)q)}(t-t_0)^{nq}. \tag{3.23}$$

On the other hand,

$$e_q(t-t_0) = \sum_{k=1}^{+\infty} \frac{(t-t_0)^{kq-1}}{\Gamma(kq)}, \tag{3.24}$$

then we can obtain

$$x_i(t) = c_i\Gamma(q)\lambda_i^{(1/q-1)} e_q\left(\lambda_i^{1/q}(t-t_0)\right). \tag{3.25}$$

Since $x_i(a) = b_i$,

$$x_i(t) = b_i e_q^{-1}\left(\lambda_i^{1/q}(a-t_0)\right) e_q\left(\lambda_i^{1/q}(t-t_0)\right), \quad (i = 1, 2, \ldots, m). \tag{3.26}$$

Hence the proof of Theorem 3.3 is complete.

The result remains valid even if $q \to 1^-$. In this case,

$$X'(t) = PX(t), \quad X(a) = B \tag{3.27}$$

has a unique solution in

$$C[t_0, T] \times C[t_0, T] \times \cdots \times C[t_0, T], \tag{3.28}$$

given by

$$X(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T, \tag{3.29}$$

with

$$x_i(t) = b_i \exp(-\lambda_i(a - t_0)) \exp(\lambda_i(t - t_0)), \quad (i = 1, 2, \ldots, m). \tag{3.30}$$

□

Theorem 3.4. *Let $0 < q < 1$. For all $(a, B) \in (t_0, T] \times R^m$ the initial value problem*

$$D_{t_0}^q X(t) = PX(t) + D, \quad X(a) = B, \tag{3.31}$$

where

$$P = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_m \end{pmatrix}, \tag{3.32}$$

and $D = (d_1, d_2, \ldots, d_m)$ has a unique solution in

$$C_{1-q}[t_0, T] \times C_{1-q}[t_0, T] \times \cdots \times C_{1-q}[t_0, T], \tag{3.33}$$

given by

$$X(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T, \tag{3.34}$$

with

$$x_i(t) = \left(b_i - d_i\lambda_i^{-1}E_q\left(\lambda_i^{1/q}(a - t_0)\right)\right)e_q^{-1}\left(\lambda_i^{1/q}(a - t_0)\right)e_q\left(\lambda_i^{1/q}(t - t_0)\right) + d_i\lambda_i^{-1}E_q\left(\lambda_i^{1/q}(t - t_0)\right) \quad (i = 1, 2, \ldots, m). \tag{3.35}$$

Proof. We can write (3.44) in the following form:

$$\begin{gathered} D_{t_0}^q x_1(t) = \lambda_1 x_1(t) + d_1, \\ D_{t_0}^q x_2(t) = \lambda_2 x_2(t) + d_2, \\ \vdots \\ D_{t_0}^q x_m(t) = \lambda_m x_m(t) + d_m, \\ x_1(a) = b_1, \ldots, x_m(a) = b_m. \end{gathered} \tag{3.36}$$

According to Lemma 2.4, the equation

$$D_{t_0}^q x_i(t) = \lambda_i x_i(t) + d_i \tag{3.37}$$

is equivalent to the following equations:

$$x_i(t) = c_i(t-t_0)^{q-1} + I_{t_0}^q(\lambda_i x_i(t)) + I_{t_0}^q(d_i), \tag{3.38}$$

for some $c_i \in R$. From (3.38) we obtain, by iteration,

$$\begin{aligned}
I_{t_0}^q \lambda_i x_i(t) &= I_{t_0}^q\left(c_i\lambda_i(t-t_0)^{q-1}\right) + I_{t_0}^{2q}\left(\lambda_i^2 x_i(t)\right) + I_{t_0}^{2q}(\lambda_i d_i),\\
x_i(t) &= c_i(t-t_0)^{q-1} + I_{t_0}^q\left(\lambda_i c_i(t-t_0)^{q-1}\right) + I_{t_0}^{2q}\left(\lambda_i^2 x_i(t)\right) + I_{t_0}^q(d_i) + I_{t_0}^{2q}(\lambda_i d_i),\\
I_{t_0}^q x_i(t) &= I_{t_0}^q\left(c_i(t-t_0)^{q-1}\right) + I_{t_0}^{2q}(\lambda_i x_i(t)) + I_{t_0}^{2q}(d_i),\\
x_i(t) &= c_i\Gamma(q)\left(\frac{(t-t_0)^{q-1}}{\Gamma(q)} + \frac{\lambda_i(t-t_0)^{2q-1}}{\Gamma(2q)} + \cdots + \frac{\lambda_i^{n-1}(t-t_0)^{nq-1}}{\Gamma(nq)}\right)\\
&\quad + d_i\left(\frac{(t-t_0)^q}{\Gamma(q+1)} + \frac{\lambda_i(t-t_0)^{2q}}{\Gamma(2q+1)} + \cdots + \frac{\lambda_i^{n-1}(t-t_0)^{nq}}{\Gamma(nq+1)}\right) + \lambda_i^n I_{t_0}^{nq} x_i(t).
\end{aligned} \tag{3.39}$$

Letting $n \to +\infty$, $\|\lambda_i^n I_{t_0}^{nq} x_i(t)\|_{1-q} \to 0$ if $x_i(t) \in C_{1-q}[t_0, T]$. Indeed,

$$\begin{aligned}
\left\|\lambda_i^n I_{t_0}^{nq} x_i(t)\right\|_{1-q} &= \max_{t\in[t_0,T]}(t-t_0)^{1-q}\left|\lambda_i^n \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{nq-1} x_i(s)ds\right|\\
&\le \left|\lambda_i^n\right| \|x_i(t)\|_{1-q} \frac{\Gamma(q)}{\Gamma((n+1)q)}(t-t_0)^{nq}.
\end{aligned} \tag{3.40}$$

On the other hand,

$$\begin{aligned}
e_q(t-t_0) &= \sum_{k=1}^{+\infty} \frac{(t-t_0)^{kq-1}}{\Gamma(kq)},\\
E_q(t-t_0) &= \sum_{k=1}^{+\infty} \frac{(t-t_0)^{kq}}{\Gamma(kq+1)}.
\end{aligned} \tag{3.41}$$

Then we can obtain

$$x_i(t) = c_i\Gamma(q)\lambda_i^{(1/q-1)} e_q\left(\lambda_i^{1/q}(t-t_0)\right) + d_i\lambda_i^{-1}E_q\left(\lambda_i^{1/q}(t-t_0)\right). \tag{3.42}$$

We know that $d_i\lambda_i^{-1}E_q(\lambda_i^{1/q}(t-t_0))$ is satisfied for the fractional nonhomogeneous linear differential equation $D_{t_0}^q x_i(t) = \lambda_1 x_i(t) + d_i$. So we can also deduce that the general solution of the fractional nonhomogeneous linear differential equation is equal to the general solution of the corresponding homogeneous linear differential equation plus the special solution of the nonhomogeneous linear differential equation. If $X(a) = B$, $x_i(a) = b_i$, then

$$x_i(t) = \left(b_i - d_i\lambda_i^{-1}E_q\left(\lambda_i^{1/q}(a-t_0)\right)\right)e_q^{-1}\left(\lambda_i^{1/q}(a-t_0)\right)e_q\left(\lambda_i^{1/q}(t-t_0)\right) + d_i\lambda_i^{-1}E_q\left(\lambda_i^{1/q}(t-t_0)\right) \quad (i = 1, 2, \ldots, m). \tag{3.43}$$

Hence the proof of Theorem 3.4 is complete. □

The result remains valid even if $q \to 1^-$. In this case,

$$X'(t) = PX(t) + D, \quad X(a) = B, \tag{3.44}$$

where

$$P = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_m \end{pmatrix}, \tag{3.45}$$

and $D = (d_1, d_2, \ldots, d_m)$ has a unique solution in

$$C[t_0, T] \times C[t_0, T] \times \cdots \times C[t_0, T], \tag{3.46}$$

given by

$$X(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T, \tag{3.47}$$

with

$$\begin{aligned} x_i(t) &= \left(b_i - d_i\lambda_i^{-1}E_1(\lambda_i(a-t_0))\right)e_1^{-1}(\lambda_i(a-t_0))e_1(\lambda_i(t-t_0)) + d_i\lambda_i^{-1}E_1(\lambda_i(t-t_0)) \\ &= \left(b_i + d_i\lambda_i^{-1}\exp(-\lambda(a-t_0))\right)\exp(\lambda_i(t-t_0)) - d\lambda^{-1}, \quad (i = 1, 2, \ldots, m). \end{aligned} \tag{3.48}$$

4. Existence and Uniqueness of the Solution

In Section 3 we have obtained the unique solution of the constant coefficient homogeneous and nonhomogeneous linear fractional differential equations for P being the diagonal matrix. In the present section we will prove the existence and uniqueness of these two kinds of equations for any $P \in L(R^m)$.

Theorem 4.1. *Let $0 < q < 1$ and $P \in L(R^m)$. If the matrix P has distinct real eigenvalues, then for all $(a, B) \in (t_0, T] \times R^m$ the initial value problem*

$$D_{t_0}^q X(t) = PX(t), \quad X(a) = B \tag{4.1}$$

has the unique solution $X(t) \in C_{1-q}[t_0, T] \times C_{1-q}[t_0, T] \times \cdots \times C_{1-q}[t_0, T]$.

Proof. Since the matrix P has distinct real eigenvalues, there exists an invertible matrix Q such that

$$Q^{-1}PQ = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_m \end{pmatrix}, \tag{4.2}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of the matrix P. If we define $Y(t) = Q^{-1}X(t)$,

$$D_{t_0}^q Y(t) = D_{t_0}^q Q^{-1}X(t) = Q^{-1}D_{t_0}^q X(t) = Q^{-1}PX(t) = RY(t), \tag{4.3}$$

with $R = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$. From the above Theorem 3.3 we know the initial value problem

$$D_{t_0}^q Y(t) = RY(t), \quad Y(a) = Q^{-1}B \tag{4.4}$$

has a unique solution $Y(t) \in C_{1-q}[t_0, T] \times C_{1-q}[t_0, T] \times \cdots \times C_{1-q}[t_0, T]$ defined on $[t_0, T]$. Then $X(t) = QY(t)$ uniquely solves the equations (4.1), where $t \in [t_0, T]$. Hence the proof of Theorem 4.1 is complete . □

Theorem 4.2. *Let $0 < q < 1$. For all $(a, B) \in (t_0, T] \times R^2$ the initial value problem*

$$D_{t_0}^q X(t) = PX(t), \quad X(a) = B, \tag{4.5}$$

where

$$P = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \tag{4.6}$$

has the unique solution $X(t) \in C_{1-q}[t_0, T] \times C_{1-q}[t_0, T]$ defined on $[t_0, T]$.

Proof. Let us define

$$Z(t) = x_1(t) + ix_2(t), \quad \mu = \alpha + i\beta. \tag{4.7}$$

We can find that (4.5) is equivalent to the following equation

$$D_{t_0}^q Z(t) = \mu Z(t), \quad Z(a) = x_1(a) + x_2(a) = b_1 + ib_2. \tag{4.8}$$

Obviously, $Z(t) \in C_{1-q}[t_0, T]$ if $x_1(t)$ and $x_2(t)$ belong to $C_{1-q}[t_0, T]$. From the above Theorem 3.3 in Section 3, we know the complex Equation (4.7) has a unique solution defined on $[t_0, T]$. Hence the proof of Theorem 4.2 is complete. □

Theorem 4.3. *Let $0 < q < 1$ and $P \in R^2$. If P has eigenvalues $\alpha \pm i\beta$, for all $(a, B) \in (t_0, T] \times R^2$ the initial value problem*

$$D_{t_0}^q X(t) = PX(t), \quad X(a) = B \tag{4.9}$$

has a unique solution $X(t) \in C_{1-q}[t_0, T] \times C_{1-q}[t_0, T]$.

Proof. Since P has eigenvalues $\alpha \pm i\beta$, there exists an invertible matrix Q such that $P = QSQ^{-1}$ where

$$S = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{4.10}$$

Define

$$Y(t) = Q^{-1}X(t), \tag{4.11}$$

then

$$D_{t_0}^q Y(t) = D_{t_0}^q Q^{-1}X(t) = Q^{-1}D_{t_0}^q X(t) = Q^{-1}PX(t) = SY(t). \tag{4.12}$$

From the above Theorem 4.2, we know the initial value problem

$$D_{t_0}^q Y(t) = SY(t), \quad Y(a) = Q^{-1}B \tag{4.13}$$

has a unique solution defined on $[t_0, T]$. Hence the proof of result is complete. □

Theorem 4.4. *Let $0 < q < 1$ and $P \in R^m$ be an elementary Jordan matrix:*

$$\begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}. \tag{4.14}$$

The initial value problem

$$D_{t_0}^q X(t) = PX(t), \quad X(a) = B \tag{4.15}$$

has a unique solution $X(t) \in C_{1-q}[t_0, T] \times C_{1-q}[t_0, T] \times \cdots \times C_{1-q}[t_0, T]$ provided $(a, B) \in (t_0, a_0] \times R^m$, where a_0 is a suitable constant depending on t_0, q, and λ.

Proof. From the (4.15), we can write the equations in the following form:

$$\begin{aligned} D_{t_0}^q x_1(t) &= \lambda x_1(t), \\ D_{t_0}^q x_2(t) &= x_1(t) + \lambda x_2(t), \\ &\vdots \\ D_{t_0}^q x_m(t) &= x_{m-1}(t) + \lambda x_m(t), \\ x_1(a) = b_1, &\ldots, x_m(a) = b_m. \end{aligned} \tag{4.16}$$

Consider the first equation

$$D_{t_0}^q x_1(t) = \lambda x_1(t), \quad x_1(a) = b_1. \tag{4.17}$$

We can obtain the solution of this equation

$$x_1(t) = b_1 e_q^{-1}\left(\lambda^{1/q}(a - t_0)\right) e_q\left(\lambda^{1/q}(t - t_0)\right). \tag{4.18}$$

Consider the second equation

$$D_{t_0}^q x_2(t) = x_1(t) + \lambda x_2(t), \quad x_2(a) = b_2, \tag{4.19}$$

where now $x_1(t) \in C_{1-q}[t_0, T]$ is a known function. Since $x_1(t), x_2(t) \in C_{1-q}[t_0, T]$, according to Lemma 2.9, (4.19) has a unique solution in $C_{1-q}[t_0, T]$. Now $x_1(t)$ and $x_2(t)$ are known functions which will be substituted in

$$D_{t_0}^q x_3(t) = x_2(t) + \lambda x_3(t), \quad x_3(a) = b_3 \tag{4.20}$$

and so on. Thus the system of equations given in (4.15) has unique solution in $C_{1-q}[t_0, T] \times C_{1-q}[t_0, T] \times \cdots \times C_{1-q}[t_0, T]$. □

Theorem 4.5. *Let $0 < q < 1$ and $P \in L(R^m)$. The initial value problem*

$$D_{t_0}^q X(t) = PX(t), \quad X(a) = B \tag{4.21}$$

has the unique solution $X(t) \in C_{1-q}[t_0, T] \times C_{1-q}[t_0, T] \times \cdots \times C_{1-q}[t_0, T]$ provided $(a, B) \in (t_0, a_0] \times R^m$, where a_0 is a suitable constant depending on t_0, q, and λ.

Proof. In view of Lemma 2.8, there exists an invertible matrix Q such that $Q^{-1}PQ$ is composed of diagonal blocks of the type J_i and $\widehat{J}_i$, as defined in the preceding Lemmas 2.7 and 2.8. Let $B = Q^{-1}PQ$ and $Y(t) = Q^{-1}X(t)$. Consider the initial value problem:

$$\begin{gathered} D_{t_0}^q Y(t) = D_{t_0}^q Q^{-1}X(t) = Q^{-1}D_{t_0}^q X(t) = Q^{-1}PX(t) = BY(t), \\ Y(a) = Q^{-1}X(a) = Q^{-1}B. \end{gathered} \tag{4.22}$$

Then in view of Theorems 4.1–4.5, (4.22) has a unique solution: $Y(t) \in C_{1-q}[t_0,T] \times C_{1-q}[t_0,T] \times \cdots \times C_{1-q}[t_0,T]$. Therefore (4.21) has a unique solution $Q^{-1}Y(t) \in C_{1-q}[t_0,T] \times C_{1-q}[t_0,T] \times \cdots \times C_{1-q}[t_0,T]$. □

Remark 4.6. All the above results are valid for $q \to 1^-$. Moreover, we can also discuss the case if $a = t_0$, in this case, we cannot consider the usual initial condition $x(t_0) = b$, but $\lim_{t\to t_0}(t-t_0)^{1-q}x(t) = b$. We can also obtain some similar results by the same method, So we did not give the detailed process and conclusion in this paper.

5. Illustrative Examples

In this section, we give some specific examples to illustrate the above results.

Example 5.1. Consider the following system, where $0 < q < 1$, $t \in [t_0,T]$, $(a,B) \in (t_0,T] \times R^3$,$B = (b_1,b_2,b_3)$,

$$\begin{gathered} D_{t_0}^q x_1(t) = 3x_1(t) - x_2(t) + x_3(t), \\ D_{t_0}^q x_2(t) = -x_1(t) + 5x_2(t) - x_3(t), \\ D_{t_0}^q x_3(t) = x_1(t) - x_2(t) + 3x_3(t), \\ x_1(a) = b_1, \quad x_2(a) = b_2, \quad x_3(a) = b_3. \end{gathered} \tag{5.1}$$

Here

$$P = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}, \tag{5.2}$$

having the eigenvalues 2, 3, and 6. Choose the eigenvectors $g_1 = (1,0,-1)^T$, $g_2 = (1,1,1)^T$, and $g_3 = (1,-2,1)^T$. Then

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} = Q^{-1}\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}Q, \tag{5.3}$$

where

$$Q^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}. \tag{5.4}$$

Define the $Y = Q^{-1}X$. Then the system of equation in Y is decoupled, namely,

$$\begin{gathered} D_{t_0}^{q} y_1(t) = 2y_1(t), \\ D_{t_0}^{q} y_2(t) = 3y_2(t), \\ D_{t_0}^{q} y_3(t) = 6y_3(t), \\ y_1(a) = \frac{1}{2}b_1 - \frac{1}{2}b_3, \\ y_2(a) = \frac{1}{3}b_1 + \frac{1}{3}b_2 + \frac{1}{3}b_3, \\ y_3(a) = \frac{1}{6}b_1 - \frac{1}{3}b_2 + \frac{1}{6}b_3. \end{gathered} \tag{5.5}$$

In view of (3.30), we can obtain

$$\begin{aligned} y_1(t) &= \left(\frac{1}{2}b_1 - \frac{1}{2}b_3\right) e_q^{-1}\left(2^{1/q}(a - t_0)\right) e_q\left(2^{1/q}(t - t_0)\right), \\ y_2(t) &= \left(\frac{1}{3}b_1 + \frac{1}{3}b_2 + \frac{1}{3}b_3\right) e_q^{-1}\left(3^{1/q}(a - t_0)\right) e_q\left(3^{1/q}(t - t_0)\right), \\ y_3(t) &= \left(\frac{1}{6}b_1 - \frac{1}{3}b_2 + \frac{1}{6}b_3\right) e_q^{-1}\left(6^{1/q}(a - t_0)\right) e_q\left(6^{1/q}(t - t_0)\right). \end{aligned} \tag{5.6}$$

Hence

$$\begin{gathered} x_1(t) = y_1(t) + y_2(t) + y_3(t), \\ x_2(t) = y_2(t) - 2y_3(t), \\ x_3(t) = -y_1(t) + y_2(t) + y_3(t). \end{gathered} \tag{5.7}$$

Example 5.2. Consider the following system, where $0 < q < 1$, $t \in [t_0, T]$, $(a, B) \in (t_0, T] \times R^2, B = (b_1, b_2)$,

$$\begin{aligned} D_{t_0}^q x_1(t) &= -2x_1(t) - x_2(t) \\ D_{t_0}^q x_2(t) &= 13x_1(t) + 4x_2(t) \\ x_1(a) &= b_1, \quad x_2(a) = b_2. \end{aligned} \tag{5.8}$$

Here

$$P = \begin{pmatrix} -2 & -1 \\ 13 & 4 \end{pmatrix} \tag{5.9}$$

having the eigenvalues $1 \pm 2i$. Choose the eigenvectors $g_1 = (1, -3-2i)^T$, and $g_2 = (1, -3+2i)^T$, Then

$$\begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix} = Q^{-1} \begin{pmatrix} -2 & -1 \\ 13 & 4 \end{pmatrix} Q, \tag{5.10}$$

where

$$\begin{aligned} Q &= \begin{pmatrix} 1 & 1 \\ -3-2i & -3+2i \end{pmatrix}, \\ Q^{-1} &= \begin{pmatrix} \dfrac{2+3i}{4} & \dfrac{i}{4} \\ \dfrac{2-3i}{4} & -\dfrac{i}{4} \end{pmatrix}. \end{aligned} \tag{5.11}$$

Define the $Y = Q^{-1}X$. Then the system of equation in Y is decoupled, namely,

$$\begin{aligned} D_{t_0}^q y_1(t) &= (1+2i) y_1(t), \quad y_1(a) = \frac{1}{2} b_1 + \left(\frac{3}{4} b_1 + \frac{1}{4} b_2 \right) i \\ D_{t_0}^q y_2(t) &= (1-2i) y_2(t), \quad y_2(a) = \frac{1}{2} b_1 - \left(\frac{3}{4} b_1 + \frac{1}{4} b_2 \right) i. \end{aligned} \tag{5.12}$$

In view of (3.30), we can obtain

$$\begin{aligned} y_1(t) &= \left(\frac{1}{2} b_1 + \left(\frac{3}{4} b_1 + \frac{1}{4} b_2 \right) i \right) e_q^{-1}\left((1+2i)^{1/q} (a - t_0) \right) e_q\left((1+2i)^{1/q} (t - t_0) \right) \\ y_2(t) &= \left(\frac{1}{2} b_1 - \left(\frac{3}{4} b_1 + \frac{1}{4} b_2 \right) i \right) e_q^{-1}\left((1-2i)^{1/q} (a - t_0) \right) e_q\left((1-2i)^{1/q} (t - t_0) \right). \end{aligned} \tag{5.13}$$

Hence

$$x_1(t) = y_1(t) + y_2(t)$$
$$x_2(t) = -3(y_1(t) + y_2(t)) - 2i(y_1(t) - y_2(t)). \tag{5.14}$$

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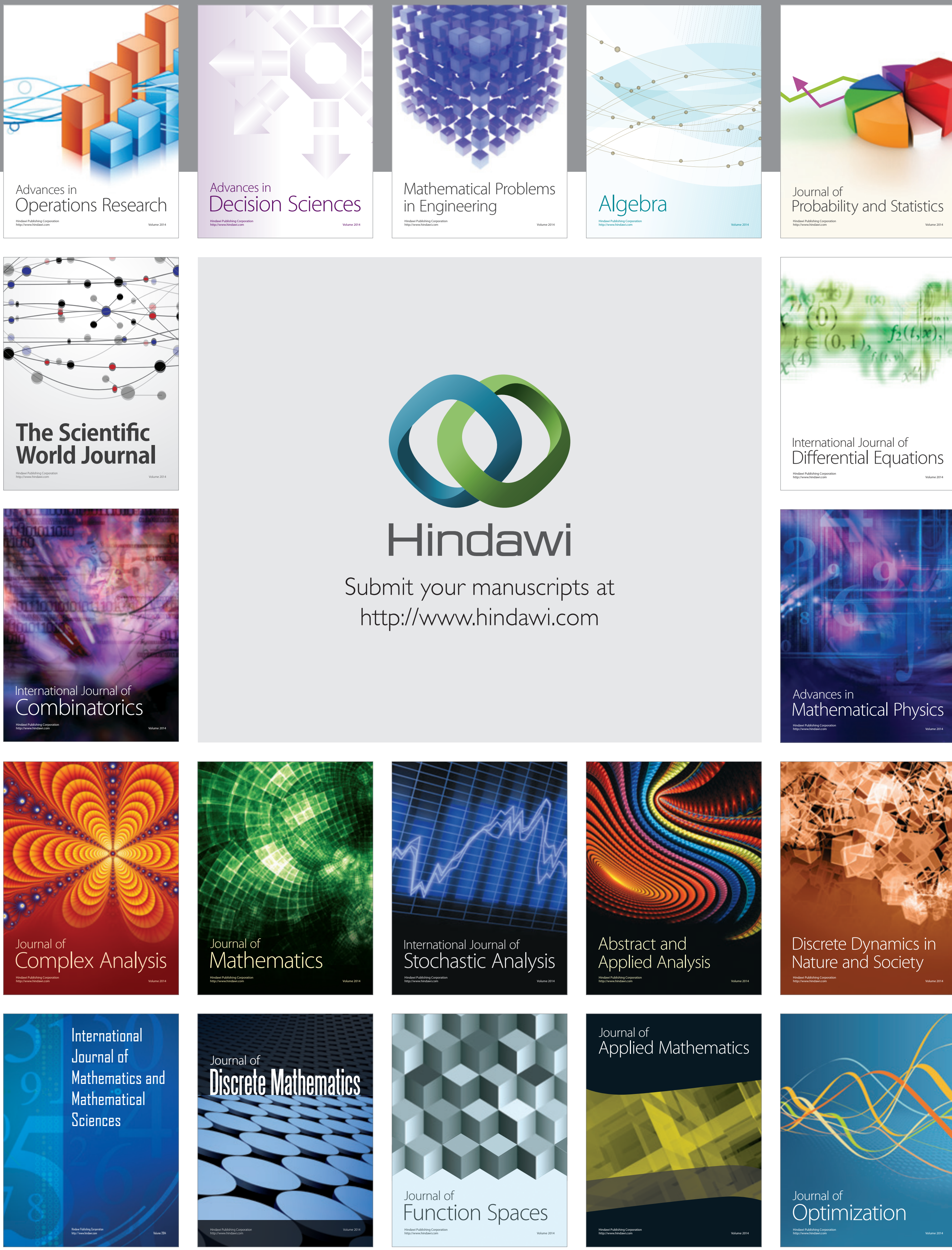
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