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*Research Article*

# Approximation of Common Fixed Points of Nonexpansive Semigroups in Hilbert Spaces

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Let $H$ be a real Hilbert space. Consider on $H$ a nonexpansive semigroup $S = \{T(s) : 0 \leq s < \infty\}$ with a common fixed point, a contraction $f$ with the coefficient $0 < \alpha < 1$, and a strongly positive linear bounded self-adjoint operator $A$ with the coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \overline{\gamma}/\alpha$. It is proved that the sequence $\{x_n\}$ generated by the iterative method $x_0 \in H$, $x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)(1/s_n)\int_0^{s_n} T(s)x_n ds$, $n \geq 0$ converges strongly to a common fixed point $x^* \in F(S)$, where $F(S)$ denotes the common fixed point of the nonexpansive semigroup. The point $x^*$ solves the variational inequality $\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0$ for all $x \in F(S)$.

## 1. Introduction and Preliminaries

Let $H$ be a real Hilbert space and $T$ be a nonlinear mapping with the domain $D(T)$. A point $x \in D(T)$ is a fixed point of $T$ provided $Tx = x$. Denote by $F(T)$ the set of fixed points of $T$; that is, $F(T) = \{x \in D(T) : Tx = x\}$. Recall that $T$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(A). \tag{1.1}$$

Recall that a family $S = \{T(s) \mid s \geq 0\}$ of mappings from $H$ into itself is called a one-parameter nonexpansive semigroup if it satisfies the following conditions:

(i) $T(0)x = x$, for all $x \in H$;

(ii) $T(s + t)x = T(s)T(t)x$, for all $s, t \geq 0$ and for all $x \in H$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$, for all $s \geq 0$ and for all $x, y \in H$;

(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(S)$ the set of common fixed points of $S$, that is, $F(S) = \bigcap_{0\le s<\infty} F(T(s))$. It is known that $F(S)$ is closed and convex; see [1]. Let $C$ be a nonempty closed and convex subset of $H$. One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping; see [2, 3]. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad x \in C, \tag{1.2}$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that $T_t$ has a unique fixed point $x_t$ in $C$. If $T$ enjoys a nonempty fixed point set, Browder [2] proved the following well-known strong convergence theorem.

**Theorem B.** *Let $C$ be a bounded closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping on $C$. Fix $u \in C$ and define $z_t \in C$ as $z_t = tu + (1-t)Tz_t$ for $t \in (0,1)$. Then as $t \to 0$, $\{z_t\}$ converges strongly to a element of $F(T)$ nearest to $u$.*

As motivated by Theorem B, Halpern [4] considered the following explicit iteration:

$$x_0 \in C, \qquad x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \ge 0, \tag{1.3}$$

and proved the following theorem.

**Theorem H.** *Let $C$ be a bounded closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping on $C$. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}$, $0 < \theta < 1$. Define a sequence $\{x_n\}$ by (1.3). Then $\{x_n\}$ converges strongly to the element of $F(T)$ nearest to $u$.*

In 1977, Lions [5] improved the result of Halpern [4], still in Hilbert spaces, by proving the strong convergence of $\{x_n\}$ to a fixed point of $T$ where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty}(\alpha_{n+1} - \alpha_n)/\alpha_{n+1}^2 = 0$.

It was observed that both Halpern's and Lions's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n+1)$. This was overcome in 1992 by Wittmann [6], who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of $T$ if $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Recall that a mapping $f : H \to H$ is an $\alpha$-contraction if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \quad \forall x, y \in H. \tag{1.4}$$

Recall that an operator $A$ is strongly positive on $H$ if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.5}$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [7–13] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping $T$ on a real Hilbert space $H$:

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.6}$$

where $A$ is a linear bounded operator on $H$ and $b$ is a given point in $H$. In [11], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in \mathrm{H}$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \quad n \geq 0, \tag{1.7}$$

strongly converges to the unique solution of the minimization problem (1.6) provided that the sequence $\{\alpha_n\}$ satisfies certain conditions.

Recently, Marino and Xu [9] studied the following continuous scheme:

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t, \tag{1.8}$$

where $f$ is an $\alpha$-contraction on a real Hilbert space $H$, $A$ is a bounded linear strongly positive operator and $\gamma > 0$ is a constant. They showed that $\{x_t\}$ strongly converges to a fixed point $\overline{x}$ of $T$. Also in [9], they introduced a general explicit iterative scheme by the viscosity approximation method:

$$x_n \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0 \tag{1.9}$$

and proved that the sequence $\{x_n\}$ generated by (1.9) converges strongly to a unique solution of the variational inequality

$$\left\langle (A - \gamma f)x^*,\ x - x^*\right\rangle \geq 0, \quad \forall x \in F(T), \tag{1.10}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax,\ x\rangle - h(x), \tag{1.11}$$

where $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In this paper, motivated by Li et al. [8], Marino and Xu [9], Plubtieng and Punpaeng [14], Shioji and Takahashi [15], and Shimizu and Takahashi [16], we consider the mapping $T_t$ defined as follows:

$$T_t x = t\gamma f(x) + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x\,ds, \tag{1.12}$$

where $\gamma > 0$ is a constant, $f$ is an $\alpha$-contraction, $A$ is a bounded linear strongly positive self-adjoint operator and $\{\lambda_t\}$ is a positive real divergent net. If $\gamma\alpha < \overline{\gamma}$ for each $0 < t < \|A\|^{-1}$, one can see that $T_t$ is a $(1 - t(\overline{\gamma} - \gamma\alpha))$-contraction. So, by Banach's contraction mapping principle, there exists an unique solution $x_t$ of the fixed point equation

$$x_t = t\gamma f(x_t) + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds. \tag{1.13}$$

We show that the sequence $\{x_t\}$ generated by above continuous scheme strongly converges to a common fixed point $x^* \in F(S)$, which is the unique point in $F(S)$ solving the variational inequality $\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0$ for all $x \in F(S)$. Furthermore, we also study the following explicit iterative scheme:

$$x_0 \in H, \qquad x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \quad n \geq 0. \tag{1.14}$$

We prove that the sequence $\{x_n\}$ generated by (1.14) converges strongly to the same $x^*$.

The results presented in this paper improve and extend the corresponding results announced by Marino and Xu [9], Plubtieng and Punpaeng [14], Shioji and Takahashi [15], and Shimizu and Takahashi [16].

In order to prove our main result, we need the following lemmas.

**Lemma 1.1** (see [16]). *Let $D$ be a nonempty bounded closed convex subset of a Hilbert space $H$ and let $S = \{T(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup on $D$. Then, for any $0 \leq h < \infty$,*

$$\limsup_{t\to\infty}\sup_{x\in D}\left\|\frac{1}{t}\int_0^t T(s)x\,ds - T(h)\frac{1}{t}\int_0^t T(s)x\,ds\right\| = 0. \tag{1.15}$$

**Lemma 1.2** (see [17]). *Let $H$ be a Hilbert space, $C$ a closed convex subset of $H$, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is demiclosed, that is, if $\{x_n\}$ is a sequence in $C$ weakly converging to $x$ and if $\{(I-T)x_n\}$ strongly converges to $y$, then $(I-T)x = y$.*

**Lemma 1.3** (see [18]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $P_C$ be the metric projection from $H$ onto $C$( i.e., for $x \in H$, $P_C x$ is the only point in $C$ such that $\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}$). Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relations*

$$\langle x - z,\ y - z\rangle \leq 0, \quad \forall y \in C. \tag{1.16}$$

**Lemma 1.4.** *Let $H$ be a Hilbert space, $f$ a $\alpha$-contraction, and $A$ a strongly positive linear bounded self-adjoint operator with the coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \overline{\gamma}/\alpha$,*

$$\langle x - y,\ (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\overline{\gamma} - \gamma\alpha)\|x - y\|^2, \quad x, y \in H. \tag{1.17}$$

*That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \alpha\gamma$.*

*Proof.* From the definition of strongly positive linear bounded operator, we have

$$\langle x - y,\ A(x - y) \rangle \geq \overline{\gamma}\|x - y\|^2. \tag{1.18}$$

On the other hand, it is easy to see

$$\langle x - y,\ \gamma f x - \gamma f y \rangle \leq \gamma\alpha\|x - y\|^2. \tag{1.19}$$

Therefore, we have

$$\begin{aligned} \langle x - y,\ (A - \gamma f)x - (A - \gamma f)y \rangle &= \langle x - y,\ A(x - y) \rangle - \langle x - y,\ \gamma f x - \gamma f y \rangle \\ &\geq (\overline{\gamma} - \gamma\alpha)\|x - y\|^2 \end{aligned} \tag{1.20}$$

for all $x, y \in H$. This completes the proof. □

*Remark 1.5.* Taking $\gamma = 1$ and $A = I$, the identity mapping, we have the following inequality:

$$\langle x - y,\ (I - f)x - (I - f)y \rangle \geq (1 - \alpha)\|x - y\|^2, \quad x, y \in H. \tag{1.21}$$

Furthermore, if $f$ is a nonexpansive mapping in Remark 1.5, we have

$$\langle x - y,\ (I - f)x - (I - f)y \rangle \geq 0, \quad x, y \in H. \tag{1.22}$$

**Lemma 1.6** (see [9]). *Assume $A$ is a strongly positive linear bounded self-adjoint operator on a Hilbert space $H$ with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

**Lemma 1.7** (see [12]). *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the following condition:*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\sigma_n, \quad \forall n \geq 0, \tag{1.23}$$

*where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence of real numbers such that*

(i) $\lim_{n\to\infty}\gamma_n = 0$ *and* $\sum_{n=0}^{\infty}\gamma_n = \infty$,

(ii) *either* $\limsup_{n\to\infty}\sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\sigma_n| < \infty$.

*Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.*

## 2. Main Results

**Lemma 2.1.** *Let $H$ a real Hilbert space and $S = \{T(s) : 0 \le s < \infty\}$ a nonexpansive semigroup on $H$ such that $F(S) \ne \emptyset$. Let $\{\lambda_t\}_{0<t<1}$ be a continuous net of positive real numbers such that $\lim_{t\to 0}\lambda_t = \infty$. Let $f : H \to H$ be an $\alpha$-contraction, $A$ a strongly positive linear bounded self-adjoint operator of $H$ into itself with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{x_t\}$ be a sequence defined by* (1.13). *Then*

(i) $\{x_t\}$ *is bounded for all* $t \in (0, \|A\|^{-1})$;

(ii) $\lim_{t\to 0}\|T(\tau)x_t - x_t\| = 0$ *for all* $0 \le \tau < \infty$;

(iii) $x_t$ *defines a continuous curve from* $(0, \|A\|^{-1})$ *into* $H$.

*Proof.* (i) Taking $p \in F(S)$, we have

$$\begin{aligned}
\|x_t - p\| &\le \left\| t\gamma f(x_t) + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - p \right\| \\
&\le t\|\gamma f(x_t) - Ap\| + (1 - t\overline{\gamma})\frac{1}{\lambda_t}\int_0^{\lambda_t} \|T(s)x_t - p\| ds \\
&\le t\|\gamma f(x_t) - Ap\| + (1 - t\overline{\gamma})\|x_t - p\| \\
&\le t\gamma\|f(x_t) - f(p)\| + t\|\gamma f(p) - Ap\| + (1 - t\overline{\gamma})\|x_t - p\| \\
&\le [1 - t(\overline{\gamma} - \gamma\alpha)]\|x_t - p\| + t\|\gamma f(p) - Ap\|.
\end{aligned} \tag{2.1}$$

It follows that

$$\|x_t - p\| \le \frac{1}{\overline{\gamma} - \alpha\gamma}\|\gamma f(p) - Ap\|. \tag{2.2}$$

This implies that $\{x_t\}$ is not only bounded, but also that $\{x_t\}$ is contained in $B(p, 1/(\overline{\gamma} - \gamma\alpha)\|\gamma f(p) - Ap\|)$ of center $p$ and radius $1/(\overline{\gamma} - \gamma\alpha)\|\gamma f(p) - Ap\|$, for all fixed $p \in F(S)$. Moreover for $p \in F(S)$ and $t \in (0, \|A\|^{-1})$,

$$\begin{aligned}
\left\| \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - p \right\| &= \left\| \frac{1}{\lambda_t}\int_0^{\lambda_t} (T(s)x_t - T(s)p) ds \right\| \\
&\le \|x_t - p\| \\
&\le \frac{1}{\overline{\gamma} - \gamma\alpha}\|\gamma f(p) - Ap\|.
\end{aligned} \tag{2.3}$$

(ii) Observe that

$$\begin{aligned}
\|T(\tau)x_t - x_t\| \le{}& \left\| T(\tau)x_t - T(\tau)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\right) \right\| \\
&+ \left\| T(\tau)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\right) - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds \right\|
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_t ds-x_t\right\|\\
&\le 2\left\|x_t-\frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_t ds\right\|\\
&\quad+\left\|T(\tau)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_t ds\right)-\frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_t ds\right\|\\
&=2t\left\|\gamma f(x_t)-A\frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_t ds\right\|\\
&\quad+\left\|T(\tau)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_t ds\right)-\frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_t ds\right\|.
\end{aligned}
\tag{2.4}
$$

Taking $B(p,1/(\overline{\gamma}-\gamma\alpha)\|\gamma f(p)-Ap\|)$ as $D$ in Lemma 1.1 and passing to $\lim_{t\to 0}$ in (2.4), we can obtain (ii) immediately.

(iii) Taking $t_1,t_2\in(0,\|A\|^{-1})$ and fixing $p\in F(S)$, we see that

$$
\begin{aligned}
&\|x_{t_1}-x_{t_2}\|\\
&\quad\le\left\|(t_1-t_2)\gamma f(x_{t_1})+t_2\gamma\big(f(x_{t_1})-f(x_{t_2})\big)-(t_1-t_2)A\frac{1}{\lambda_{t_1}}\int_0^{\lambda_{t_1}}T(s)x_{t_1}ds\right.\\
&\qquad\left.+(I-t_2A)\left(\frac{1}{\lambda_{t_1}}\int_0^{\lambda_{t_1}}T(s)x_{t_1}ds-\frac{1}{\lambda_{t_2}}\int_0^{\lambda_{t_2}}T(s)x_{t_2}ds\right)\right\|\\
&\quad\le|t_1-t_2|\gamma\left\|f(x_{t_1})\right\|+t_2\gamma\alpha\|x_{t_1}-x_{t_2}\|+|t_1-t_2|\|A\|\left\|\frac{1}{\lambda_{t_1}}\int_0^{\lambda_{t_1}}T(s)x_{t_1}ds\right\|\\
&\qquad+(1-t_2\overline{\gamma})\left\|\frac{1}{\lambda_{t_1}}\int_0^{\lambda_{t_1}}T(s)x_{t_1}ds-\frac{1}{\lambda_{t_2}}\int_0^{\lambda_{t_1}}T(s)x_{t_2}ds-\frac{1}{\lambda_{t_2}}\int_{\lambda_{t_1}}^{\lambda_{t_2}}T(s)x_{t_2}ds\right\|\\
&\quad\le|t_1-t_2|\gamma\left\|f(x_{t_1})\right\|+t_2\gamma\alpha\|x_{t_1}-x_{t_2}\|+|t_1-t_2|\|A\|\left\|\frac{1}{\lambda_{t_1}}\int_0^{\lambda_{t_1}}T(s)x_{t_1}ds\right\|\\
&\qquad+(1-t_2\overline{\gamma})\left(\|x_{t_1}-x_{t_2}\|+\left|\frac{1}{\lambda_{t_1}}-\frac{1}{\lambda_{t_2}}\right|\left\|\int_0^{\lambda_{t_1}}T(s)x_{t_2}ds\right\|+\frac{1}{\lambda_{t_2}}\left\|\int_{\lambda_{t_1}}^{\lambda_{t_2}}T(s)x_{t_2}ds\right\|\right).
\end{aligned}
\tag{2.5}
$$

Thus applying (2.3), we arrive at

$$
\begin{aligned}
&\|x_{t_1}-x_{t_2}\|\\
&\quad\le|t_1-t_2|\gamma\left\|f(x_{t_1})\right\|+t_2\gamma\alpha\|x_{t_1}-x_{t_2}\|+|t_1-t_2|\|A\|\left(\frac{1}{\overline{\gamma}-\gamma\alpha}\|\gamma f(p)-Ap\|+\|p\|\right)\\
&\qquad+(1-t_2\overline{\gamma})\left(\|x_{t_1}-x_{t_2}\|+\frac{2}{\lambda_{t_2}}|\lambda_{t_2}-\lambda_{t_1}|\left(\frac{1}{\overline{\gamma}-\gamma\alpha}\|\gamma f(p)-Ap\|+\|p\|\right)\right)
\end{aligned}
$$

$$\begin{aligned}
&\leq |t_1 - t_2|\left(\gamma\|f(x_{t_1})\| + \|A\|\left(\frac{1}{\overline{\gamma}-\gamma\alpha}\|\gamma f(p) - Ap\| + \|p\|\right)\right) \\
&\quad + (1 - t_2(\overline{\gamma}-\gamma\alpha))\|x_{t_1} - x_{t_2}\| + \frac{2}{\lambda_{t_2}}|\lambda_{t_2} - \lambda_{t_1}|\left(\frac{1}{\overline{\gamma}-\gamma\alpha}\|\gamma f(p) - Ap\| + \|p\|\right).
\end{aligned} \tag{2.6}$$

It follows that

$$\|x_{t_1} - x_{t_2}\| \leq M_1|t_1 - t_2| + M_2|\lambda_{t_2} - \lambda_{t_1}|, \tag{2.7}$$

where

$$M_1 = \frac{\gamma(\overline{\gamma}-\gamma\alpha)\|f(x_{t_1})\| + \|A\|\|\gamma f(p) - Ap\| + (\overline{\gamma}-\gamma\alpha)\|A\|\|p\|}{t_2(\overline{\gamma}-\gamma\alpha)^2} \tag{2.8}$$

and

$$M_2 = \frac{2(\|\gamma f(p) - Ap\| + (\overline{\gamma}-\gamma\alpha)\|p\|)}{\lambda_{t_2} t_2(\overline{\gamma}-\gamma\alpha)^2}. \tag{2.9}$$

This inequality, together with the continuity of the net $\{\lambda_t\}$, gives the continuity of the curve $\{x_t\}$. □

**Theorem 2.2.** *Let $H$ be a real Hilbert space $H$ and $S = \{T(s) : 0 \leq s < \infty\}$ a nonexpansive semigroup such that $F(S) \neq \emptyset$. Let $\{\lambda_t\}_{0<t<1}$ be a net of positive real numbers such that $\lim_{t\to 0}\lambda_t = \infty$. Let $f$ be an $\alpha$-contraction and let $A$ be a strongly positive linear bounded self-adjoint operator on $H$ with the coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Then sequence $\{x_t\}$ defined by* (1.13) *strongly converges as $t \to 0$ to $x^* \in F(S)$, which solves the following variational inequality:*

$$\langle(\gamma f - A)x^*,\ p - x^*\rangle \leq 0, \quad \forall p \in F(S). \tag{2.10}$$

*Equivalently, one has*

$$P_{F(S)}(I - A + \gamma f)x^* = x^*. \tag{2.11}$$

*Proof.* The uniqueness of the solution of the variational inequality (2.10) is a consequence of the strong monotonicity of $A - \gamma f$ (Lemma 1.4) and it was proved in [9]. Next, we will use $x^* \in F(S)$ to denote the unique solution of (2.10). To prove that $x_t \to x^*$ $(t \to 0)$, we write, for a given $p \in F(S)$,

$$x_t - p = t(\gamma f(x_t) - Ap) + (I - tA)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - p\right). \tag{2.12}$$

Using $x_t - p$ to make inner product, we obtain that

$$\begin{aligned}\|x_t - p\|^2 &= \left\langle (I - tA)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - p\right),\ x_t - p\right\rangle + t\langle \gamma f(x_t) - Ap,\ x_t - p\rangle \\ &\leq (1 - t\overline{\gamma})\|x_t - p\|^2 + t\langle \gamma f(x_t) - Ap,\ x_t - p\rangle.\end{aligned} \tag{2.13}$$

It follows that

$$\begin{aligned}\|x_t - p\|^2 &\leq \frac{1}{\overline{\gamma}}(\gamma\langle f(x_t) - f(p), x_t - p\rangle + \langle \gamma f(p) - Ap, x_t - p\rangle) \\ &\leq \frac{\gamma\alpha}{\overline{\gamma}}\|x_t - p\|^2 + \frac{1}{\overline{\gamma}}\langle \gamma f(p) - Ap,\ x_t - p\rangle,\end{aligned} \tag{2.14}$$

which yields that

$$\|x_t - p\|^2 \leq \frac{1}{\overline{\gamma} - \alpha\gamma}\langle \gamma f(p) - Ap, x_t - p\rangle. \tag{2.15}$$

Since $H$ is a Hilbert space and $\{x_t\}$ is bounded as $t \to 0$, we have that if $\{t_n\}$ is a sequence in $(0,1)$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup \overline{x}$. By (2.15), we see $x_{t_n} \to \overline{x}$. Moreover, by (ii) of Lemma 2.1 we have $\overline{x} \in F(S)$. We next prove that $\overline{x}$ solves the variational inequality (2.10). From (1.13), we arrive at

$$(A - \gamma f)x_t = -\frac{1}{t}(I - tA)\left[x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\right]. \tag{2.16}$$

For $p \in F(S)$, it follows from (1.22) that

$$\begin{aligned}\langle (A - \gamma f)x_t, x_t - p\rangle &= -\frac{1}{t}\left\langle (I - tA)\left[x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\right], x_t - p\right\rangle \\ &= -\frac{1}{t}\left\langle \frac{1}{\lambda_t}\int_0^{\lambda_t}[(I - T(s))x_t - (I - T(s))p]ds, x_t - p\right\rangle \\ &\quad + \left\langle A\frac{1}{\lambda_t}\int_0^{\lambda_t}(I - T(s))x_t ds,\ x_t - p\right\rangle \\ &= -\frac{1}{t\lambda_t}\int_0^{\lambda_t}\langle (I - T(s))x_t - (I - T(s))p,\ x_t - p\rangle ds \\ &\quad + \left\langle A\frac{1}{\lambda_t}\int_0^{\lambda_t}(I - T(s))x_t ds,\ x_t - p\right\rangle \\ &\leq \left\langle A\frac{1}{\lambda_t}\int_0^{\lambda_t}(I - T(s))x_t ds,\ x_t - p\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \left\langle A\left(t\gamma f(x_t) - tA\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\right), x_t - p\right\rangle \\
&= t\left\langle A\left(\gamma f(x_t) - A\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\right), x_t - p\right\rangle.
\end{aligned}
\tag{2.17}
$$

Passing to $\lim_{t\to 0}$, since $\{x_t\}$ is a bounded sequence, we obtain

$$\langle (A - \gamma f)\overline{x},\ \overline{x} - p\rangle \le 0, \tag{2.18}$$

that is, $\overline{x}$ satisfies the variational inequality (2.10). By the uniqueness it follows $\overline{x} = x^*$. In a summary, we have shown that each cluster point of $\{x_t\}$ (as $t \to 0$) equals $x^*$. Therefore, $x_t \to x^*$ as $t \to 0$. The variational inequality (2.10) can be rewritten as

$$\langle [(I - A + \gamma f)x^*] - x^*,\ x^* - p\rangle, \quad p \in F(S). \tag{2.19}$$

This, by Lemma 1.3, is equivalent to

$$P_{F(S)}(I - A + \gamma f)x^* = x^*. \tag{2.20}$$

This completes the proof. □

*Remark 2.3.* Theorem 2.2 which include the corresponding results of Shioji and Takahashi [15] as a special case is reduced to Theorem 3.1 of Plubtieng and Punpaeng [14] when $A = I$, the identity mapping and $\gamma = 1$.

**Theorem 2.4.** *Let $H$ be a real Hilbert space $H$ and $S = \{T(s) : 0 \le s < \infty\}$ a nonexpansive semigroup such that $F(S) \ne \emptyset$. Let $\{s_n\}$ be a positive real divergent sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ satisfying the following conditions $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Let $f$ be an $\alpha$-contraction and let $A$ be a strongly positive linear bounded self-adjoint operator with the coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Then sequence $\{x_n\}$ defined by (1.14) strongly converges to $x^* \in F(S)$, which solves the variational inequality (2.10).*

*Proof.* We divide the proof into three parts.

*Step 1.* Show the sequence $\{x_n\}$ is bounded.

Noticing that $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$, we may assume, with no loss of generality, that $\alpha_n/(1-\beta_n) < \|A\|^{-1}$ for all $n \ge 0$. From Lemma 1.6, we know that $\|(1-\beta_n)I - \alpha_n A\| \le (1 - \beta_n - \alpha_n\overline{\gamma})$. Picking $p \in F(S)$, we have

$$
\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \left\|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)\left(\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - p\right)\right\|
\end{aligned}
$$

$$\begin{aligned}
&\le \alpha_n \|\gamma f(x_n) - Ap\| + \beta_n \|x_n - p\| + (1-\beta_n - \alpha_n \overline{\gamma}) \left\| \frac{1}{s_n} \int_0^{s_n} T(s) x_n ds - p \right\| \\
&\le \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\ \| + \beta_n \|x_n - p\| + (1-\beta_n - \alpha_n \overline{\gamma}) \|x_n - p\| \\
&\le [1 - \alpha_n(\overline{\gamma} - \gamma\alpha)] \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|.
\end{aligned} \tag{2.21}$$

By simple inductions, we see that

$$\|x_n - p\| \le \max\left\{ \|x_0 - p\|, \frac{\|Ap - \gamma f(p)\|}{\overline{\gamma} - \gamma\alpha} \right\}, \tag{2.22}$$

which yields that the sequence $\{x_n\}$ is bounded.

*Step 2.* Show that

$$\limsup_{n\to\infty} \langle (\gamma f - A)x^*, y_n - x^* \rangle \le 0, \tag{2.23}$$

where $x^*$ is obtained in Theorem 2.2 and $y_n = (1/s_n)\int_0^{s_n} T(s)x_n ds$.

Putting $z_0 = P_{F(S)}x_0$, from (2.22) we see that the closed ball $M$ of center $z_0$ and radius $\max\{\|z_0 - p\|, \|Az_0 - \gamma f(z_0)\|/(\overline{\gamma} - \gamma\alpha)\}$ is $T(s)$-invariant for each $s \in [0, \infty)$ and contain $\{x_n\}$. Therefore, we assume, without loss of generality, $S = \{T(s) : 0 \le s < \infty\}$ is a nonexpansive semigroup on $M$. It follows from Lemma 1.1 that

$$\lim_{n\to\infty} \|y_n - T(h)y_n\| = 0 \tag{2.24}$$

for all $0 \le h < \infty$. Taking a suitable subsequence $\{y_{n_i}\}$ of $\{y_n\}$, we see that

$$\limsup_{n\to\infty} \langle (\gamma f - A)x^*,\ y_n - x^* \rangle = \lim_{i\to\infty} \langle (\gamma f - A)x^*,\ y_{n_i} - x^* \rangle. \tag{2.25}$$

Since the sequence $\{y_n\}$ is also bounded, we may assume that $y_{n_i} \rightharpoonup \overline{x}$. From the demiclosedness principle, we have $\overline{x} \in F(S)$. Therefore, we have

$$\limsup_{n\to\infty} \langle (\gamma f - A)x^*,\ y_n - x^* \rangle = \langle (\gamma f - A)x^*,\ \overline{x} - x^* \rangle \le 0. \tag{2.26}$$

On the other hand, we have

$$\|x_{n+1} - y_n\| \le \alpha_n \|\gamma f(x_n) - Ax_n\| + \beta_n \|x_n - y_n\|. \tag{2.27}$$

From the assumption $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$, we see that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0, \tag{2.28}$$

which combines with (2.26) gives that

$$\limsup_{n\to\infty}\langle(\gamma f-A)x^*,x_{n+1}-x^*\rangle\le 0. \tag{2.29}$$

*Step 3.* Show $x_n \to x^*$ as $n \to \infty$.

Note that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\quad=\langle\alpha_n(\gamma f(x_n)-Ax^*)+\beta_n(x_n-x^*)+((1-\beta_n)I-\alpha_nA)(y_n-x^*),\ x_{n+1}-x^*\rangle\\
&\quad=\alpha_n\langle\gamma f(x_n)-Ax^*,\ x_{n+1}-x^*\rangle+\beta_n\langle x_n-x^*,\ x_{n+1}-x^*\rangle\\
&\qquad+\langle((1-\beta_n)I-\alpha_nA)(y_n-x^*),\ x_{n+1}-x^*\rangle\\
&\quad\le\alpha_n(\gamma\langle f(x_n)-f(x^*),\ x_{n+1}-x^*\rangle+\langle\gamma f(x^*)-Ax^*,\ x_{n+1}-x^*\rangle)\\
&\qquad+\beta_n\|x_n-x^*\|\|x_{n+1}-x^*\|+\|(1-\beta_n)I-\alpha_nA\|\|y_n-x^*\|\|x_{n+1}-x^*\|\\
&\quad\le\alpha_n\alpha\gamma\|x_n-x^*\|\|x_{n+1}-x^*\|+\alpha_n\langle\gamma f(x^*)-Ax^*,\ x_{n+1}-x^*\rangle\\
&\qquad+\beta_n\|x_n-x^*\|\|x_{n+1}-x^*\|+(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-x^*\|\|x_{n+1}-x^*\|\\
&\quad=[1-\alpha_n(\overline{\gamma}-\gamma\alpha)]\|x_n-x^*\|\|x_{n+1}-x^*\|+\alpha_n\langle\gamma f(x^*)-Ax^*,\ x_{n+1}-x^*\rangle\\
&\quad\le\frac{1-\alpha_n(\overline{\gamma}-\gamma\alpha)}{2}\left(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\right)+\alpha_n\langle\gamma f(x^*)-Ax^*,\ x_{n+1}-x^*\rangle.\\
&\quad\le\frac{1-\alpha_n(\overline{\gamma}-\gamma\alpha)}{2}\|x_n-x^*\|^2+\frac{1}{2}\|x_{n+1}-x^*\|^2+\alpha_n\langle\gamma f(x^*)-Ax^*,\ x_{n+1}-x^*\rangle.
\end{aligned}\tag{2.30}$$

It follows that

$$\|x_{n+1}-x^*\|^2\le[1-\alpha_n(\overline{\gamma}-\gamma\alpha)]\|x_n-x^*\|^2+2\alpha_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle. \tag{2.31}$$

By using Lemma 1.7, we can obtain the desired conclusion easily. □

*Remark 2.5.* If $\gamma=1$ and $A=I$, the identity mapping, then Theorem 2.4 is reduced to Theorem 3.3 of Plubtieng and Punpaeng [14].

If the sequence $\{\beta_n\}\equiv 0$, then Theorem 2.4 is reduced to the following.

**Corollary 2.6.** *Let $H$ be a real Hilbert space $H$ and $S=\{T(s):0\le s<\infty\}$ a nonexpansive semigroup such that $F(S)\ne\emptyset$. Let $\{s_n\}$ be a positive real divergent sequence and let $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying the following conditions $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$. Let $f$ be a $\alpha$-contraction and let $A$ be a strongly positive linear bounded self-adjoint operator with the coefficient $\overline{\gamma}>0$. Assume that $0<\gamma<\overline{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0\in H,\qquad x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_nA)\frac{1}{s_n}\int_0^{s_n}T(s)x_n\,ds,\quad n\ge 0. \tag{2.32}$$

*Then the sequence $\{x_n\}$ defined by above iterative algorithm converges strongly to $x^*\in F(S)$, which solves the variational inequality* (2.10).

*Remark 2.7.* Corollary 2.6 includes Theorem 2 of Shioji and Takahashi [15] as a special case.

*Remark 2.8.* Theorem 2.2 and Corollary 2.6 improve Theorem 3.2 and Theorem 3.4 of Marino and Xu [9] from a single nonexpansive mapping to a nonexpansive semigroup, respectively.

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## References

[1] F. E. Browder, "Nonexpansive nonlinear operators in a Banach space," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 54, pp. 1041–1044, 1965.
[2] F. E. Browder, "Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces," *Archive for Rational Mechanics and Analysis*, vol. 24, pp. 82–90, 1967.
[3] S. Reich, "Strong convergence theorems for resolvents of accretive operators in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 75, no. 1, pp. 287–292, 1980.
[4] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.
[5] P.-L. Lions, "Approximation de points fixes de contractions," *Comptes Rendus de l'Académie des Sciences*, vol. 284, no. 21, pp. A1357–A1359, 1977.
[6] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," *Archiv der Mathematik*, vol. 58, no. 5, pp. 486–491, 1992.
[7] F. Deutsch and I. Yamada, "Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 19, no. 1-2, pp. 33–56, 1998.
[8] S. Li, L. Li, and Y. Su, "General iterative methods for a one-parameter nonexpansive semigroup in Hilbert space," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 9, pp. 3065–3071, 2009.
[9] G. Marino and H.-K. Xu, "A general iterative method for nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.
[10] G. Marino, V. Colao, X. Qin, and S. M. Kang, "Strong convergence of the modified Mann iterative method for strict pseudo-contractions," *Computers & Mathematics with Applications*, vol. 57, no. 3, pp. 455–465, 2009.
[11] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.
[12] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.
[13] I. Yamada, N. Ogura, Y. Yamashita, and K. Sakaniwa, "Quadratic optimization of fixed points of nonexpansive mappings in Hilbert space," *Numerical Functional Analysis and Optimization*, vol. 19, no. 1-2, pp. 165–190, 1998.
[14] S. Plubtieng and R. Punpaeng, "Fixed-point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces," *Mathematical and Computer Modelling*, vol. 48, no. 1-2, pp. 279–286, 2008.
[15] N. Shioji and W. Takahashi, "Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 34, no. 1, pp. 87–99, 1998.
[16] T. Shimizu and W. Takahashi, "Strong convergence to common fixed points of families of nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 211, no. 1, pp. 71–83, 1997.
[17] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28, Cambridge University Press, Cambridge, UK, 1990.
[18] S.-S. Chang, Y. J. Cho, and H. Zhou, *Iterative Methods for Nonlinear Operator Equations in Banach Spaces*, Nova Science Publishers, Huntington, NY, USA, 2002.